The 7th International Symposium on Operations Research and Its Applications (ISORA'08)
Lijiang, China, October 31–Novemver 3, 2008

Analysis of a Machine Repair System with Warm Spares and N-Policy Vacations

Dequan Yue[1] Wuyi Yue[2] Hongjuan Qi[1]

[1]College of Science, Yanshan University, Qinhuangdao 066004, China
[2]Department of Intelligence and Informatics, Konan University, Kobe 658-8501, Japan

Abstract In this paper, we analyze a machine repairable system with warm spares and two repairmen. The first repairman never takes vacations. The second repairman leaves for a vacation of random length when the number of the failed units is less than N. At the end of a vacation period, this repairman returns back if there are N or more failed units accumulated in the system. Otherwise this repairman goes for another vacation. The vacation time is exponentially distributed. By using the Markov process theory, we develop the difference equations of the steady-state probabilities and solve the equations iteratively. We present derivations of some queueing and reliability measures. A cost model is developed to determine the optimum N while the system availability is maintained at a certain level. Sensitivity analysis is also investigated.

Keywords Repairable system; warm spares; vacation; availability; failure frequency

1 Introduction

The machine repairable systems with spares have been used in many situations. For example, to avoid any loss of production, the plant always keeps spare machines so that a spare machine can immediately act as a substitute when an operating machine fails. Another example of a repairable system with spares can be found in the operating room of a hospital, where standby power equipments are needed since the operating on a patient can not be stopped when the power is breakdown. Similar examples can be found in many fields such as power stations, manufacturing systems and industrial systems.

Wang and Sivazlian [1] considered the reliability characteristics of a repairable system with m operating units, s warm spares and R repairmen. They obtained the expressions of the reliability and the mean time to system failure. Wang and Ke [2] extended this model to consider the balking and reneging of the failed units. They obtained the steady-state availability and mean time to system failure. Hsieh and Wang [3] considered the reliability characteristics of a repairable system with m operating units, s warm spares and one removable repairman in the facility, where the repairman applies the N policy. They obtained the expressions of the reliability and the mean time to system failure. Jain and Maheshwari [4] extended the model of Hsieh and Wang [3] to analyze the repairable system in transient state by incorporating reneging behavior of the failed units.

The models mentioned above are all assumed that the repairmen are always available, but in many real world repairable systems, repairmen may become unavailable for a random period of time when there are no failed units in the system at a service completion instant. In this random period, repairmen can perform some secondary tasks which may reduce the burden on the system in terms of cost. However, there are only a few works that take into consideration the vacation of repairmen in the machine repair models with spares.

Gupta [5] first studied machine interference problem with warm spares and single server in which the server takes a vacation of random duration every time the repair facility becomes empty. They gave an algorithm to compute the steady-state probability distribution of the number of failed machines in the system.

Jain and Singh [6] studied a machine repair model with warm standbys, setup and vacation from the view point of the queueing theory. They considered (N, L) switch-over policy for the two repairmen. The first repairman turns on for repair only when N-failed units are accumulated and starts repair after a set up time. As soon as the system becomes empty, the repairman leaves for a vacation and returns back when this repairman finds the number of failed units in the system greater than or equal to a threshold value N. The second repairman turns on when there are L failed units in the system and goes for a vacation if there are less than L failed units.

Ke and Wang [7] studied machine repair problem with two type spares and multi-server vacations. They solved the steady-state probabilities equations iteratively and derived the steady-state probabilities in matrix form.

In many practical multiple server systems, only some servers perform secondary jobs (take vacation) when they become idle and the other servers are always available for serving the arriving units. In queueing system, this type of vacation is called “the partial server vacation" (see Zhang and Tian [8]). This motivates us to study the machine repair system with warm spares and a “partial server vacation" policy.

In this paper, we consider a machine repairable system with warm spares and two repairmen where “the partial server vacation" is applied. In our system, the first repairman is always available for serving the failed units. The second repairman leaves for a vacation of random length which is distributed exponentially when the number of the failed units is less than N. At the end of a vacation period, the repairman returns back if there are N or more failed units accumulated in the system. Otherwise, this repairman goes for another vacation. Moreover, in this paper, we not only analyze the queueing problems but also analyze the reliability characteristics of the system, while the papers presented in [5]-[7] only considered the queueing problems.

The rest of the paper is organized as follows. In Section 2, we describe the system model. In Section 3, using Markov process, we develop the difference equations of the state probabilities of the system and solve the equations iteratively. The expressions of some steady-state performance measures of queueing and reliability are obtained. In Section 4, we develop a cost model and investigate sensitivity analysis. Conclusions are given in Section 5.

