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An Approximation Solution to the ELS Model for Perishable Inventory with Backlogging*

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Abstract An Economic Lot-sizing(ELS) problem with perishable inventory has been studied extensively over the years and plays a fundamental role in the inventory management. In this paper, we consider the problem where backlogging is allowed with the general economies of scale cost functions. Since the special case without backlogging is NP-hard, the considered problem is also NP-hard. The main contributions of this paper is to explore the properties of the optimal solution and propose an approximation solution with the cost no more than $\frac{4\sqrt{2}+5}{7}$ times the optimal cost. Our results generalize a study on an ELS model for perishable inventory but without backlogging.

Keywords Economic lot-sizing problem; Perishable inventory; Backlogging; Economies of scale function.

1 Introduction

The classical Economic Lot-sizing (ELS) problem was first introduced by Wagner and Whitin [1], and it has become one of the most studied problems in the area of production planning and inventory control. Chan et al.[2] consider an ELS problem with a modified all-unit discount freight cost structure. They demonstrate the NP-hardness and analyze the worst case performance of an easy-to-implement approximation solution which satisfies ZIO policy. Chan et al.[3] extend it to a single-warehouse multi-retailer setting. In both of the papers, they prove that the cost of the approximation solution is no more than $\frac{4}{3}(\frac{5.6}{4.6}$ if cost is stationary) times the optimal one. In addition, a lot of research has been devoted to considering an ELS problem with perishable inventory(see [4],[5]). However, to summarize the above papers, most of the cost structures are concave functions.

Few theoretical results have been published on the ELS problem for perishable inventory with economies of scale functions, which are fairly general cost structures. For the case that backlogging is not allowed that has been proved NP-hard problem, Chu et al. [6]propose an approximation solution in polynomial time by the the consecutive-cover-ordering (CCO) policy and show the cost is no more than $\frac{4\sqrt{2}+5}{7}(\frac{3}{2}$ if the ordering cost is stationary) times the optimal cost.

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In this paper, we extend the model studied by Chu et al.[6] to the case that backlogging is allowed and we explicitly prove that our model is similar in the optimal properties to that proposed by them. Motivated by technique used in their paper, we propose an approximation solution with the cost not more than $\frac{4\sqrt{2}+5}{7}$ times the optimal cost and prove that the bound is tight.

2 Formulations and the optimality properties

2.1 Notations and formulations

We begin this section with some notations used in the following paper.

d_t = demand in period $t, t = 1, 2, \ldots, n$;

x_t = production quantity in period t. If $x_t > 0$, we call period t a production period;

I_{it} = the amount of inventory produced in period i at the beginning of period t, $1 \leq i \leq t \leq n$;

$\alpha_{i,t}$ = the proportion of I_{it} that is lost in period t, $\alpha_{i,t} \in [0,1]$, $1 \leq i \leq t \leq n$;

Z_{kt} = the amount of unfilled period k demand at the end of period t, $1 \leq k \leq t \leq n$;

M_{kt} = the amount of demand in period t that is satisfied by the products in production period k, $k \in [1,n]$;

$C_t(x_t)$ = the cost of producing x_t units in period t;

$H_{it}(I_{it})$ = the cost of holding I_{it} units in period t, $1 \leq i \leq t \leq n$;

$B_{kt}(Z_{kt})$ = the penalty cost of leaving Z_{kt} units unfilled in period t, $1 \leq k \leq t \leq n$;

Hence, our problem denoted as BP can be formulated as

$$(BP) \quad \min \quad \sum_{t=1}^{n} \{C_t(x_t) + \sum_{k=1}^{t} [H_{kt}(I_{kt}) + B_{kt}(Z_{kt})]\}$$

$$s.t. \quad x_t = I_{tt} + \sum_{l=1}^{t} M_{tl}, \qquad t \in [1,n] \qquad (1)$$

$$I_{it} = (1 - \alpha_{i,t-1}) I_{i,t-1} - M_{it}, \qquad 1 \leq i < t \leq n \qquad (2)$$

$$Z_{kt} = Z_{k,t-1} - M_{tk}, \qquad 1 \leq k < t \leq n \qquad (3)$$

$$d_t = \sum_{i=1}^{n} M_{it}, \qquad t \in [1,n] \qquad (4)$$

$$x_t,\ I_{it},\ Z_{it} \geq 0,\ M_{kt} \geq 0, \qquad 1 \leq i \leq t \leq n,\ k \in [1,n] \qquad (5)$$

In many practical situation, the longer a perishable product is held, the faster it may deteriorate and the higher its inventory holding costs. So we make the following assumptions.

