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Determining Arc Segment Significance in Two Idealized Road Networks

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Abstract As road networks become increasingly congested, accurate information regarding their vulnerability to an amalgamation of situations becomes more vital. Understanding how accidents, routine closures (due to construction, public events, etc) and even premeditated attacks on individual arcs influence the road network may assist policy makers in decisions on network expansion or planning, helping to ensure the development of a more robust, reliable, and protected network. This paper discusses a method for determining arc segment significance within two idealized networks using SPCP (Shortest Path Counting Problem) and DSPCP (Directed Shortest Path Counting Problem) as an analytical basis.

Keywords arc analysis; SPCP; DSPCP; road network design

1 Introduction

Transportation systems are key infrastructure lifelines of many contemporary civilizations. The accessibility and mobility enabled through open use of a transportation system is a vital and necessary freedom contributing to the fluidity and stability of national operability. The major proportion of transportation modes (i.e. automobile, bus, trucking) rely on the connectivity and accessibility of roadways and road networks. As populations grow, these road networks remain largely unchanged, resulting in the natural congestion of segments of the network. Further contributing to congestion are the planned and unexpected closures of often vital segments of the transportation network due to a variety of causes (i.e. planned construction, planned public events, motor vehicle accidents, natural disasters, etc). When building or augmenting a transportation network (or planning municipal repairs or events), it would be very useful to know which arcs would cause the most dramatic effects to the flow and congestion of the network.

Methods for quantitatively evaluating the crowdedness of network arcs have been shown to generalize very well when attempting to describe and analyze real-world situations. The two methods examined in this paper are the Shortest Path Counting Problem (SPCP) and Directed Shortest Path Counting Problem (DSPCP) .(Oyama and Taguchi

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1991a; Oyama and Taguchi 1991b; Li 1994). The SPCP calculates the crowdedness (i.e. number of shortest paths containing) each arc of an undirected transportation network, while the DSPCP examines networks where arc direction is explicitly considered. In analyzing networks using SPCP and DSPCP, it is possible not only to gain knowledge of the crowdedness, or weight, of individual arcs, but also the maximum values, mean values, and deviation of such values .(Li and Zhang 2007). In this paper, the effect that the removal of a single arc has on the weights of all other network arcs is examined. The distribution of such weight changes may then be used in the identification of vital arc segments whose destruction or impedance would most drastically affect the flow and operability of the network.

The remainder of this paper is organized as follows. The next section introduces two general networks which are used in the development of quantitative expressions for the change in arc weights given the removal of a single network arc. In the third section, a brief discussion of SPCP and DSPCP and their results will familiarize the reader with the methodology off which this work is based (important results of the SPCP and DSPCP will be shown). Section four extends the DSPCP in order to calculate the change in arc weights. Determining these weight changes will enable examination that may result in the identification of critical arcs of importance (those whose removal causes the most drastic weight changes). In the fifth section, we describe some simple implications pertaining to such arc identification and present some situations where such knowledge would be useful. Section six summarizes the results of this study.

2 Two Idealized Networks

In this work, two idealized networks are presented for analysis. The two network types are best described as grid type and circular-radial (circular). Figure 1 shows the idealized grid network $G(m,n)$, with m horizontal roads, n vertical roads, the set of grid points expressed as $\left\{x_{yz}|1 \leq y \leq m+1, 1 \leq z \leq n+1\right\}$, and Z_{kl} and H_{kl} vertical and horizontal edges of the network respectively (Z_{kl} connecting x_{kl} with $x_{k+1,l}$ and H_{kl} connecting x_{kl} with $x_{k,l+1}$). Figure 2 illustrates the idealized circular-radial network $T(m,n)$, with m circular roads, n radial roads (angles between the radial roads being equal), the set of points in T expressed as $\left\{x_{yz}|1 \leq y \leq m, 1 \leq z \leq n\right\}$, and R_{kl} and C_{kl} radial and circular arcs respectively (R_{kl} connecting x_{kl} with $x_{k+1,l}$ and C_{kl} connecting x_{kl} with $x_{k,l+1}$).

These two types of idealized networks were chosen for their wide-ranging applicability to real-world networks and their structural simplicity which allows for a variety of quantitative results to be obtained. Using networks of this type provides the potential for derived results to be applied to authentic traffic networks. Examples of cities arranged in a grid-type fashion are far reaching and include such congested areas as Philadelphia (USA), New York City (USA), Mannheim (Germany), Kyoto (Japan), among many others, while circular-radial networks include Hamadan (Iran), Manchester (England), and Paris (France), as well as many of the highway super-structures that surround large urban areas such as Washington, DC (USA) (Zhang 2004; Cyburbia 2007; Alsford 2007; Initiative 2007).