2 System Model

In this paper, we consider a machine repairable system with m-operating units, s-standby units and two repairmen. The first repairman is always available for serving the failed units, while the second repairman leaves for vacations under some conditions. We call the first repairman "Repairman 1" and call the second repairman "Repairman 2". The assumptions of the system model are as follows:

(a) For the functioning of the system, m-operating units are required. However, the system may also work in short mode, i.e., if all spare units are exhausted and there are less than m but more than k operating units in operation. In other words, the system breakdowns if and only if $L = s + m - k + 1$ or more units fail. The life times of operating and spare units are exponentially distributed with rates λ and α, respectively.

(b) As soon as an operating unit fails, it is replaced by a spare unit if available and is immediately sent for repair. When a spare unit turns to be an operating unit, its failure characteristics will be the same as the operating unit. When a failed unit is repaired, it is as good as new one. The repaired unit goes to be operating if there are less than m operating units, otherwise the unit joins the standby group. The repair times of Repairmen 1 and Repairman 2 are exponentially distributed with rates μ_1 and μ_2, respectively.

(c) Repairman 1 is always available for serving the failed units. However, if there are less than N failed units, Repairman 2 goes for a vacation of random length. On return from a vacation if Repairman 2 finds more than or equal to N failed units accumulated in the system, Repairman 2 will start to repair the failed units, otherwise Repairman 2 goes for another vacation. The vacation time is exponentially distributed with rate θ.

(d) The switch is perfect and switch-over time is instantaneous. When a spare moves into an operating state, its failure characteristics will be that of an operating machine.

3 Steady-state Analysis

In this section, we develop the difference equations of the steady-state probabilities of the system by using Markov process. Then, we derive the expressions of the steady state probabilities by recursive method. Finally, we give the explicit expressions of some steady-state performance measures.

Let $L(t)$ be the number of failed units in the repair facility at time t and let

$$J(t) = \begin{cases} 0, & \text{Repairman 2 is on vacation at time } t \\ 1, & \text{Repairman 2 is not on vacation at time } t. \end{cases}$$

Then, $\{L(t), J(t)\}$ is a Markov process with state space

$$E = \{(n,0) : n = 0,1,...,L\} \cup \{(n,1) : n = N, N+1, ..., L\}.$$

Define the steady-state probabilities of the system as follows:

$$P_{n,0} = \lim_{t\to\infty} P\{L(t) = n, J(t) = 0\}, \quad 0 \le n \le L,$$
$$P_{n,1} = \lim_{t\to\infty} P\{L(t) = n, J(t) = 1\}, \quad N \le n \le L.$$

Then, the steady state probabilities equations governing the model are obtained as follows:

$$-\lambda_0 P_{0,0} + \mu_1 P_{1,0} = 0, \tag{1}$$

$$\lambda_{n-1} P_{n-1,0} - (\lambda_n + \mu_1) P_{n,0} + \mu_1 P_{n+1,0} = 0, \quad 1 \le n \le N-2, \tag{2}$$

$$\lambda_{N-2} P_{N-2,0} - (\lambda_{N-1} + \mu_1) P_{N-1,0} + \mu_1 P_{N,0} + \mu_2 P_{N,1} = 0, \tag{3}$$

$$\lambda_{n-1} P_{n-1,0} - (\lambda_n + \mu_1 + \theta) P_{n,0} + \mu_1 P_{n+1,0} = 0, \quad N \le n \le L-1, \tag{4}$$

$$\lambda_{L-1} P_{L-1,0} - (\mu_1 + \theta) P_{L,0} = 0, \tag{5}$$

$$\theta P_{N,0} - (\lambda_N + \mu_2) P_{N,1} + \mu_2 P_{N+1,1} = 0, \tag{6}$$

$$\theta P_{n,0} + \lambda_{n-1} P_{n-1,1} - (\lambda_n + \mu_2) P_{n,1} + \mu_2 P_{n+1,1} = 0, \quad N+1 \le n \le L-1, \tag{7}$$

$$\theta P_{L,0} + \lambda_{L-1} P_{L-1,1} - \mu_2 P_{L,1} = 0 \tag{8}$$

with the normalizing condition

$$\sum_{n=0}^{L} P_{n,0} + \sum_{n=N}^{L} P_{n,1} = 1 \tag{9}$$

where

$$\lambda_n = \begin{cases} m\lambda + (s-n)\alpha, & n = 0,1,\ldots,s \\ (m+s-n)\lambda, & n = s+1, s+2, \ldots, L-1 \\ 0, & n = L. \end{cases}$$

