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On Integrated Convex Optimization in Normed Linear Space

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Abstract In this paper, the concept of generalized saddle point(GSP) is employed to discuss the optimization problems of a set of convex functions on a normed linear space X, which presents an equivalence under a special condition between GSP and its optimum solution. A study on integrated convex optimization problem by using Gâteaux and Fréchet differentiability respectivly, and the equivalent relationships among GSP, Gâteaux and Fréchet differentiability respectively, and optimum solution are concerned in this paper.

Keywords Convex optimization; Sun point; Gâteaux-differentiable; Fréchet-differentiable; Generalized saddle point; Normed linear space

1 Introduction

It is well known that convex optimization in Banach space has been much interested by people because of its wide range of application in mathematics and engineering. The book written by Th. Precupanu and Viorel Barbu [1] shows that convexity is very essential and useful in optimization theories. The characterization theorems of the theory of best approximation, given in the book written by Ivan singer [2], can be generalized to convex optimization. It shows the relationship between best approximation and convex optimization in locally convex space using subdifferentials and directional derivatives. The content of best approximation and optimization also can be found in [8]. Nonlinear optimization in normed linear spaces has been discussed by S. Y. Xu [3], but not in depth.

In this paper, concept of generalized saddle point (GSP) introduced by J. Li [12] is employed to study the relationship between integrated optimum solution of a set of convex function on a normed linear space and the GSP, i.e., the equivalence under a special condition. Integrated convex optimization in normed linear space is still studied by employing the Gâteaux and Fréchet differentiability, and relationship between GSP and optimum solution is also concerned in this paper.

The rest of this paper is organized as follows. Section 2 presents a study on integrated convex optimization problem (Γ, G) by using the tools of Gâteaux and Fréchet differentiability respectivly. In section 3, we integrated the content of section 2 with the concept of the generalized saddle point to discuss the equivalence between optimum solution and the GSP.

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2 Integrated Optimization Problem in Normed Linear Space

Assume X be a normed linear space, and X^* dual space of X, namely, the set of all the linear functional on X, and H be a set of convex real-valued functions ϕs on X, i.e., $\phi : X \to \mathbb{R}$, where $\mathbb{R}$ is the real number field. We define $\Gamma = \sup\limits_{\phi \in H} \phi$, that is to say, $\Gamma(x) = \sup\limits_{\phi \in H} \phi(x), x \in X$. Let G be a subset of X, we concern optimization problem

$$(\Gamma, G): \qquad \inf_{g \in G} \Gamma(g).$$

Let $g_0 \in G$, if it satisfies $\Gamma(g_0) = \inf\limits_{g \in G} \Gamma(g)$, then we call g_0 be an optimum solution of (Γ, G), all of which are denoted as a set $P_{(\Gamma,G)}$, namely,

$$P_{(\Gamma,G)} = \{g_0 \in G : \quad \Gamma(g_0) = \inf_{g \in G} \Gamma(g)\}.$$

Let $B^* = \{f \in X^* : \quad ||f||_{X^*} \leq 1\}$ (the unit ball of X^*), and $||\cdot||_X$ and $||\cdot||_{X^*}$ be the norms of X and X^* respectively, of which all the concepts can be found in [10,16] . When ϕ is defined as $\phi(\cdot) = |f(x - \cdot)|$, where $x \in X$, $f \in B^*$, the optimization problem (Γ, G) can be changed to be

$$\inf_{g \in G} \Gamma(g) = \inf_{g \in G} \sup_{f \in B^*} |f(x-g)| = \inf_{g \in G} ||x-g||_X,$$

which, in fact, is the best approximation of x by the elements of G, which has been investigated in many literatures and books such as [2,4,5,6,7,8,11] etc..