Assumption 1. $\alpha_{i,t} \geq \alpha_{j,t}$, $1 \leq i \leq j \leq t \leq n$.

Assumption 2. $H_{it}(y+\triangle) - H_{it}(y) \geq H_{jt}(x+\triangle) - H_{jt}(x)$, and $B_{it}(y+\triangle) - B_{it}(y) \geq B_{jt}(x+\triangle) - B_{jt}(x)$, where $\triangle,\ x,\ y \geq 0$ and $1 \leq i \leq j \leq t \leq n$.

Functions $C_i(\bullet), H_{ij}(\bullet)$ and $B_{ij}(\bullet)$ for $i \leq j$, are assumed to be general economies of scale functions. Here the definition and properties of economies of scale functions proposed are omitted due to space limitation and they can be found in Chu et al. [6]. We assume that the initial inventory and its final inventory at the end of the horizon are zero.

Production and demand fulfillments occur at the beginning of each period. Backlogging is allowed and there is no backlogging at the beginning or at the end of planning horizon.

Let $A^i_{kt} = \frac{1}{\prod_{l=k}^{t-1}(1-\alpha_{il})}$ for $i \le k < t$ with the conventions that $A^i_{ii} = 1$ and $A^i_{kt} = +\infty$ for $k \le l < t$ if $\alpha_{i,l} = 1$. We easily have $A^i_{jt} = A^i_{jk}A^i_{kt}$ for $i \le j < k < t$, and $A^i_{kt} \ge A^j_{kt}$ for $1 \le i \le j < k < t \le n$.

2.2 The optimality properties

Backlogging that is allowed in our problem may result in that the demand in any period t can be fulfilled by the one earlier production period to t but also by one future period instead of only by one earlier production period if backlogging is not allowed. Hence our analytical problem focus on how the optimal properties change. In this section, we first prove the FIFO policy that proposed by Chu et al.[6] for their model still holds for our model.

Property 1. *For any optimal solution $\Omega^+(X^+, I^+, M^+, Z^+)$ to BP problem, the following holds: if for two production periods i and j with $i < j$ we have that $M^+_{jt} > 0$ for some period t, then for any period $k > t$, $M^+_{ik} = 0$.*

Proof. Suppose that in an optimal solution $\Omega^+(X^+, I^+, M^+, Z^+)$, there exists some t such that $M^+_{jt} > 0$ and $M^+_{ik} > 0$ for all $k > t$. Since backlogging is allowed in our model, property 1 can be proved by considering the four cases: 1) $i < j \le t < k$, 2) $i < t \le j < k$ or $i \le t < j \le k$, 3) $i \le t < k \le j$, or $t < i \le k < j$, or $t \le i < k \le j$, or $t < k \le i < j$, 4) $t \le i < j \le k$ or $t < i \le k < j$ and the only case 1) is considered by Chu et al.[6] for their model. For these cases, we modify solution Ω^+ to a new feasible solution without increasing the total cost and easily have $M^+_{jt} = 0$ or $M^+_{ik} = 0$. Hence we have contradictions.

Next, we will establish another structural property (the similar proof is in Chu et al.[6]) of optimal solutions to *BP* from the properties of economies of scale functions and Property 1.

Property 2. *There is an optimal FIFO solution to BP problem such that each demand d_t, $t \in [1,n]$, is satisfied by at most three production periods.*

3 The worst case performance analysis

It has been shown that the special case of *BP* problem, the case that backlogging is not allowed, is NP-hard, the *BP* problem is also NP-hard. In this section, our primary aim is to develop an approximation solution and explore its value away from the optimal value. Firstly, we define a problem as the *SBP* if it is the same as the *BP* problem except that the demand in any period is fulfilled entirely by exactly one production period in a feasible solution. Since the *SBP* problem can be solved in polynomial time via a DP algorithm proved by Hsu [5], we can take the optimal solution to *SBP* for an approximation solution to *BP*. Now, we assume that the optimal solution to *BP* exists and shift it into a feasible solution to *SBP*, then we will show how far the optimal value of the *SBP* problem away from that of the *BP* problem.