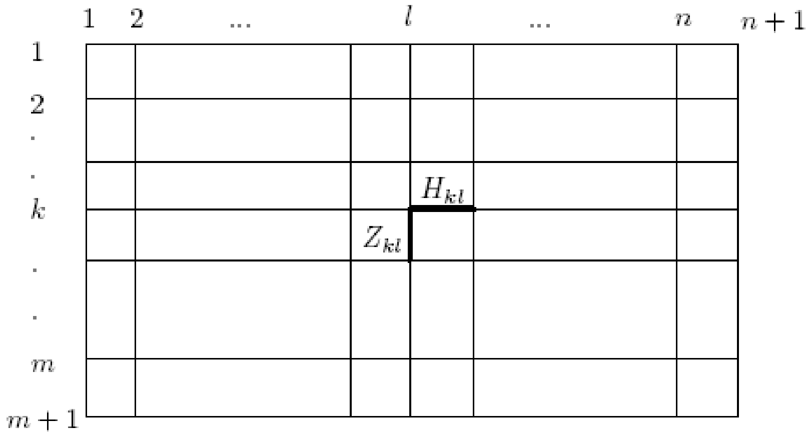

Figure 1: Idealized Grid Network

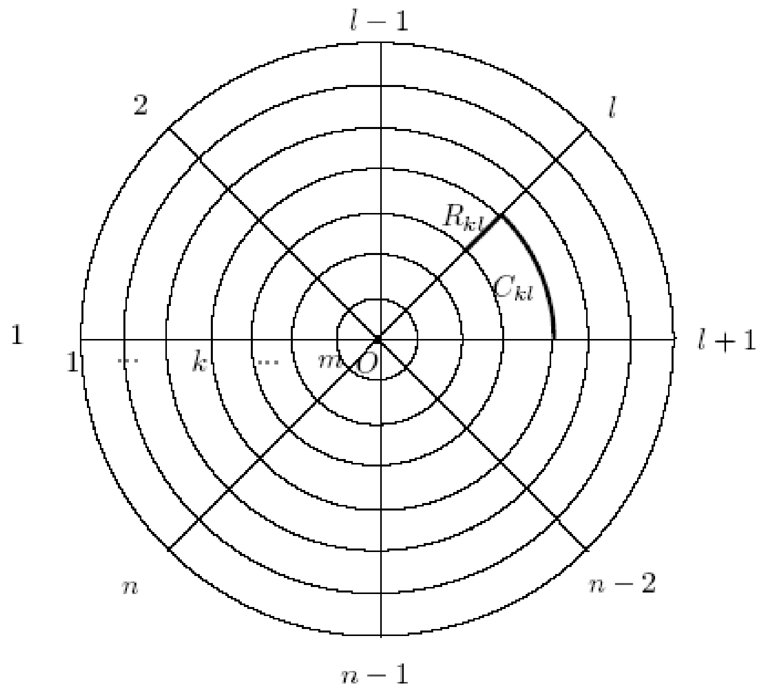

Figure 2: Idealized Circular-Radial Network

3 SPCP and DSPCP

Both SPCP and DSPCP count the number of shortest paths that pass through each arc of the designated network and assign a corresponding weight $\omega(\varepsilon)$ to the arc ε (recall that the SPCP does not consider directional arcs). When determining the shortest path in a network, both SPCP and DSPCP assume the following two rules .(Oyama and Taguchi 1991a). It is assumed that these two rules, when applied, will always yield one unique shortest path.

1. The number of turns is minimized.

2. The number of left turns is maximized when shortest paths with equal number of turns exist.

In the case of SPCP, the following theorems regarding the weight of network edges may be obtained (Oyama and Taguchi 1991a).