We have known that $\Gamma(x)$ is a convex function on X [12]. It is well known that convex function on X is continuous, so is the function $\Gamma(x)$. Now we introduce the following notations for conciseness of the discussion.

$$M_{(\Gamma,x)} = \{\phi \in H : \ \Gamma(x) = \phi(x), \quad x \in X\}$$

$$U(x_0, \delta) = \{x : \ \|x - x_0\| < \delta, \quad x \in X\}$$

$$\tilde{G}_{g_0} = \bigcup_{g \in G} \{g_\alpha : \quad g_\alpha = (1-\alpha)g_0 + \alpha g, \quad \alpha \in [0,1], g_0 \in G\},$$

Definition 1. We call ϕ to be Fréchet differentiable [14] at x_0 if there exists a $f_x \in X^*$ such that

$$\lim_{\|h\| \to 0^+} \frac{|\phi(x_0+h) - \phi(x_0)) - f_{x_0}(h)|}{\|h\|} = 0$$

for all $h \in X$, and its Fréchet derivative is denoted by $\partial\phi(x_0, h)$, namely $\partial\phi(x_0, h) = f_{x_0}(h)$.

In a more familiar case, $\phi : \mathbb{R}^n \to \mathbb{R}^m$, the derivative $\partial\phi$ is simply the Jacobian of ϕ.

Definition 2. Assume $x_0 \in X$, $\phi \in M_{(\Gamma,x_0)}$, we call ϕ be a locally maximal function of Γ at x_0, if there exists a positive real number δ, such that $\phi(x) = \Gamma(x)$ for any $x \in U(x_0, \delta)$.

It is evident to see that when ϕ is a locally maximal function of Γ at x_0, we know there exists a positive real number δ, such that $\phi \in M_{(\Gamma,x)}$ for any $x \in U(x_0, \delta)$.

On the basis of the above discussion, it is evident to know the following lemma.

Lemma 1. *Assume $h \in X$, $\phi_0 \in M_{(\Gamma,x_0)}$, if ϕ_0 is a locally maximal function of Γ and Fréchet differentiable at x_0, then $\Gamma(\cdot)$ is also Fréchet differentiable at x_0 and $\partial\Gamma(x_0,h) = \partial\phi(x_0,h)$.*

Definition 3. Assume $x_0, h \in X$, ϕ_0 is called to be Gâteaux differentiable [15,2,16] at x_0 if the limit

$$\phi_0'(x_0,h) = \lim_{\alpha \to 0} \frac{\phi_0(x_0+\alpha h) - \phi_0(x_0)}{\alpha}$$

exists for $h \in X$, and $\phi_0'(x_0,h)$ is called the Gâteaux derivative at x_0 with respect to increment h.

For conciseness of the discussion, we introduce the following lemma which is easy to obtain.

Lemma 2. *Assume $h \in X$, $\phi_0 \in M_{(\Gamma,x_0)}$, if ϕ_0 is a locally maximal function of Γ and Gâteaux differentiable at x_0 with respect to h, then $\Gamma(\cdot)$ is also Gâteaux differentiable at x_0 with respect to increment h and $\Gamma'(x_0,h) = \phi'(x_0,h)$.*

It is well known that the concept of fréchet derivative is stronger than that of the Gâteaux derivative, namely if $\partial\phi(x_0,h)$ exists, then $\phi'(x_0,h)$ is necessary to exist.

Theorem 3. *Assume there exists a $\phi_0 \in M_{(\Gamma,g_0)}$ which is locally maximal function of Γ and Fréchel differentiable at g_0, then $g_0 \in P_{(\Gamma,G)}$ if and only if*

$$\partial\phi_0(g_0, g-g_0) \geq 0 \tag{1}$$

for any $g \in G$.

Proof. Assume $g_0 \in P_{(\Gamma,G)}$, then $\Gamma(g) \geq \Gamma(g_0)$. Because there exists a $\phi_0 \in M_{(\Gamma,g_0)}$ which is a locally maximal function of Γ and Fréchel differentiable at g_0, $\Gamma(\cdot)$ is also Fréchel differentiable at g_0 by lemma 1, that is to say, for an arbitrary positive number ε, there exists a $\delta > 0$, we get that when $\|g-g_0\| < \delta$,

$$\partial\phi_0(g_0,g-g_0) - \varepsilon\|g-g_0\| \leq \Gamma(g) - \Gamma(g_0) \leq \partial\phi_0(g_0,g-g_0) + \varepsilon\|g-g_0\|, \tag{2}$$

which implies $\phi_0(g_0, g-g_0) \geq 0$ by the right inequality of (2) .