Here we generalize the following definitions proposed by Chu et al. in which still be used in our paper. Let $R(\Omega^+)$ be the set of periods whose demand is satisfied by two or three production periods in the optimal FIFO solution Ω^+. Without loss of generality, for any period $j \in R(\Omega^+)$, we assume that there are three successive production periods

$u(j)$, $v(j)$ and $w(j)$ fulfilling $\alpha_j d_j$, $\beta_j d_j$ and $(1-\alpha_j-\beta_j)d_j$ units of period j demand, respectively, where $\alpha_j \in (0,1)$, $\beta_j \in [0,1)$. If there is a demand fulfilled by two production periods, we can regard $v(j)$ as a pseudo-period and $d_k = 0$, $C_k(\bullet) = \infty$, $H_{lk}(\bullet) = 0$, $\alpha_{lk} = 0$, $B_{lk}(\bullet) = 0$ for any l. In addition, for the case that the backlogging is not allowed, S_{ij} for $i \le j$ is denoted by the average cost to satisfy one unit demand in period j from production period i, that is, $S_{ij} = \overline{C}_i(X_i^+)A_{ij}^i + \sum_{l=i}^{j-1} \overline{H}_{il}(I_{il}^+)A_{lj}^i$. In our model, Let P_{ij} be the average cost to satisfy one unit demand in period j by production period i when backlogging occurs, that is, $P_{ij} = \overline{C}_i(X_i^+) + \sum_{l=j}^{i-1} \overline{B}_{jl}(Z_{jl}^+)$ for $i \ge j$.

With these notations, it is clear that there are four cases among j, $u(j)$, $v(j)$ and $w(j)$ instead of one case $u(j) < v(j) < w(j) \le j$ considered by Chu et al.[6]. Let $m(j) = 1,2,3,4$ denote the following four cases, respectively, $j \le u(j) < v(j) < w(j)$, $u(j) \le j < v(j) < w(j)$, $u(j) < v(j) \le j < w(j)$ and $u(j) < v(j) < w(j) \le j$. Then, V_{mj}, the average cost of fulfilling period j demand, according to different values of m can be formulated as follows.

$V_{1j} = [\alpha_j P_{uj} + \beta_j P_{vj} + (1-\alpha_j-\beta_j)P_{wj}]d_j$ for $m(j) = 1$.
$V_{2j} = [\alpha_j S_{uj} + \beta_j P_{vj} + (1-\alpha_j-\beta_j)P_{wj}]d_j$ for $m(j) = 2$.
$V_{3j} = [\alpha_j S_{uj} + \beta_j S_{vj} + (1-\alpha_j-\beta_j)P_{wj}]d_j$ for $m(j) = 3$.
$V_{4j} = [\alpha_j S_{uj} + \beta_j S_{vj} + (1-\alpha_j-\beta_j)S_{wj}]d_j$ for $m(j) = 4$.
Obviously, $V(\Omega^+) \ge \sum_{j \in R(\Omega^+)} V_{m(j)j} \quad m(j) \in [1,4]$.

Let Ω_{1j}, Ω_{2j} and Ω_{3j} be the solutions via combining three production periods $u(j)$, $v(j)$ and $w(j)$ of the optimal solution into production period $u(j)$, $v(j)$ and $w(j)$, respectively, and $V(\Omega_{ij})$ be the corresponding cost of solution Ω_{ij}. Obviously, $\Omega_{ij} (i = 1,2,3)$ is the same as Ω^+ except that d_j is satisfied entirely by one of them. Note that in an optimal solution, once the period j is fixed, its corresponding three production periods are known and the value of $m(j)$ is fixed. Let $\Delta_{ij}^{m(j)} (i = 1,2,3)$ be the upper bounds of the incremental cost $V(\Omega_{ij}) - V(\Omega^+)$, then we have the following results:

$\Delta_{1j}^1 = [(1-\alpha_j)P_{uj} - \beta_j P_{vj}]d_j$, $\Delta_{2j}^1 = (1-\beta_j P_{vj})d_j$, $\Delta_{3j}^1 = [(\alpha_j+\beta_j)P_{wj} - \beta_j P_{vj}]d_j$
.
$\Delta_{1j}^2 = [(1-\alpha_j)S_{uj} - \beta_j P_{vj}]d_j$, $\Delta_{2j}^2 = (1-\beta_j P_{vj})d_j$, $\Delta_{3j}^2 = [(\alpha_j+\beta_j)P_{wj} - \beta_j P_{vj}]d_j$.
$\Delta_{1j}^3 = [(1-\alpha_j)S_{uj} - \beta_j S_{vj}]d_j$, $\Delta_{2j}^3 = (1-\beta_j S_{vj})d_j$, $\Delta_{3j}^3 = [(\alpha_j+\beta_j)P_{wj} - \beta_j S_{vj}]d_j$.
$\Delta_{1j}^4 = [(1-\alpha_j)S_{uj} - \beta_j S_{vj}]d_j$, $\Delta_{2j}^4 = (1-\beta_j S_{vj})d_j$, $\Delta_{3j}^4 = [(\alpha_j+\beta_j)S_{wj} - \beta_j S_{vj}]d_j$.

We improve the algorithm in Chu et al.[6] and provide the following Shift algorithm to establish a feasible solution $\widehat{\Omega}$ to *SBP* from the optimal solution Ω^+ to *BP*.

Shift algorithm

Step 0. Set $p = 0$ and $\Omega^0 = \Omega^+$.

Step 1. Define j_p as the smallest index in $R(\Omega^+)$ for solution Ω^p. Combining the three production periods $u(j), v(j)$ and $w(j)$ into one of these production periods according to the value of $\min\{\Delta_{1j}^{m(j)}, \Delta_{2j}^{m(j)}, \Delta_{3j}^{m(j)}\}$, $m(j) \in [1,4]$ and we get a new solution Ω'.

Step 2. Set $R(\Omega') = R(\Omega^p) \backslash \{j_p\}$. If $R(\Omega') = \emptyset$, set $\widehat{\Omega} := \Omega'$ stop; otherwise, set $p := p+1$, $\Omega^p := \Omega'$, $R(\Omega^p) := R(\Omega')$ and go to Step 1.

We easily have a feasible solution $\widehat{\Omega}$ for problem *SBP* via the above algorithm in at most n iterations. For each p, we denote δ_p be 0 if $\min\{\Delta_{1j}^{m(j)},\ \Delta_{2j}^{m(j)},\ \Delta_{3j}^{m(j)}\} = \Delta_{3j}^{m(j)}$, otherwise, 1. Next, define $L_p^1 = C_{w(j_p)}(X_{w(j_p)}^p) - C_{w(j_p)}(X_{w(j_p)}^+)$ and $L_p^2 = C_{w(j_p)}(X_{w(j_p)}^p) - C_{w(j_p)}(X_{w(j_p)}^+) + \sum_{l=w(j_p)}^{j_p-1}[H_{w(j_p)l}(I_{w(j_p)l}^p) - H_{w(j_p)l}(I_{w(j_p)l}^+)]$, we have $\delta_p L_p^{i_p} \le 0$ for $i_p = 1,2$, and $i_p = 2$ if and only if $m(j_p) = 4$.

Based on the above preparations, we show inductively the main results in our paper hold. The following theorem will show that the results proposed by Chu et al. are special cases of our problem.

Theorem 1. *Consider the objective function value of an feasible solution to SBP problem that obtained by Shift algorithm for* $1 \le p \le |R(\Omega^+)|$ *and the optimal objective function value of BP problem such that increment total cost is at most* $\sum_{l=1}^{p}\min\{\Delta_{1j_l}^{m(j_l)},\ \Delta_{2j_l}^{m(j_l)},\ \Delta_{3j_l}^{m(j_l)}\} + \delta_p L_p^{i_p}$ *for* $m(j_l) \in [1,4]$ *and* $i_p = 1,2$.

Proof. Let $V(\Omega^+)$ and $V(\Omega_p)$ be the optimal objective function value and the approximation objective function value obtained by Shift algorithm for $1 \le p \le |R(\Omega^+)|$, respectively. Clearly, to prove the theorem, we only prove the following inequality holds for $i_p = 1,2$.

$$V(\Omega_p) - V(\Omega^+) \le \sum_{l=1}^{p}\min\{\Delta_{1j_l}^{m(j_l)},\ \Delta_{2j_l}^{m(j_l)},\ \Delta_{3j_l}^{m(j_l)}\} + \delta_p L_p^{i_p},\ \ m(j_l) \in [1,4]. \tag{6}$$

For simplicity, we replace $j_p, u_{j_p}, v_{j_p}, w_{j_p}, \alpha_{j_p}, \beta_{j_p}$by $j, u, v, w, \alpha, \beta$.