In order to solve the steady-state probability equations (1)-(9), we define

$$\varphi_n = \begin{cases} \left(\dfrac{1}{\mu_1}\right)^{n} \displaystyle\prod_{j=0}^{n-1} \lambda_j, & 1 \le n \le N-1 \\ \left(\dfrac{1}{\mu_1}\right)^{N-1} \displaystyle\prod_{j=N}^{n} \beta_j \prod_{j=0}^{N-2} \lambda_j, & N \le n \le L \end{cases} \tag{10}$$

where $\beta_j,\ j = N, N+1, \ldots, L$, is defined iteratively as follows:

$$\beta_L = \frac{\lambda_{L-1}}{\mu_1 + \theta}, \tag{11}$$

$$\beta_n = \frac{\lambda_{n-1}}{\mu_1 + \theta \left[1 + \displaystyle\sum_{i=n+1}^{L} \prod_{j=n+1}^{i} \beta_j \right]}, \quad N \le n \le L-1. \tag{12}$$

The following theorem gives the solutions of the steady-state probabilities equations (1)-(9).

Theorem 1. The steady-state probabilities are given by

$$P_{n,0} = \varphi_n P_{0,0}, \quad 1 \le n \le L, \tag{13}$$

$$P_{n,1} = \frac{\theta}{\mu_2} \psi_n P_{0,0}, \quad N \le n \le L \tag{14}$$

and

$$P_{0,0} = \left(1 + \sum_{n=1}^{L} \varphi_n + \frac{\theta}{\mu_2} \sum_{n=N}^{L} \psi_n\right)^{-1} \tag{15}$$

where φ_n is defined by Eq. (10) and ψ_n is defined iteratively as follows:

$$\psi_N = \sum_{i=N}^{L} \varphi_i, \tag{16}$$

$$\psi_n = \frac{\lambda_{n-1}}{\mu_2} \psi_{n-1} + \sum_{i=n}^{L} \varphi_i, \quad N+1 \le n \le L. \tag{17}$$

Proof. By Eqs. (1) and (2), we get

$$P_{n,0} = \frac{\lambda_{n-1}}{\mu_1} P_{n-1,0}, \quad 1 \le n \le N-1. \tag{18}$$

The recursive relation (18) gives

$$P_{n,0} = \varphi_n P_{0,0}, \quad 1 \le n \le N-1 \tag{19}$$

where φ_n, for $1 \le n \le N-1$, is defined by Eq. (10). By Eqs. (4) and (5), we get

$$\lambda_{n-1} P_{n-1,0} - (\mu_1 + \theta) P_{n,0} = \theta \sum_{i=n+1}^{L} P_{i,0}, \quad N \le n \le L-1. \tag{20}$$

Eq. (5) yields

$$P_{L,0} = \beta_L P_{L-1,0} \tag{21}$$

where β_L is defined by Eq. (11). Repeating the use of Eqs. (20) and (21), we give

$$P_{n,0} = \beta_n P_{n-1,0}, \quad N \le n \le L-1 \tag{22}$$

where β_n is defined by Eq. (12). From Eqs. (19), (21) and (22), we get Eq. (13). Eq. (7) yields

$$\mu_2 P_{n,1} - \lambda_{n-1} P_{n-1,1} = \mu_2 P_{n+1,1} - \lambda_n P_{n,1} + \theta P_{n,0}, \quad N+1 \le n \le L-1. \tag{23}$$

Repeating the use of Eq. (23) and Eq. (8), we get

$$P_{n,1} = \frac{\lambda_{n-1}}{\mu_2} P_{n-1,1} + \frac{\theta}{\mu_2} \sum_{i=n}^{L} P_{i,0}, \quad N+1 \le n \le L. \tag{24}$$

In Eq. (24), letting $n = N+1$ and then substituting Eq. (24) into Eq. (6), we get

$$P_{N,1} = \frac{\theta}{\mu_2} \sum_{i=N}^{L} P_{i,0}. \tag{25}$$

Substituting Eq. (13) into Eq. (24) and (25), we get Eq. (14), where ψ_n is defined by Eqs. (16) and (17). Substituting Eqs. (13) and (14) into Eq. (9), we get $P_{0,0}$ as shown in Eq. (15). This proves Theorem 1.

Using the steady state probabilities presented in Theorem 1, we can easily obtain the following performances of queueing and reliability.