Conversely, assume that there exists a $\phi_0 \in M_{(\Gamma,g_0)}$ satisfying

$$\partial\phi_0(g_0, g-g_0) \geq 0.$$

When $\partial\phi_0(g_0,g-g_0) = 0$, we can obtain $\Gamma(g) = \Gamma(g_0)$ for all $g \in G$ by using the inequality (2). If $\partial\phi_0(g_0,g-g_0) > 0$, there exists an $\varepsilon > 0$ such that $\partial\phi_0(g_0,g-g_0) \geq \varepsilon\|g-g_0\|$, which implies that

$$\Gamma(g) - \Gamma(g_0) \geq \partial\phi_0(g_0,g-g_0) - \varepsilon\|g-g_0\| \geq 0$$

by the Fréchel differentiability of ϕ_0 at g_0. □

Definition 4. We say that g_0 is a sun-point (also see [3]) of Γ in G , if $g_0 \in P_{(\Gamma,G)}$ implies $g_0 \in P_{(\Gamma,\tilde{G}_{g_0})}$, where $P_{(\Gamma,\tilde{G}_{g_0})}$ is the optimum solution set of optimization problem $(\Gamma,\tilde{G}_{g_0})$. If every point $g \in G$ is a sun-point of Γ , we refer G to be a sun-set of Γ.

Sun point plays an important role in nonlinear best approximation which has been introduced by Efimov and Stechkin [13], and were concerned in many literatures, some of which have been collected in the monograph written by Braess [5].

Let

$$\gamma(t) = \frac{\phi(g_0 + t(g - g_0)) - \phi(g_0)}{t}, \qquad \forall t \in (0,1],$$

where ϕ is a convex function. For conciseness of the narration, we give a lemma which is easily obtained.

Lemma 4. *Let ϕ is a convex function, then $\gamma(t)$ is an increasing function on* $(0,1]$

Theorem 5. *Assume G is a sun-set of Γ and there exists a $\phi_0 \in M_{(\Gamma,g_0)}$ which is a locally maximal function of Γ and Gâteaux differentiable at g_0, then $g_0 \in P_{(\Gamma,G)}$ if and only if*

$$\phi_0'(g_0, g - g_0) \geq 0$$

for all $g \in G$.

Proof. Assume that there exists a $\phi_0 \in M_{(\Gamma,g_0)}$ which is a locally maximal function of Γ and Gâteaux differentiable at g_0. We get there exists a $\delta > 0$, such that

$$\Gamma(g) \quad = \quad \phi_0(g) \qquad \forall g \in U(g_0, \delta),$$

and

$$\phi_0'(g_0, g - g_0) = \lim_{t \to 0^+} \frac{\phi_0(g_0 + t(g - g_0)) - \phi_0(g_0)}{t},$$

which implies

$$\phi_0'(g_0, g - g_0) = \lim_{t \to 0^+} \frac{\Gamma(g_0 + t(g - g_0)) - \Gamma(g_0)}{t}, \tag{3}$$

through employing lemma 2. By the assumption that G is a sun-set of Γ and $g_0 \in P_{(\Gamma,G)}$, we have $\Gamma(g_0 + t(g - g_0)) \geq \Gamma(g_0)$, $0 < t \leq 1$, which infers

$$\phi_0'(g_0, g - g_0) \geq 0$$

by the equality (3). This is end of the proof of necessity of the theory.

Conversely, if $\phi_0'(g_0, g - g_0) \geq 0$, where $\phi_0 \in M_{(\Gamma,g_0)}$, which is locally maximal function of Γ at g_0. We have known that $\Gamma(x)$ is a convex function on normed linear space X, hence

$$\gamma(t) = \frac{\Gamma(g_0 + t(g - g_0)) - \Gamma(g_0)}{t}$$

is an increasing function on $(0,1]$ by Lemma 4. Consequently we get $\gamma(1) \geq \lim_{t\to 0^+}\gamma(t)$, namely

$$\begin{aligned} & \Gamma(g)-\Gamma(g_0) \\ \geq & \lim_{t\to 0^+}\frac{\Gamma(g_0+t(g-g_0)-\Gamma(g_0)}{t} \\ = & \phi_0'(g_0,g-g_0)\geq 0, \end{aligned}$$

which implies that $g_0 \in P_{(\Gamma,G)}$. □

3 Generalized Saddle Point Solution of (Γ, G)

Let X be a normed linear space, $\tilde{H}$ be a set of all real-valued and convex functions on X. Now we define a functional $\Psi : (\tilde{H},X) \to R$, that is