For $p = 1$, that is, in the first iteration, we have $M_{uj}^+ = \alpha d_j,\ \ M_{vj}^+ = \beta d_j,\ \ M_{wj}^+ = (1-\alpha-\beta)d_j$, and obtain a solution Ω^1 by executing one of the three combinations in step 1 in Shift algorithm.

Case 1. If $m(j_1) = 1$, we have $i_1 = 1$ from the definition of δ_p. In this case, we have $x_v^+ = \beta d_j;\ \ z_{jl}^+ = d_j$ for all $l \in [j,u)$; $z_{jl}^+ = (1-\alpha)d_j$ for all $l \in [u,v)$; $z_{jl}^+ = (1-\alpha-\beta)d_j$ for all $l \in [v,w)$.

Subcase 1.1. If $\min\{\Delta_{1j}^1,\ \Delta_{2j}^1,\ \Delta_{3j}^1\} = \Delta_{1j}^1$, we have the following equations, $M_{uj}' = d_j,\ M_{vj}' = M_{wj}' = 0,\ x_u' = x_u^+ + (1-\alpha)d_j,\ x_u' = 0,\ x_w' = x_w^+ - (1-\alpha-\beta)d_j,\ z_{jl}' = 0$ for all $l \in [u,w)$. Thus, the following inequalities hold.

$$V(\Omega') - V(\Omega^+) \le (1-\alpha)d_jP_{uj} - \beta d_jP_{vj} + C_w(x_w') - C_w(x_w^+) - \sum_{l=j}^{w-1}\overline{B}_{jl}(Z_{jl}^+)(1-\alpha-\beta)d_j$$
$$\le \Delta_{1j}^1 + L_1^1\delta_1,\quad \delta_1 = 1$$

The last inequality holds because $C_u(\bullet), H_{uj}(\bullet)$ and $B_{uj}(\bullet)$ $(u \le j)$ are assumed to be general economies of scale functions.

Subcase 1.2. If $\min\{\Delta_{1j}^1,\ \Delta_{2j}^1,\ \Delta_{3j}^1\} = \Delta_{2j}^1$, we have $M_{uj}' = M_{wj}' = 0,\ \ M_{vj}' = d_j,\ \ x_u' = x_u^+ - \alpha d_j,\ \ x_v' = x_v^+ + (1-\beta)d_j,\ \ x_w' = x_w^+ - (1-\alpha-\beta)d_j,\ \ z_{jl}' = z_{jl}^+$ for all $l \in [j,u)$, $z_{jl}' = z_{jl}^+ + \alpha d_j$ for all $l \in [u,v)$, $z_{jl}' = 0$ for all $l \in [v,w)$. Thus,

$V(\Omega')-V(\Omega^+)\leq(1-\beta)d_jP_{vj}$+$C_w(x'_w)$-$C_w(x^+_w)$-$\sum_{l=j}^{u-1}\overline{B}_{jl}(Z^+_{jl})(1-\beta)d_j$-$\sum_{l=u}^{w-1}\overline{B}_{jl}(Z^+_{jl})(1-\alpha-\beta)d_j\leq\Delta^1_{2j}+L^1_1\delta_1,\quad\delta_1=1$