Theorem 1. *For a given grid type network G(m,n), the weights of the arc elements Z_{kl} and H_{kl} with respect to the shortest paths can be expressed as*

$$\omega(Z_{kl}) = 2k(m+1-k)(n+1) \quad 1 \le k \le m, \quad 1 \le l \le n+1,$$
$$\omega(H_{kl}) = 2l(n+1-l)(m+1) \quad 1 \le k \le m+1 \quad 1 \le l \le n.$$

Theorem 2. *For a given circular-radial type network T(m,n), the weights of the arc elements R_{kl} and C_{kl} with respect to the shortest paths can be expressed as*

$$\omega(R_{kl}) = 2kmn - 2k^2(2p_0+1) \quad 1 \le k \le m, \quad 1 \le l \le n,$$
$$\omega(C_{kl}) = (2k-1)p_0(p_0+1) \quad 1 \le k \le m, \quad 1 \le l \le n,$$

where $p_0 = \lfloor n/\pi \rfloor$. Additional analysis of SPCP allows for the definition of maximum weight values, expected values and variances to be calculated .(Li and Zhang 2007).

DSPCP expands SPCP by considering directional arcs (arcs that are separated into their positive, ε^+, and negative, ε^-, components). In an idealized grid network, the positive arc component represents a directed arc from left to right or in the downward direction. In the circular-radial network, a positive arc describes an outward pointed (center to circumference) or counter-clockwise arc .(Li and Zhang 2007). The negative arc component of DSPCP represents those arcs of reverse direction to the positive arc segments.

In the case of DSPCP, the following theorems regarding the weight of network edges may be obtained .(Li and Zhang 2007).

Theorem 3 *For a given grid type network G(m,n), the weights of the vertical arc elements Z^+_{kl} and Z^-_{kl} and horizontal arc elements H^+_{kl} and H^-_{kl} with respect to the shortest paths can be expressed as*

$$\omega(Z^+_{kl}) = k(m+1-k)(2n+3-2l) \quad 1 \le k \le m, 1 \le l \le n+1,$$

$$\omega(Z^-_{kl}) = k(m+1-k)(2l-1) \quad 1 \le k \le m, 1 \le l \le n+1,$$

$$\omega(H^+_{kl}) = l(n+1-l)(2k-1) \quad 1 \le k \le m+1, 1 \le l \le n,$$

$$\omega(H^-_{kl}) = l(n+1-l)(2m+3-2k) \quad 1 \le k \le m+1, 1 \le l \le n.$$

Theorem 4 *For a given circular-radial type network T(m,n), the weights of the radial arc elements* R_{kl}^{+} *and* R_{kl}^{-} *and circular arc elements* C_{kl}^{+} *and* C_{kl}^{-} *with respect to the shortest paths can be expressed as*

$$\omega(R_{kl}^{+}) = \omega(R_{kl}^{-}) = k(mn+1) - k^2(2p_0+1) \quad 1 \le k \le m, 1 \le l \le n,$$

$$\omega(C_{kl}^{+}) = \omega(C_{kl}^{-}) = \frac{1}{2}(2k-1)p_0(p_0+1) \quad 1 \le k \le m, 1 \le l \le n.$$

4 Determining Arc Segment Significance

A natural question regarding network composition is that of arc significance with relation to the network structure. In other words, how important is the arc to the flow of travel over the network, or to what extent would the destruction or removal of a specific arc hinder transportation within the network (if at all). Given the results of the DSPCP described above, it is possible to answer such questions very directly for the grid network structure and the circular-radial network structure. To accomplish this, an arc is selected to be removed from the given network. Resolving the DSPCP with the removal of this arc will give a new set of weights $\varpi(\varepsilon)$. Comparing $\varpi(\varepsilon)$ these new arc weights to the original weights of the unaltered network, $\omega(\varepsilon)$, yields the value $\Delta\omega$ for each arc $(\varpi(\varepsilon) - \omega(\varepsilon))$. Given the inherent properties of the idealized networks, it is possible to derive directly the new arc weight values $\varpi(\varepsilon)$ without resolving the DSPCP.

In both networks, the derived value $\Delta\omega$ of is the maximum value which $\Delta\omega$ may take. Deriving the maximal weight change, while not the only method of analysis, yields the worst-case value, or influence, resulting from an arc's destruction. As will be discussed in section 5, this type of analysis is useful in many network applications.

4.1 Results for the Idealized Grid Network

Theorem 5 and theorem 6 express $\Delta\omega$ in terms of the grid type network parameters *m, n, k,* and *l*.