$$\Psi(\phi,x)=\phi(x), \qquad \forall(\phi,x)\in(\tilde{H},X).$$

Let H be a subset of $\tilde{H}$, and G be a subset of X, we also define

$$\Gamma(x)=\sup_{\phi\in H}\phi(x)=\sup_{\phi\in H}\Psi(\phi,x),$$

then the optimization problem (Γ,G) changes to be

$$\inf_{g\in G}\Gamma(g)=\inf_{g\in G}\sup_{\phi\in H}\Psi(\phi,g).$$

Definition 5. Let $(\bar{\phi},\bar{g}) \in (H,G)$, we call $(\bar{\phi},\bar{g})$ a generalized saddle point (GSP) of Ψ in (H,G), if it satisfies the following condition

$$\Psi(\phi,\bar{g})\leq\Psi(\bar{\phi},\bar{g})\leq\Psi(\bar{\phi},g), \qquad (\phi,g)\in(H,G).$$

The concept of GSP had been introduced in [12]. The notion of saddle point is a fundamental concept in many areas of science and economics. A classical instance is the famous saddle point theorem for a zero-sum matrix game due to J. Von Neumann and O. Morgenstern [9].

Theorem 6. *Let $\bar{\phi} \in M_{(\Gamma,\bar{g})}$ be Gâteaux differentiable at $\bar{g}$, then $(\bar{\phi},\bar{g})$ is a GSP of Ψ in (H,G) if and only if*

$$\bar{\phi}'(\bar{g},g-\bar{g})\geq 0, \quad g\in G.$$

Proof. Assume $(\bar{\phi},\bar{g})$ is a GSP of Ψ in (H,G), we have

$$\Psi(\phi,\bar{g})\leq\Psi(\bar{\phi},\bar{g})\leq\Psi(\bar{\phi},g),$$

namely

$$\phi(\bar{g})\leq\bar{\phi}(\bar{g})\leq\bar{\phi}(g). \tag{4}$$

Therefor, by the inequality (4) and convexity of $\bar{\phi}$, we can get

$$\begin{aligned}\bar{\phi}'(\bar{g}, g-\bar{g}) &= \lim_{\alpha\to 0^+} \frac{\bar{\phi}(\bar{g}+\alpha(g-\bar{g}))-\bar{\phi}(\bar{g})}{\alpha} \\ &\geq \lim_{\alpha\to 0^+} \frac{(1-\alpha)\bar{\phi}(\bar{g})+\alpha\bar{\phi}(g)-\bar{\phi}(\bar{g})}{\alpha} \\ &= \lim_{\alpha\to 0^+} \frac{\alpha[\bar{\phi}(g)-\bar{\phi}(\bar{g})]}{\alpha} \\ &= \bar{\phi}(g)-\bar{\phi}(\bar{g}) \geq 0.\end{aligned}$$

Conversely, we assume that $\bar{\phi}'(\bar{g}, g-\bar{g}) \geq 0$. Let

$$\gamma(\alpha) = \frac{\bar{\phi}(\bar{g}+\alpha(g-\bar{g}))-\bar{\phi}(\bar{g})}{\alpha},$$

where $\alpha \in (0,1]$. When ϕ is a convex function, $\gamma(\alpha)$ is an increasing function on $(0,1]$ by virtue of Lemma 4. Consequently we have $\gamma(1) \geq \lim_{\alpha\to 0^+} \gamma(\alpha)$, which implies

$$\bar{\phi}(g) \geq \bar{\phi}(\bar{g}) \tag{5}$$

Furthermore,

$$\phi(\bar{g}) \leq \sup_{\phi\in H} \phi(\bar{g}) = \bar{\phi}(\bar{g}), \tag{6}$$

because of $\bar{\phi} \in M_{(\Gamma,\bar{g})}$. By the inequalities (5)(6), we have

$$\Psi(\phi,\bar{g}) \leq \Psi(\bar{\phi},\bar{g}) \leq \Psi(\bar{\phi},g), \qquad (\phi,g) \in (H,G).$$