Subcase 1.3. Suppose that $\min\{\Delta^1_{1j},\ \Delta^1_{2j},\ \Delta^1_{3j}\}=\Delta^1_{3j}$. we have $M'_{uj}=M'_{vj}=0,\ \ M'_{wj}=d_j$čň $x'_u=x^+_u-\alpha d_j,\ x'_v=0,\ x'_w=x^+_w+(\alpha+\beta)d_j, z'_{jl}=z^+_{jl}+\alpha d_j$ for all $l\in[u,v),\ z'_{jl}=z^+_{jl}+(\alpha+\beta)d_j$ for all $l\in[v,w)$. After the combination, we have $V(\Omega')-V(\Omega^+)\leq[\overline{C}_w(x^+_w)+\sum_{l=j}^{w-1}\overline{B}_{jl}(Z^+_{jl})](\alpha+\beta)d_j-[\overline{C}_v(x^+_v)+\sum_{l=j}^{v-1}\overline{B}_{jl}(Z^+_{jl})]\beta d_j-\sum_{l=j}^{w-1}\overline{B}_{jl}(Z^+_{jl})(\alpha+\beta)d_j+\sum_{l=j}^{v-1}\overline{B}_{jl}(Z^+_{jl})\beta d_j+\sum_{l=u}^{v-1}\overline{B}_{jl}(Z^+_{jl})\alpha d_j+\sum_{l=v}^{w-1}\overline{B}_{jl}(Z^+_{jl})(\alpha+\beta)d_j=\Delta^1_{3j}-\sum_{l=j}^{u-1}\overline{B}_{jl}(Z^+_{jl})\alpha d_j\leq\Delta^1_{3j}+L^1_1\delta_1,\quad\delta_1=0.$

Case 2. Similar to the proof for the case that $m(j_1)=1$, it holds for $m(j_1)=2$ or $m(j_1)=3$ or $m(j_1)=4$.

Thus, the statement holds for $p=1$. Assume now that the statement holds for $p=k-1$.

we show that it holds for p, that is, the following inequality holds.

$$V(\Omega^k)-V(\Omega^+)\leq\sum_{l=1}^{k}\min\{\Delta^{m(j_l)}_{1j_l},\ \Delta^{m(j_l)}_{2j_l},\ \Delta^{m(j_l)}_{3j_l}\}+\delta_kL^{i_p}_k,\ \ m(j_l)\in[1,4].\tag{7}$$

Firstly, we have $w(j_{k-1})\leq u(j_k)$ by property 1. We consider the following two cases.

Case 1. $w(j_{k-1})<u(j_k)$. In this case, we have $x^{k-1}_u=x^+_u,\ \ I^{k-1}_{ul}=I^+_{ul}$. By the same argument as the case $p=1$, we obtain the following inequality: for $m(j_k)\in[1,4]$,

$$V(\Omega^k)-V(\Omega^{k-1})\leq\min\{\Delta^{m(j_k)}_{1j_k},\ \Delta^{m(j_k)}_{2j_k},\ \Delta^{m(j_k)}_{3j_k}\}+\delta_kL^{i_k}_k,\ \ i_k=1,2.\tag{8}$$

Hence it easily to obtain the inequality 7 holds.

Case 2. $w(j_{k-1})=u(j_k)$. We consider the following subcases.

Subcase 2.1. $m(j_k)=1$. It is easy to verify that $m(j_{k-1})$ is 1, 2 or 3 and $i_k=1, i_{k-1}=1$. Now we further consider the following two subcases.

2.1.1. Suppose in the kth iteration, $\min\{\Delta^{m(j_k)}_{1j_k},\ \Delta^{m(j_k)}_{2j_k},\ \Delta^{m(j_k)}_{3j_k}\}=\Delta^{m(j_k)}_{1j_k}$ that is, $\Delta^{m(1)}_{1j_k}$.

If $\delta_{k-1}=1$, there is $\min\{\Delta^{m(j_{k-1})}_{1j_{k-1}},\ \Delta^{m(j_{k-1})}_{2j_{k-1}},\ \Delta^{m(j_{k-1})}_{3j_{k-1}}\}=\Delta^{m(j_{k-1})}_{1j_{k-1}}$ or $\Delta^{m(j_{k-1})}_{2j_{k-1}}$ in the $(k-1)$th iteration. Since $\delta_k=1$, $x^{k-1}_u\leq x^+_u$, and the other variables are the same as those in the optimal solution, we have $L^{i_{k-1}}_{k-1}=L^1_{k-1}=C_{u(j_k)}(X^{k-1}_{u(j_k)})-C_{u(j_k)}(X^+_{u(j_k)})=C_u(X^{k-1}_u)-C_u(X^+_u)$. Similar to the combination policy in the first iteration, we prove that the inequality 8 for $m(j_k)=1$ holds , then 7 holds. Otherwise, $\delta_{k-1}=0$, that is, in the $(k-1)$th iteration there is $\min\{\Delta^{m(j_{k-1})}_{1j_{k-1}},\ \Delta^{m(j_{k-1})}_{2j_{k-1}},\ \Delta^{m(j_{k-1})}_{3j_{k-1}}\}=\Delta^{m(j_{k-1})}_{3j_{k-1}}$. Since $\delta_k=1,\ \ \delta_{k-1}=0$, and the other variables are the same as those in the optimal solution, we have $C_u(x^k_u)-C_u(x^{k-1}_u)\leq\overline{C}_u(x^{k-1}_u)(1-\alpha)d_j\leq\overline{C}_u(x^+_u)(1-\alpha)d_j$. Similar to the discussion in the first iteration, we prove that the inequalities 8 and 7 hold.

