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Global Optimality Conditions for Optimization Problems*

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Abstract We establish new necessary and sufficient optimality conditions for optimization problems. In particular, we establish tractable optimality conditions for the problems of minimizing a weakly convex or concave function subject to standard constraints, such as box constraints, binary constraints, and simplex constraints. Our main theoretical tool for establishing these optimality conditions is abstract convexity.

Keywords Global optimization; optimality conditions; abstract convexity.

1 Introduction

We study optimality conditions for some classes of global minimization problems. References [1]-[3] and [6, 7, 12], etc. study global optimality conditions for the problems with quadratic objective function subject to either box constraints, binary constraints, quadratic constraints, or linear constraints. In the present paper, we follow the approach of references [2, 3, 9, 12, 13], where abstract convexity is used in order to obtain new optimality conditions for global optimization problems. These optimality conditions are expressed in terms of abstract subdifferential (L-subdifferential) and abstract normal cone (L-normal cone). The present paper extends the existing optimality conditions to the necessary and sufficient optimality condition for a general global optimization problem:

$$(P) \qquad \min_{x\in U} f(x), \qquad \text{where } f:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}, U\subset \operatorname{dom} f.$$

This necessary and sufficient condition given in this paper is expressed in terms of (ε,L)-subdifferential and (ε,L)-normal set (see Theorem 1), which extends classical results expressed in terms of ε-subdifferential and ε-normal cone in the sense of convex analysis for concave minimization problems, which can be found in [1, 11] (see Corollary 3.1).

Our analysis allows us to obtain some tractable optimality conditions for the classes of problems where the objective function is a weakly convex or concave function subject to arbitrary constraint set. A function f is a weakly convex (or concave) function if

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and only if it can be represented as the sum of a quadratic function and a convex (or concave) function. These classes of functions are broad enough (see Section 4). So it always not easy to get some tractable global optimality conditions for weakly convex (or concave) problems. Reference [13] gives some sufficient conditions for some special classes of weakly convex programming problems. The present paper gives more stronger sufficient optimality conditions for general weakly convex problems, which extends the results given in [13]. The present paper also gives some necessary optimality conditions for general weakly concave problems (see Theorem 13), which are new results given in this paper.

The lay-out of the paper is as follows. Section 2 provides a necessary and sufficient condition in terms of (ε,L)-subdifferential and (ε,L)-normal set. Some sufficient and/or necessary conditions in terms of L-subdifferential and L-normal cone are also presented in Section 2. Section 3 provides sufficient global optimality conditions for the class of problems where the objective function a weakly convex function. Section 4 provides necessary optimality conditions for the class of problems where the objective function is a weakly concave function.

We use the following notation:

$\mathbb{R}^n$ is the n-dimensional Euclidean space;

$\mathbb{R}^n_+$ is the n-dimensional non-negative Euclidean space, i.e., $\mathbb{R}^n_+ := \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_i \geq 0, i = 1,\dots,n\}$;

$\langle u,v\rangle$ stands for the inner product of vectors u and v;

S^n is the set of symmetric $n\times n$ matrices;

$\mathrm{Diag}\,(q)$ is a diagonal matrix with the diagonal q;

$A \succeq B$ means that the matrix $A-B$ is positive semidefinite.

$\partial f(x)$ and $\partial_\varepsilon f(x)$ are the subdifferential and ε-subdifferential respectively of a convex function f at a point x in the sense of convex analysis.

$N_U(x)$ and $N_U^\varepsilon(x)$ are the normal cone and ε-normal set respectively of a convex set U at a point x in the sense of convex analysis.