Theorem 5 *For a given grid type network G(m,n) with arc* Z_{kl} *destructed, the maximum change in weight of affected arcs may be expressed as*

$$Z_{kl}^{+} \quad \text{destructed}: \quad \Delta\omega(\mathrm{Z}_{k,l-1}^{+})_{\max} = \Delta\omega(H_{k,l-1}^{-})_{\max} = k(m+1-k)(n+2-l)$$

$$Z_{kl}^{-} \quad \text{destructed}: \quad \Delta\omega(\mathrm{Z}_{k,l+1}^{-})_{\max} = \Delta\omega(H_{k+1,l}^{+})_{\max} = k(m+1-k)l$$

Theorem 6 *For a given grid type network G(m,n) with arc* H_{kl} *destructed, the maximum change in weight of affected arcs may be expressed as*

$$H_{kl}^{+} \quad \text{destructed}: \quad \Delta\omega(\mathrm{H}_{k+1,l}^{+})_{\max} = \Delta\omega(Z_{k,l}^{+})_{\max} = l(n+1-l)k$$

$$H_{kl}^{-} \quad \text{destructed}: \quad \Delta\omega(\mathrm{H}_{k-1,l}^{-})_{\max} = \Delta\omega(Z_{k-1,l+1}^{-})_{\max} = l(n+1-l)(m+2-k)$$

Figure 3 illustrates the case of theorem 5 above. It is shown that, in an idealized grid network where ties in shortest path value are broken by choosing the path with minimal right turns (assuming travel occurs on the left side of any roadway), the removal of the directed arc Z_{kl}^{+}(Figure 3a) will yield the largest DSPCP weight change for the arcs $H_{k,l-1}^{-}$

and $Z^+_{k,l-1}$due to the necessary alteration of all paths originating in s_2 and s_3 and arriving in s_5 (Figure 3b illustrates a similar situation when arc Z^-_{kl} is destructed).

Using a grid type network with m = 70 and n= 120, Figure 4 and Figure 5 illustrate the effective weight change described above should a vertical arc or horizontal arc be removed from the network.

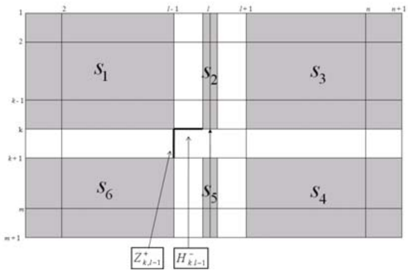

Figure 3a: Destruction of arc Z^+_{kl}

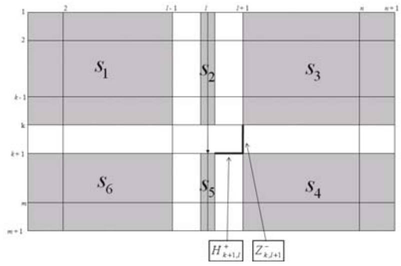

Figure 3b: Destruction of arc Z^-_{kl}

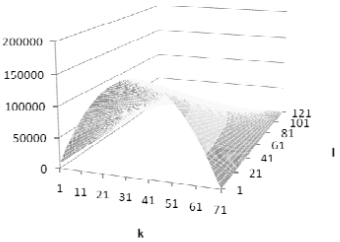

Figure 4a: Removal of vertical arc Z^+_{kl}

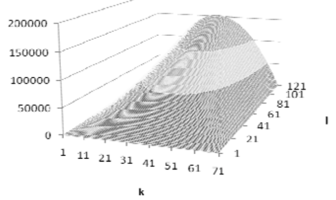

Figure 4b: Removal of vertical arc Z^-_{kl}

In both cases, the direction of the arc (positive or negative) has an inverse effect on arc weight. The shape of the surface in all cases is attributable to the location of the removed arc and the relationship between the alternative segmentation of the network (refer back to Figure 3). As an example, the removal of Z^+_{kl} from the network has the largest effect when $k$is at the vertical midpoint of the network and l is at the leftmost edge. In this situation, the regions of s_2, s_3, and s_5 are at their largest combined volumes and cause the highest weight change (from this position, any shift of l to the right will reduce the size of region s_3 and enlarge region s_1). As there are less arcs in region , there are less destinations s_3 in and less O-D pairs from s_3 to s_5 for which a shortest path must be found. This results in lower values of $\Delta\omega$ as the removed arc progresses towards the right side of the network.

4.2 Results for the Idealized Circular-Radial Network

Theorem 7 and theorem 8 express in terms of the circular type network parameters *m, n, k,* and *l*.