Corollary 1.
(1) The average number of failed units in the system is given by

$$L_f = \left(\sum_{n=1}^{N-1} n\varphi_i + \frac{\theta}{\mu_2} \sum_{n=N}^{L} n\psi_n \right) P_{0,0}. \tag{26}$$

(2) The average number of units that function as spares is given by

$$L_s = \begin{cases} \left(\sum_{n=N}^{s-1} (s-n)\varphi_n + \frac{\theta}{\mu_2} \sum_{n=N}^{s-1} (s-n)\psi_n \right) P_{0,0}, & N < s \\ \sum_{n=0}^{s-1} (s-n)\varphi_n P_{0,0}, & N > s. \end{cases} \tag{27}$$

(3) The probability that Repairman 1 is busy is given by

$$P_b^1 = 1 - P_{0,0}. \tag{28}$$

(4) The probability that Repairman 2 is busy is given by

$$P_b^2 = \frac{\theta}{\mu_2} \sum_{n=N}^{L} \psi_n P_{0,0}. \tag{29}$$

(5) The steady-state availability of the system is given by

$$A = 1 - \left(\varphi_L + \frac{\theta}{\mu_2} \psi_L \right) P_{0,0}. \tag{30}$$

(6) The steady-state failure frequency is given by

$$M = \lambda_{L-1} \left(\varphi_{L-1} + \frac{\theta}{\mu_2} \psi_{L-1} \right) P_{0,0}. \tag{31}$$

4 Numerical Analysis

We develop a steady-state expected cost function per unit time, and impose a constraint on the availability of the system in which N is a decision variable. Our objective is to determine the optimum N^*, so that the cost is minimized and the availability of the system is maintained at a certain level.

Following Ke and Wang [7], let C_f be the cost per unit time of one failed unit in the repair facility, let C_s be the cost per unit time of one unit that functions as a spare, let C_b^1 be the cost per unit time that Repairman 1 is busy, let C_b^2 be the cost per unit time that

Repairman 2 is busy, let C_I be the cost per unit time that Repairman 1 is idle, and let C_v be the reward per unit time that Repairman 2 is on vacation.

Using the definition of the cost parameters listed above, the total expected cost function per unit time is given by

$$F(N) = C_f L_f + C_s L_s + C_b^1 P_b^1 + C_b^2 P_b^2 + C_I P_I - C_v P_v \tag{32}$$

where $P_I = P_{0,0}$ is the probability that Repairman 1 is idle, $P_v = 1 - P_b^2$ is the probability that Repairman 2 is on vacation, L_f, L_s, P_b^1 and P_b^2 are given by Corollary 1.

In order to maintain the availability of the system at a certain level, we present the cost minimization problem as follows:

$$\begin{array}{ll} \text{Min} & F(N) = \quad C_f L_f + C_s L_s + C_b^1 P_b^1 + C_b^2 P_b^2 + C_I P_I - C_v P_v \\ \text{S.t.} & A \geq A_0 \end{array} \tag{33}$$

where A is the steady-state availability of the system given by Eq. (30) and A_0 is the given level of the availability of the system as a system parameter.

The analytic study of the behavior of the expected cost function would have been an arduous task to undertake since the decision variable N is discrete and appear in an expression which is a highly nonlinear and complex. Thus, one may use a heuristic approach to obtain the optimum value N^* which is determined by satisfying inequalities as follows:

$$F(N^* - 1) > F(N^*) < F(N^* + 1)$$

and $A \geq A_0$.

We provide an example to perform a sensitivity analysis for changes in the optimum value N^* along with changes in specific values of the system parameters λ, α, μ_1, μ_2 and θ.

Example. We set $m = 10$, $s = 6$, $k = 7$, and $A_0 = 0.8$, the following cost elements are used:

$$C_f = 80, \quad C_s = 50, \quad C_b^1 = 70, \quad C_b^2 = 60, \quad C_I = 40, \quad C_v = 70.$$

The numerical results of the optimum value N^*, the optimum cost $F(N^*)$ and the availability of the system at the optimum value N^* are illustrated in Tables 1-3.

In Table 1, we fix $\mu_1 = 6$, $\mu_2 = 12$, $\theta = 4$, and vary the value of N from 1 to 10, and choose different values of (λ, α). Table 1 shows that the optimum cost $F(N^*)$ increases significantly and the optimum value N^* decreases as λ increases, while the optimum cost $F(N^*)$ increases slightly as α increases. From the last four columns of Table 1, we observe that the optimum value N^* dose not vary at all when α varies from 2 to 5. Intuitively, this seems too insensitive to changes in α. This is because that the number of working units is more than units as spares. Hence, the cost due to failure of working units is larger than the cost due to failure of units as spares.