2 Necessary and sufficient conditions for global optimality

In this section we first give some preliminaries from abstract convexity (see [5, 8, 10]). Let X be a set and H be a set of functions $h: X\to\mathbb{R}$. Let L be a set of functions defined on $\mathbb{R}^n$. For a function $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ and a point $\bar{x}\in \mathrm{dom}\, f = \{x\in\mathbb{R}^n : f(x) < +\infty\}$ define L-subdifferential $\partial_L f(\bar{x})$ of f at $\bar{x}$ by

$$\partial_L f(\bar{x}) = \{l\in L : f(x)\geq f(\bar{x}) + l(x) - l(\bar{x}),\ \forall x\in\mathbb{R}^n\}.$$

For $\varepsilon\geq 0$, define (ε,L)-subdifferential $\partial_{\varepsilon,L} f(\bar{x})$ of f at $\bar{x}$ by

$$\partial_{\varepsilon,L} f(\bar{x}) = \{l\in L : f(x)\geq f(\bar{x}) + l(x) - l(\bar{x}) - \varepsilon,\ \forall x\in\mathbb{R}^n\}.$$

Obviously, $\partial_L f(\bar{x}) \subset \partial_{\varepsilon,L} f(\bar{x})$, for any $\varepsilon\geq 0$.

Let $U \subset \mathbb{R}^n$ and let $\bar{x} \in U$. The L-normal set and the $(\varepsilon, L)-$normal set of U at $\bar{x}$ are defined as follows:

$$N_{L,U}(\bar{x}) = \{l \in L : l(x) \leq l(\bar{x}), \quad \forall x \in U\}$$

and

$$N^{\varepsilon}_{L,U}(\bar{x}) = \{l \in L : l(x) \leq l(\bar{x}) + \varepsilon, \quad \forall x \in U\}.$$

If L is a cone, that is, $(l \in L, \lambda > 0) \implies (\lambda l \in L)$ then $N_{L,U}(\bar{x})$ is also a cone. Since the set L we consider is always a subspace(in particular, a cone), we will use the usual term L-normal cone instead of L-normal set. Properties of (ε, L)-subdifferential (L-subdifferential) and (ε, L)-normal set (L-normal set) have been investigated in ([4, 8]).

L-subdifferentials ((ε, L)-subdifferentials) and L-normal cones ((ε, L)-normal sets) are main tools for deriving global optimality conditions for a global minimizer. In other words

$$\bar{L} := \left\{ l : l = \frac{1}{2}\langle Qx, x\rangle + \langle \beta, x\rangle \text{ with } Q = \text{Diag}(q);\ q, \beta \in \mathbb{R}^n \right\}. \tag{1}$$

We call $\widehat{L}$ the subset of $\bar{L}$ which consists of functions $l(x) = \alpha\|x\|^2 + \langle \beta, x\rangle$ with $\alpha \in \mathbb{R}$.

We now give a necessary and sufficient global optimality condition with (ε,L)-subdifferential and (ε,L)-normal set for the following general global optimization problem (P), which extends the necessary and sufficient global optimality conditions given with ε-subdifferential of convex functions and ε-normal set of convex sets for concave minimization problems (see [1, 11]):

$$(P) \qquad \text{minimize } f(x) \text{ subject to } x \in U,$$

where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $U \subset \text{dom} f = \{x \in \mathbb{R}^n : f(x) < +\infty\}$.

Theorem 1. *Let L be a subspace of real-valued functions defined on $\mathbb{R}^n$. Let $\bar{x} \in U$. Suppose that $-f$ is H_L-convex on U. Then $\bar{x}$ is a global minimizer of problem (P) if and only if*

$$\partial_{\varepsilon,L}(-f)(\bar{x}) \subset N^{\varepsilon}_{L,U}(\bar{x}), \quad \forall \varepsilon > 0, \tag{2}$$

where

$$H_L = \{l(x) + c \mid l \in L, c \in \mathbb{R}\}. \tag{3}$$

Here we give a necessary and sufficient global optimality condition for the general optimization problem (P) in terms of (ε, L)-subdifferentials and (ε, L)-normal set. If U is convex and $f(x)$ is concave on U, then the following results which can be found in [1, 11] can be reduced from Theorem 1.