2.1.2. Suppose in the kth iteration, $\min\{\Delta^{m(j_k)}_{1j_k},\ \Delta^{m(j_k)}_{2j_k},\ \Delta^{m(j_k)}_{3j_k}\}=\Delta^{m(j_k)}_{2j_k}$ or $\Delta^{m(j_k)}_{3j_k}$ In this case, no matter how it combine in the $(k-1)$th iteration, the cost in period $i(j_m)$ is not greater than that before combining. Thus, inequalities 8 and 7 hold.

Subcase 2.2. $m(j_k) = 2$. In this case, $i_k = 1$, then the following two subcases are considered.

2.2.1. If $m(j_{k-1}) = 1,2,3$, similar to the discussion in the case 2.1.

2.2.2. If $m(j_{k-1}) = 4$, we have $u \le j_{k-1} < j_k$ and we further consider the following two subcases.

2.2.2.1. Suppose in kth iteration, $\min\{\Delta_{1j_k}^{m(j_k)},\ \Delta_{2j_k}^{m(j_k)},\ \Delta_{3j_k}^{m(j_k)}\} = \Delta_{1j_k}^{m(j_k)}$, then $\delta_k = 1$.

If $\min\{\Delta_{1j_{k-1}}^{m(j_{k-1})},\ \Delta_{2j_{k-1}}^{m(j_{k-1})},\ \Delta_{3j_{k-1}}^{m(j_{k-1})}\} = \Delta_{1j_{k-1}}^{m(j_{k-1})}$ or $\Delta_{2j_{k-1}}^{m(j_{k-1})}$ holds in the $(k-1)$th iteration, we have $\delta_{k-1} = 1$, $x_u^{k-1} \le x_u^+$ and $I_{ul}^{k-1} \le I_{ul}^+$ for any $l \in [u, j_{k-1})$. Thus,

$$\delta_{k-1} L_{k-1}^{i_{k-1}} = L_{k-1}^2 = C_u(X_u^{k-1}) - C_u(X_u^+) + \sum_{l=u}^{j_{k-1}-1}[H_{ul}(I_{ul}^{k-1}) - H_{ul}(I_{ul}^+)]$$

and $C_u(x_u^k) - C_u(x_u^{k-1}) + \sum_{l=u}^{j-1}[H_{ul}(I_{ul}^k) - H_{ul}(I_{ul}^{k-1})] \le S_{u,j}(1-\alpha)d_j - \delta_{k-1}L_{k-1}^{i_{k-1}}$

We easily show that $V(\Omega^k) - V(\Omega^{k-1}) \le \min\{\Delta_{1j_k}^2,\ \Delta_{2j_k}^2,\ \Delta_{3j_k}^2\} + \delta_k L_k^{i_k} - \delta_{k-1}L_{k-1}^{i_{k-1}}$ and the inequality 7 holds.

If $\min\{\Delta_{1j_{k-1}}^{m(j_{k-1})},\ \Delta_{2j_{k-1}}^{m(j_{k-1})},\ \Delta_{3j_{k-1}}^{m(j_{k-1})}\} = \Delta_{3j_{k-1}}^{m(j_{k-1})}$ holds in the $(k-1)$th iteration, we have $\delta_{k-1} = 0$. Since $x_u^{k-1} \ge x_u^+$, and $I_{ul}^{k-1} \ge I_{ul}^+$ for any $l \in [u, j_{k-1})$, we have

$C_u(x_u^k)$-$C_u(x_u^{k-1})$+$\sum_{l=u}^{j-1}[H_{ul}(I_{ul}^k)$-$H_{ul}(I_{ul}^{k-1})] \le \overline{C}_u(x_u^{k-1})(1-\alpha)d_jA_{uj}^u$+$\sum_{l=u}^{j_{k-1}-1}\overline{H}_{ul}(I_{ul}^{k-1})(1-\alpha)d_jA_{lj}^u$+$\sum_{l=j_{k-1}}^{j-1}[H_{ul}(I_{ul}^k) - H_{ul}(I_{ul}^+)] \le S_{u,j}(1-\alpha)d_j$

Thus, it is easy to verify that the inequalities 8 and 7 hold.