In Table 2, we fix $\lambda = 1$, $\alpha = 2$, $\theta = 4$, and vary the value of N from 1 to 10, and choose different values of (μ_1, μ_2). Table 2 shows that the optimum cost $F(N^*)$ decreases and the optimum value N^* increases significantly as μ_1 increases, while the optimum cost $F(N^*)$ increases and the optimum value N^* decreases significantly as μ_2 increases. This is because that the number of failed units decreases as μ_1 increases. This results in the optimum cost decreasing as μ_1 increases. However, since Repairman 2 takes N policy vacation, the reward due to the vacation of Repairman 2 decreases as μ_2 increases. It results in the increasing of the optimum cost.

In Table 3, we fix $\lambda = 1$, $\alpha = 2$, $\mu_1 = 6$, $\mu_2 = 12$, and vary the value of N from 1 to 10, and choose different values of θ. Table 3 shows that the optimum cost $F(N^*)$ decreases slightly as θ increases. This is because that the reward due to the vacation of Repairman 2 increases as θ increases. This results in decreasing of the optimum cost. From the last five columns of Table 3, we observe that N^* are the same even though θ varies from 4 to 8. Intuitively, this seems too insensitive to changes in θ.

Table 1. The optimum cost, the optimum value and the system's availability at the optimum value ($\mu_1 = 6$, $\mu_2 = 12$, $\theta = 4$).

$(\lambda,\ \alpha)$	(0.4, 2)	(0.6, 2)	(0.8, 2)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
N^*	9	8	8	8	8	8	8
$F(N^*)$	85.85	152.05	258.83	378.23	382.50	385.71	388.24
$A(N^*)$	0.99	0.97	0.92	0.86	0.86	0.86	0.85

Table 2. The optimum cost, the optimum value and the system's availability at the optimum value ($\lambda = 1$, $\alpha = 2$, $\theta = 4$).

$(\mu_1,\ \mu_2)$	(4, 12)	(6, 12)	(8, 12)	(10, 12)	(6, 10)	(6, 12)	(6, 14)
N^*	7	8	8	9	8	8	7
$F(N^*)$	452.54	378.23	283.48	203.42	421.50	378.23	336.44
$A(N^*)$	0.86	0.86	0.90	0.93	0.83	0.86	0.91

Table 3. The optimum cost, the optimum value and the system's availability at the optimum value ($\lambda = 1$, $\alpha = 2$, $\mu_1 = 6$, $\mu_2 = 12$).

(θ)	2	3	4	5	6	7	8
N^*	6	7	8	8	8	8	8
$F(N^*)$	383.69	384.64	378.23	364.86	356.18	350.39	346.49
$A(N^*)$	0.89	0.88	0.86	0.87	0.88	0.88	0.89

5 Conclusions

In this paper, we analyzed a machine repair system with warm spares and N-policy vacation of a repairman from view point of queueing and reliability. We derived the expressions of the steady-state probabilities iteratively. Some performance measures of queueing and reliability were obtained. We developed a cost model to determine the optimum N while the system availability is maintained at a certain level. Sensitivity analysis was also investigated.

Acknowledgements

This work was supported in part by the National Natural Science Foundation of China (No. 70671088) and the Natural Science Foundation of Hebei Province (No.

A2004000185), China, and was supported in part by GRANT-IN-AID FOR SCIENTIFIC RESEARCH (No. 19500070) and MEXT.ORC (2004-2008), Japan.

References

[1] K. H. Wang, B. D. Sivazlian, Reliability of a system with warm standbys and repairmen,Microelectrons and Reliability, 29 (1989), 849-860.

[2] K. H. Wang, J. Ke, Probabilistic analysis of a repairable system with warm standbys plus balking and reneging, Applied Mathematical Modelling, 27 (2003), 327-336.

[3] Y. C. Hsieh, K. H. Wang, Reliability of a repairable system with spares and a removable repairman, Microelectronics and Reliability, 35 (1995), 197-208.

[4] M. Jain, Rakhee, S. Maheshwari, N-policy for a machine repair system with spares and reneging, Applied Mathematical Modelling, 28 (2004), 513-531.

[5] S. M. Gupta, Machine interference problem with warm spares, server vacations and exhaustive service, Performance Evaluation, 29 (1997), 195-211.

[6] M. Jain, M. Singh, Bilevel control of degraded machining system with warm stangdbys, setup and vacation, Applied Mathematical Modelling, 28 (2004), 1015-1026.

[7] J. C. Ke, K. H. Wang, Vacation policies for machine repair problem with two type spares, Applied Mathematical Modelling, 31 (2007), 880-894.

[8] Z. G. Zhang, N. Tian, Analysis on queueing systems with synchronous vacations of partial servers, Performance Evaluation, 52 (2003), 269-282.