Corollary 2. *Let $\bar{x} \in U$. Suppose that U is a convex set and f is a concave function on U. Then $\bar{x}$ is a global minimizer of problem (P) if and only if*

$$\partial_{\varepsilon}(-f)(\bar{x}) \subset N^{\varepsilon}_{U}(\bar{x}), \quad \forall \varepsilon > 0. \tag{4}$$

Remark 1. From Theorem 1, we can also obtain the following results:

Let L be a subspace of real-valued functions defined on $\mathbb{R}^n$*. Let* $\bar{x} \in U$*.*

(i)*. If* $\bar{x}$ *is a global minimizer of problem* (P)*, then*

$$\partial_{\varepsilon,L}(-f)(\bar{x}) \subset N^{\varepsilon}_{L,U}(\bar{x}), \quad \forall \varepsilon \geq 0; \tag{5}$$

(ii)*. If for any* $x \in U$*,* $\partial_L(-f)(x) \neq \emptyset$*, then* $\bar{x}$ *is a global minimizer of problem* (P) *if and only if (5) holds.*

Indeed, from the first part of the proof of Theorem 1 (for the necessary condition), we know that if $\bar{x}$ is a global minimizer of problem (P), then (5) holds, where $-f$ does not require to be H_L-convex on U. Since $\partial_L(-f)(x) \neq \emptyset$, $\forall x \in U$ implies that $-f$ is H_L-convex on U, (ii) can be obtained directly from Theorem 1.

In general, it is very difficult to calculate the (ε, L)-subdifferential and the (ε, L)-normal set for general function f and general set U, and even if $-f$ and U are convex, it is also very difficult to calculate the ε-subdifferential and the ε-normal set in the sense of convex analysis. But, for some kinds of functions f and for some kinds of sets U, we can calculate the L-subdifferential and the L-normal cone. The following simple results will be useful in the sequel.

Corollary 3. ***(Necessary Condition)*** *Let L be a subspace of real-valued functions defined on* $\mathbb{R}^n$*. Let* $\bar{x} \in U$*. If* $\bar{x}$ *is a global minimizer of problem* (P)*, then*

$$\partial_L(-f)(\bar{x}) \subset N_{L,U}(\bar{x}). \tag{6}$$

The following sufficient condition has been presented in [2, 3], which study sufficient conditions for quadratic functions over binary or/and box sets.

Proposition 4. ***(Sufficient Condition)*** *Let L be a subspace of real-valued functions defined on* $\mathbb{R}^n$*. Let* $\bar{x} \in U$*. If*

$$\big(-\partial_L f(\bar{x})\big) \cap N_{L,U}(\bar{x}) \neq \emptyset \tag{7}$$

then $\bar{x}$ *is a global minimizer of problem* (P)*.*

Corollary 3 and Proposition 4 imply that

$$\big(-\partial_L f(\bar{x})\big) \cap N_{L,U}(\bar{x}) \neq \emptyset \implies \partial_{\varepsilon,L}(-f)(\bar{x}) \subset N^{\varepsilon}_{L,U}(\bar{x}), \forall \varepsilon > 0.$$

Generally, there is a gap between the sufficient condition (7) and the necessary condition (6). But in some cases, both of (7) and (6) are necessary and sufficient conditions. In Section 7, we will see that if f is a quadratic function, then the sufficient condition (7) and the necessary condition (6) are equivalent in some sense.

Proposition 5. *Let L be a subspace of real-valued functions defined on* $\mathbb{R}^n$*,* $\bar{x} \in U$*. If* $f \in L$*, then*

$$\Big(-\partial_L f(\bar{x})\Big) \cap N_{L,U}(\bar{x}) \neq \emptyset \iff -f \in N_{L,U}(\bar{x}) \iff \partial_L(-f)(\bar{x}) \subset N_{L,U}(\bar{x}). \tag{8}$$

Condition (8) means that if $f \in L$, both of (7) and (6) are necessary and sufficient conditions.

3 Sufficient Conditions for Weakly Convex Problems

Consider problem

$$(P_1) \qquad \text{minimize } \ f(x) \ \text{ subject to } \ x \in U,$$

where $U \subset \operatorname{dom} f$,

$$f(x) = \frac{1}{2}\langle A_0 x, x\rangle + p(x), \tag{9}$$

with $A_0 \in S_n$ and p is a proper convex function.