Theorem 7 *For a given circular-radial type network T(m,n) with arc* R_{kl} *destructed,*

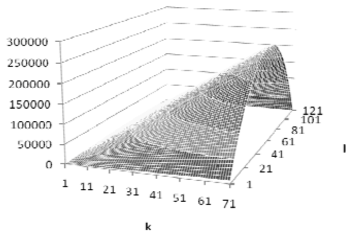

Figure 5a: Removal of horizontal arc H_{kl}^{+}

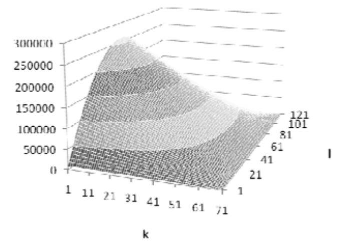

Figure 5b: Removal of horizontal arc H_{kl}^{-}

the maximum change in weight of affected arcs may be expressed as

$$\begin{array}{ll} R_{kl}^{+} & \text{destructed :} \\ & \Delta\omega(C_{k+1,l}^{-})_{\max} = \Delta\omega(R_{k,l+1}^{+})_{\max} = \Delta\omega(C_{k,l}^{+})_{\max} = k(mn+1) - k^2(2p_0+1) \\ R_{kl}^{-} & \text{destructed :} \\ & \Delta\omega(C_{k,l-1}^{+})_{\max} = \frac{1}{2}\left[k(mn+1) - k^2(2p_0+1)\right] + \frac{1}{2}k(m-k+1) \end{array}$$

Theorem 8 *For a given circular-radial type network T(m,n) with arc* C_{kl} *destructed, the maximum change in weight of affected arcs may be expressed as*

$$C_{kl}^{+} \quad \text{destructed :} \quad \Delta\omega(R_{k,l+1}^{-})_{\max} = \Delta\omega(C_{k-1,l}^{+})_{\max} = \tfrac{1}{2}(2k-1)p_0(p_0+1)$$

$$C_{kl}^{-} \quad \text{destructed :} \quad \Delta\omega(R_{k,l}^{-})_{\max} = \Delta\omega(C_{k-1,l}^{-})_{\max} = \tfrac{1}{2}(2k-1)p_0(p_0+1)$$

In the idealized circular-radial network, the location of the removed arc significantly impacts the resulting $\Delta\omega$ values. Figure 6 illustrates the case of theorem 7 above. By removing a radial arc in the negative direction (towards the network center as in Figure 6b), only the circumferential arc immediately adjacent to the removed arc ($C_{k,l-1}^{+}$) sees a dramatic weight change. Removing a radial arc in the positive direction (Figure 6a), however, causes an increase in weight on the three arcs which form the shortest path around the impasse (in this case $C_{k+1,l}^{-}$, $R_{k,l+1}^{+}$, and $C_{k,l}^{+}$).

Figure 7 illustrates the case of theorem 8 where a circumferential arc is removed. As arcs on the circumference of any of the prescribed circles are less influential than radial arcs (i.e. lower DSPCP weights), they are more rarely utilized and more easily avoided. This leads to a distribution of $\Delta\omega$ values which is more stable and consistent.

Using a circular type network with m = 30 and n= 12, Figure 8 and Figure 9 illustrate the effective weight change described above should a radial arc R_{kl} or circumferential arc C_{kl} be removed from the network.

As illustrated in Figure 8 and Figure 9, the symmetry of the circular-radial network eliminates the effect of circumferential location on $\Delta\omega$. This results in the value of $l$being a non-factor in the calculation of arc weight change. Alternately, the radial location of the removed arc plays a significant role.

From Figure 6, analysis on the removal of a radial arc (R_{kl}^{+} or R_{kl}^{-}) results in the circular-radial network decomposition into three segments s_1,s_2 , and s_3. The interaction between these three segments (and the network nodes contained within them) as the radial value k changes determines the extent to which a removed arc effects $\Delta\omega$. According