□

It is obvious to know the following corollary from theorem 5 and 6,

Corollary 7.
Let G be a sunset of Γ*, and* $\bar{\phi} \in M_{(\Gamma,\bar{g})}$ *be a locally maximal function and Gâteaux differentiable at* $\bar{g}$ *, then the following statements are equivalent,*

(1) $\bar{g} \in P_{(\Gamma,G)}$*;*
(2) $(\bar{\phi},\bar{g})$ *is GSP of* Ψ *in (H,G);*
(3) $\bar{\phi}'(\bar{g}, g-\bar{g}) \geq 0, \quad g \in G.$

Now we discuss the relation between GSP and Fréchet differentiability in the following,

Theorem 8. *Let* $\bar{\phi} \in M_{(\Gamma,\bar{g})}$ *be Fréchet differentiable at* $\bar{g}$*, then* $(\bar{\phi},\bar{g}))$ *is a GSP of* Ψ *in* (H,G)*, if and only if*

$$\partial\bar{\phi}(\bar{g}, g-\bar{g}) \geq 0, \quad g \in G.$$

Proof. Assume $\bar{\phi} \in M_{(\Gamma,\bar{g})}$ to be Fréchet differentiable at $\bar{g}$, then there exists a $f_{\bar{g}} \in X^*$ such that

$$\lim_{\|h\|\to 0^+} \frac{|\bar{\phi}(\bar{g}+h)-\bar{\phi}(\bar{g}))-f_{\bar{g}}(h)|}{\|h\|} = 0 \tag{7}$$

for any $h \in X$. Let $h = g-\bar{g}$, by the equality (7), we have that for any $\varepsilon > 0$ there exists a $\delta > 0$, when $g \in U(\bar{g},\delta) \cap G$ such that

$$f_{\bar{g}}(g-\bar{g}) - \varepsilon\|g-\bar{g}\| \leq \bar{\phi}(g) - \bar{\phi}(\bar{g}) \leq f_{\bar{g}}(g-\bar{g}) + \varepsilon\|g-\bar{g}\|. \tag{8}$$

We know that

$$\phi(\bar{g}) \leq \bar{\phi}(\bar{g}) \leq \bar{\phi}(g) \tag{9}$$

by the assumption of $(\bar{\phi},\bar{g}))$ being a GSP of Ψ in (H,G). Hence we can have

$$f_{\bar{g}}(g-\bar{g}) \geq 0$$

which is established from the second inequalities of (8) ,(9), and the arbitrariness of ε, which, furthermore, implies $\partial\bar{\phi}(\bar{g}, g-\bar{g}) \geq 0, g \in G$.

On the other hand, from the assumption $\partial\bar{\phi}(\bar{g}, g-\bar{g}) \geq 0, g \in G$, and the left inequality of (8), the inequality

$$\bar{\phi}(\bar{g}) \leq \bar{\phi}(g) \tag{10}$$

is obviously obtained. Moreover, as a result of $\bar{\phi} \in M_{(\Gamma,\bar{g})}$, we get

$$\phi(\bar{g}) \leq \sup_{\phi\in H} \phi(\bar{g}) = \bar{\phi}(\bar{g}). \tag{11}$$

Integrating (10) with (11),

$$\phi(\bar{g}) \leq \bar{\phi}(\bar{g}) \leq \bar{\phi}(g)$$

is established, which infers that $(\bar{\phi},\bar{g})$ is a GSP of Ψ in (H,G). □

It is obvious to know the following corollary by theorem 3 and 8,

Corollary 9.
Let $\bar{\phi} \in M_{(\Gamma,\bar{g})}$ be a locally maximal function and Fréchet differentiable at $\bar{g}$, then the following statements are equivalent,

(1) $\bar{g} \in P_{(\Gamma,G)}$;
(2) $\partial\bar{\phi}(\bar{g}, g-\bar{g}) \geq 0, \quad g \in G$;
(3) $(\bar{\phi},\bar{g})$ *is GSP of* Ψ *in* (H,G).

From the proofs of the above theorems, we realize that the optimum solution of (Γ,G) is equivalent to Fréchet differentiability , but not true to Gâteaux differentiability without G being a sunset of Γ, which displays that Fréchet differentiability is stronger than that of Gâteaux. But the condition that G is a sun set of Γ is not necessary in the proofs of theorem 6 and 8.

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