Proposition 6. *Let f be a function defined by (9). Let $\bar{x} \in$ dom p, the domain of p. Then*

$$\partial_{\bar{L}} f(\bar{x}) \supset \{l \in \bar{L} \mid l(x) = \frac{1}{2}\langle Qx, x\rangle + \langle \beta, x\rangle : A_0 - Q \succeq 0, (Q - A_0)\bar{x} + \beta \in \partial p(\bar{x})\}. \tag{10}$$

Corollary 7. *Let $f(x) = \frac{1}{2}\langle A_0 x, x\rangle + \langle a_0, x\rangle$ be a quadratic function. Then*

$$\partial_{\bar{L}} f(\bar{x}) \quad = \quad \{l \in \bar{L} \mid l(x) = \frac{1}{2}\langle Qx, x\rangle + \langle \beta, x\rangle : A_0 - Q \succeq 0, \beta = a_0 + (A_0 - Q)\bar{x}\} \tag{11}$$

Theorem 8. *Let $U \subset \mathbb{R}^n$ and $\bar{x} \in U$. If*

$$[SC1] \qquad \exists q,\, b \in \mathbb{R}^n \text{ such that } A_0 \succeq \operatorname{Diag}(q),\ b - A_0\bar{x} \in \partial p(\bar{x}) \quad \text{and} \quad -l_{q,b,\bar{x}} \in N_{\bar{L},U}(\bar{x}).$$

Then $\bar{x}$ is a global minimizer of problem (P_1).

Corollary 9. *Let f be a quadratic function, $f(x) = \frac{1}{2}\langle A_0 x, x\rangle + \langle a_0, x\rangle$. If*

$$[SCQ1] \qquad \exists q \in \mathbb{R}^n \text{ such that } A_0 \succeq \operatorname{Diag}(q),\ b = a_0 + A_0\bar{x} \quad \text{and} \quad -l_{q,b,\bar{x}} \in N_{\bar{L},U}(\bar{x}),$$

then $\bar{x}$ is a global minimizer of problem (P_1). Furthermore,

$$[SCQ1] \iff \Big(-\partial_{\bar{L}} f(\bar{x})\Big) \cap N_{\bar{L},U}(\bar{x}) \neq \emptyset.$$

We can obtain tractable sufficient conditions if the normal cone $N_{\bar{L},U}(\bar{x})$ can be easily calculated, in particular, if U is one of the following sets, then we have the following results:

Corollary 10. *Let $U = \prod_{i=1}^n \{u_i, v_i\}$ and $\bar{x} \in U$. If there exist $q, b \in \mathbb{R}^n$ such that $A_0 \succeq$ Diag(q) and*

$$b - A_0\bar{x} \in \partial p(\bar{x}) \ \text{ and } \ \widehat{X} b \le \frac{1}{2}\operatorname{Diag}(q)(v - u),$$

where

$$\begin{aligned} \widehat{x}_i : &= \begin{cases} 0 & \text{if } \bar{x}_i = u_i = v_i \\ -1 & \text{if } \bar{x}_i = u_i \\ 1 & \text{if } \bar{x}_i = v_i \end{cases} \\ \widehat{X} : &= \operatorname{Diag}(\widehat{x}_1, \ldots, \widehat{x}_n) \end{aligned} \tag{12}$$

and $v = (v_1, \ldots, v_n), u = (u_1, \ldots, u_n)$, then $\bar{x}$ is a global minimizer of problem (P_1).

Corollary 11. *Let $U = \{x = (x_1,\ldots,x_n) : u_i \le x_i \le v_i\}$ and $\bar{x} \in U$. If there exist $q,b \in \mathbb{R}^n$ such that $A_0 \succeq \mathrm{Diag}\,(q)$ and*

$$b - A_0\bar{x} \in \partial p(\bar{x}) \ \ \textit{and} \ \ \widetilde{X}_b b \le \frac{1}{2}Q^{-}(v-u)\},$$

where

$$\begin{aligned} \widetilde{x}_i(b) : &= \begin{cases} 0 & \text{if } \bar{x}_i = u_i = v_i \\ -1 & \text{if } \bar{x}_i = u_i < v_i \\ 1 & \text{if } \bar{x}_i = v_i > u_i \\ b_i & \text{if } \bar{x}_i \in (u_i, v_i) \end{cases} \\ \widetilde{X}_b : &= \mathrm{Diag}\,(\widetilde{x}_1(b),\ldots,\widetilde{x}_n(b)) \end{aligned} \tag{13}$$

and $v = (v_1,\ldots,v_n), u = (u_1,\ldots,u_n)$, then $\bar{x}$ is a global minimizer of problem (P_1).