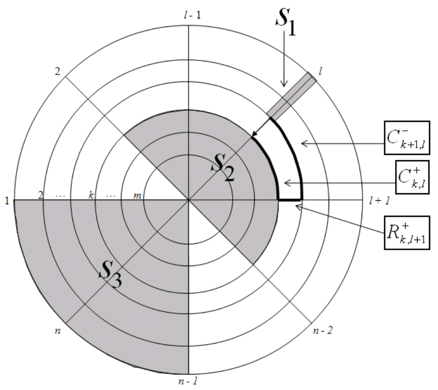

Figure 6a: Destruction of arc $R^+_{k,l}$

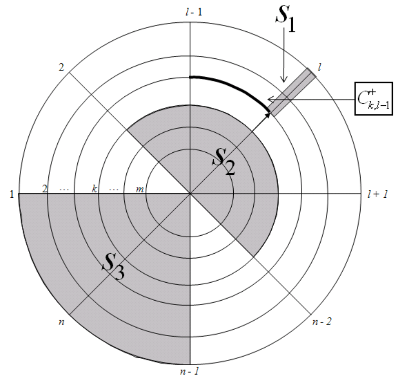

Figure 6b: Destruction of arc $R^-_{k,l}$

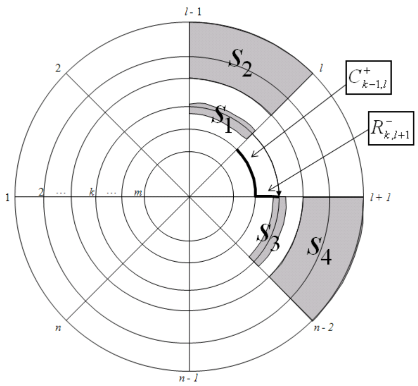

Figure 7a: Destruction of arc $C^+_{k,l}$

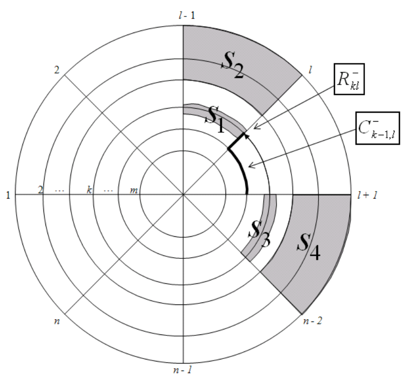

Figure 7b: Destruction of arc $C^-_{k,l}$

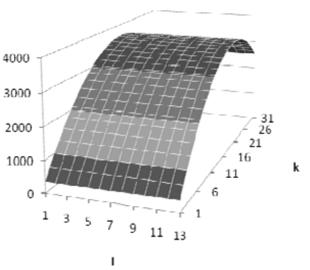

Figure 8a: Removal of radial arc R^+_{kl}

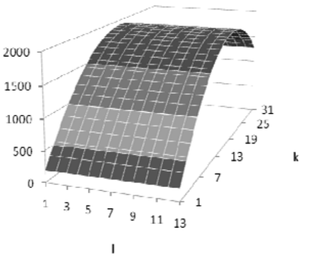

Figure 8b: Removal of radial arc R^-_{kl}

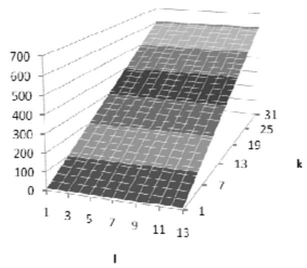

Figure 9a: Removal of circular arc C_{kl}^{+}

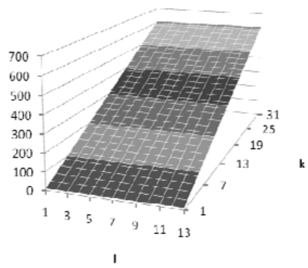

Figure 9b: Removal of circular arc C_{kl}^{-}

to the shortest path selection criterion for circular-radial networks (Oyama and Taguchi, 1991a), the segment s_3 remains unchanged as k moves along the radial arc of the network while s_1 increases with k (as k approaches the center of the network). s_2, however, decreases as the removed arc closes in on the central region. This interaction between s_1 with s_3 and s_1 with s_2 creates the parabolic shape exhibited in Figure 8 (at a critical point, the added contribution resulting from s_1 and s_3 is not enough to overcome the shrinking dimensions of s_2 and the maximal arc weight change $\Delta\omega$, while still positive, is reduced). In the case where a circumferential arc is removed (C_{kl}^{+} or C_{kl}^{-}), the symmetry of the network and the path selection criterion result in similar network segmentation (Figure 7) which leads to the increasing trend illustrated in Figure 9.

5 Discussion of Arc Segment Significance

The theoretical results presented in section 4 illustrate how the structure of a road network may be exploited to obtain a concise quantification of individual arc importance ($\omega(\varepsilon)$ of SPCP and DSPCP). The proposed approach has significant advantages over alternative arc evaluation methods, one such method being the concept of network interdiction.