2.2.2.2. Suppose that $\min\{\Delta_{1j_k}^{m(j_k)},\ \Delta_{2j_k}^{m(j_k)},\ \Delta_{3j_k}^{m(j_k)}\} = \Delta_{2j_k}^{m(j_k)}$ or $\Delta_{3j_k}^{m(j_k)}$ holds in the kth iteration, we have $\delta_k = 0$. In this case, no matte how it combine in the $(k-1)$th iteration, we have $C_u(x_u^k) - C_u(x_u^{k-1}) + \sum_{l=u}^{j-1}[H_{ul}(I_{ul}^k) - H_{ul}(I_{ul}^{k-1})] \le 0$. Thus, the inequalities 8 and 7 hold.

Case 2.3. $m(j_k) = 3$. In this case, similar to case 2.1 for $m(j_{k-1}) = 1,2,3$ and similar to the discussion of case 2.2.2 for $m(j_{k-1}) = 4$, we conclude that the theorem holds.

Case 2.4. $m(j_k) = 4$. In this case, similar to the cases 2.1 and 2.2.2, we have that the theorem holds for $m(j_{k-1}) = 1,2,3$ and $m(j_{k-1}) = 4$, respectively.

In summary, in the kth iteration, we have

$$V(\Omega^k) - V(\Omega^+) \le \sum_{l=1}^{k}\min\{\Delta_{1j_l}^{m(j_l)},\ \Delta_{2j_l}^{m(j_l)},\ \Delta_{3j_l}^{m(j_l)}\} + \delta_k L_k^{i_k},\ \ m(j_l) \in [1,4],\ \ i_k = 1,2.$$

Thus, the theorem holds.

The following lemma presented by Chu et al. [6] contributes to our results.

Lemma 1. *For arbitrary* α, β *and* c_i, $i = 1,2,3$, *where* $\alpha > 0, \beta \ge 0, \alpha+\beta < 1$ *and* $c_i \ge 0$, *we have* $\min\{(1-\alpha)c_1 - \beta c_2,\ (1-\beta)c_2, (\alpha+\beta)c_3\text{-}\beta c_2\} \le \frac{4\sqrt{2}-2}{7}[\alpha c_1 + \beta c_2 + (1-\alpha-\beta)c_3]$.

By Lemma 2, the following inequalities to *BP* problem holds

$$\min\{\Delta_{1j}^{m(j)},\ \Delta_{2j}^{m(j)},\ \Delta_{3j}^{m(j)}\} \leq \frac{4\sqrt{2}-2}{7}V_{m(j)j}, \quad m(j) \in [1,4].$$

Let Ω^* and $V(\Omega^*)$be the optimal solution of problem *SBP* and the corresponding optimal value. As mentioned above discussions, we can solve Ω^* in polynomial time by a method in Hsu [5] and we have $V(\Omega^*) \leq V(\widehat{\Omega})$ $(\widehat{\Omega} = \Omega_{|R(\Omega^+)|})$. Clearly Ω^* is a feasible solution of *BP* problem. Now, we will show how far the value of a feasible solution, Ω^*, away form that the optimal solution, Ω^+. Combining Theorem 1 with Lemma 1, we easily get the following result in which is the same as the case without backlogging and it also shows that our results have generality for the model that backlogging is allowed or not allowed.

Theorem 2.*Consider the cost of the approximation solution, $V(\Omega^*)$, and the optimal cost, $V(\Omega^+)$, such that $V(\Omega^*) \leq \frac{4\sqrt{2}+5}{7}V(\Omega^+)$ for any instance of problem (BP) and the bound is tight.*

The instance that shows the bound is tight is omitted here due to the page limitation.

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