Corollary 12. *Let $U = \mathrm{co}\{v_1,\ldots,v_k\}, k \le n+1$ be a simplex in $\mathbb{R}^n$ and $\bar{x} \in U$. If there exist $q,b \in \mathbb{R}^n$ such that $A_0 \succeq \mathrm{Diag}\,(q)$, $\quad b - A_0\bar{x} \in \partial p(\bar{x})$ and*

$$\max_{i=1,\ldots,k} h^{-}_{q,b}(v_i) \le \min_{x\in U} h^{+}_{q,b}(x), \tag{14}$$

then $\bar{x}$ is a global minimizer of problem (P).

4 Necessary Conditions for Weakly Concave Problems

Consider the problem

$$(P_2) \qquad \text{minimize } \ f(x) \ \text{ subject to } \ x \in U,$$

where $U \subset \mathrm{dom}\, f$, $f(x) = \frac{1}{2}\langle A_0x, x\rangle + p(x)$, with $A_0 \in S_n$ and p is a proper concave function.

A function f is called to be a weakly concave function if $-f$ is a weakly convex function.

Theorem 13. *Let $U \subset \mathbb{R}^n$, $\bar{x} \in U$. If $\bar{x}$ is a global minimizer of problem (P_2), then*

$$[NC1] \qquad -l_{q,b,\bar{x}} \in N_{\bar{L},U}(\bar{x}) \textit{ for any } q,b \in \mathbb{R}^n \ \textit{ with } \ \mathrm{Diag}\,(q) \succeq A_0, A_0\bar{x} - b \in \partial(-p)(\bar{x}),$$

where

$$l_{q,b,\bar{x}}(x) := \frac{1}{2}\sum_{i=1}^{n} q_i x_i^2 + \sum_{i=1}^{n} b_i x_i - \sum_{i=1}^{n} q_i \bar{x}_i x_i = \frac{1}{2}\langle Qx, x\rangle + \langle b - Q\bar{x}, x\rangle, \tag{15}$$

$Q = \mathrm{Diag}\,(q)$.

We can derive some necessary conditions for the following optimization problems (QP) with quadratic objective functions:

$$(QP) \qquad \text{minimize } \ f(x) \ \text{ subject to } \ x \in U, \tag{16}$$

where

$$f(x) := \frac{1}{2}\langle A_0x, x\rangle + \langle a_0, x\rangle, \text{ and } U \subset \mathbb{R}^n.$$

Corollary 14. *Let $U \subset \mathbb{R}^n$, $\bar{x} \in U$. If $\bar{x}$ is a global minimizer of problem (QP), then*

$$[NCQ1] \qquad -l_{q,b,\bar{x}} \in N_{\bar{L},U}(\bar{x}) \quad \textit{for any} \quad q \in \mathbb{R}^n \quad \textit{with} \;\; \mathrm{Diag}\,(q) \succeq A_0, b = a_0 + A_0\bar{x}.$$

Furthermore, $[NCQ1] \iff \partial_{\bar{L}}(-f)(\bar{x}) \subset N_{\bar{L},U}(\bar{x})$.

Remark 2. For quadratic problem (QP), conditions $[SCQ1], [NCQ1]$ and $[NCQ2]$ have the following relationships:

$$[SCQ1] \Longrightarrow [NCQ1], \quad \text{and} \quad [NCQ1] \overset{\text{from Remark 7.1}}{\Longrightarrow} [SCQ1]$$

$$[SCQ1] \Longrightarrow [NCQ2], \quad \text{but} \quad [NCQ2] \not\Longrightarrow [SCQ1]$$

$$[NCQ1] \overset{\text{from Remark 7.1}}{\Longrightarrow} [NCQ2], \quad \text{but} \quad [NCQ2] \not\Longrightarrow [NCQ1].$$

If $U = U(\bar{x}) = \prod_{i=1}^{n} U_i(\bar{x})$ for $\bar{x} \in U$, then

$$[NCQ2] \implies [NCQ1].$$

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