Network interdiction is described as the intentional destruction, by force, of a network to impede or cease enemy use. Within the realm of optimization, especially within the military community, the study of interdiction problems has been given significant attention. Considering network flow, the interdiction problem may be represented as a multi-commodity flow problem with two players ..(Lim and Smith 2007). The first player, the follower, makes profit by delivering commodities to designated destinations. The leader attempts to minimize the followers profit by selectively destroying arcs (the destruction of which costs the leader by subtracting from the leaders' interdiction budget). By exploiting the network structure and the properties of DSPCP, a quantification of arc importance may be derived which is significantly easier to implement and interpret, and which requires significantly less data handling and solution time than the network interdiction problem.

Another way to identify network vulnerability is through the identification of critical arcs. An arc is deemed 'weak' if the probability of an incident is high, 'important' if the consequence of an incident is large, and 'critical' if it is both weak and important .(Jenelius and Mattsson 2006). The model emphasizes the importance and exposure of arcs. These values are derived through the observation of how arc absence affects path travel time, using multiple, predetermined, O-D paths to evaluate .(Jenelius and Mattsson 2006). The approach of this paper, which augments the DSPCP of section 3, does not require the interpretation and quantification of arc incident probability or consequence. Inclusion of such ambiguous measures also convolutes results and strictly limits the applicability of the model to the level of specificity and accuracy of the data obtained. It is thus more preferable to utilize a model which is directly tied to the network and makes few assumptions.

6 Conclusion

Based on the results of SPCP and DSPCP as described in section 3, we theoretically studied the influence on the network that the destruction of one road segment would yield. This was accomplished through an extension of the DSPCP theorems and results which allowed for the calculation of the maximum weight change of designated arcs. In analyzing these changes, the most dramatic weight increases occur in the arcs surrounding the destructed road segment (this is true for both road networks). Network composition was also shown to have an impact on arc weight change, with the grid and circular type networks contributing differently to the calculated changes.

Determining arc segment significance as above could assist in the quick assessment and analysis of networks when faced with policy-making decisions (i.e. the effect that major road construction would have on the network). Such an approach would prove extremely useful in the identification of those arcs whose absence yields the highest weight changes. These arcs may require fortification or increased observance in order to ensure that their functionality is not inhibited in any way due to malicious attacks on the network, which could disrupt not only civilian travel but have huge implications on commerce and emergency response capabilities. The approach above represents a worse-case analysis that is quicker to perform than a standard network interdiction problem and it has been shown that the solutions found on the idealized networks of this paper generalize very well to real-world traffic networks. These final points illustrate the importance for having a quick and easy means of calculating the significance of arc segments within a network and this paper provides such a method.

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References

[1] Cyburbia: The Planning Community. "Circular Road Network versus Grid System." Retrieved December 23, 2007, from http://www.cyburbia.org.

[2] Alsford, S. (2007). "Introduction to the History of Medieval Boroughs." Retrieved December 22, 2007, from http://www.trytel.com/∼tristan/towns/townint5.html.

[3] Initiative, E. C. A. (2007). "The Near East in Late Antiquity." Retrieved December 22, 2007, from http://ecai.org/sasanianweb/index.html.

[4] Jenelius, E. and L.-G. Mattsson (2006). Developing a Methodology for Road Network Vulnerability Analysis, Molde University College.

[5] Li, M. (1994). Studies on Road Traffic Congestion Analysis and Control. Mathematical Engineering and Information Physics. Tokyo, University of Tokyo. **Masters:** 77.

[6] Li, M. and Y. Zhang (2007). "Optimal Design of Two Road Networks Based on their Properties." Acta Mathematicae Applicatae Sinica, English Series **23**(2): 337-352.

[7] Lim, C. and J. C. Smith (2007). "Algorithms for Discrete and Continuous Multicommodity Flow Network Interdiction Problems." IIE Transactions **39**: 15-26.

[8] Oyama, T. and A. Taguchi (1991a). "On some results of the shortest path counting problem." Proceedings of the Spring Meeting of the Operations Research Society of Japan: 102-103.

[9] Oyama, T. and A. Taguchi (1991b). "Further results on the shortest path counting problem." Proceedings of the Fall Meeting of the Operations Research Society of Japan: 166-167.

[10] Zhang, Q. (2004). Modeling Structure and Patterns in Road Network Generalization. ICA Workshop on Generalisation and Multiple Representation, Leicester.