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Optimal Control Policy for Stochastic Inventory Systems with Multiple Types of Reverse Flows

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Abstract We study a single-product periodic-review inventory system with multiple types of returns. The serviceable products used to fulfill stochastic customer demand can be either manufactured/ordered, or remanufactured from the returned products, and the objective is to minimize the expected total discounted cost over a finite planning horizon. We show that, under some circumstances but not all, the optimal policy has a simple form and can be completely characterized by a sequence of constant control parameters. However, in some other scenarios, the optimal policy can be quite complicated and control parameters are state-dependent. We present a partial characterization on the optimal control policy for the general case when there are only two types of returns.

Keywords Inventory system; product returns; reverse logistics; remanufacturing; control policy; and base-stock levels.

1 Introduction

In this paper, we study a periodic-review single-product inventory system with multiple types of returns. Due to differences in returned products, prior to remanufacturing, the company has to diagnose and sort the returned products based on their operational conditions (See examples and discussion on the returned products condition variability in Guide et al. 2003 and Galbreth and Blackburn 2006). Different returned products usually require different actions and incur different remanufacturing costs, e.g., the returned products at worse conditions incur a higher remanufacturing cost. For convenience, in this paper the returns of different physical conditions are referred to as *different types of returns*.

Following Simpson (1978), Inderfurth (1997), DeCroix (2006) and DeCroix and Zipkin (2005), the demands and returns in different periods are independent random variables but they could be correlated in the same period. The assumption of independence between returns and past sales will be reasonable if the sold products are widely spread and there

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are usually large quantities of sold products in the market (See also Fleishmann 2000 for a detailed justification of this assumption). The serviceable products used to fulfill customer demand can be either manufactured/ordered, or remanufactured from any type of the returned products. The objective is to characterize the optimal inventory replenishment and remanufacturing policy so that the total discounted production, remanufacturing and inventory cost over a finite planning horizon is minimized.

This work is motivated by a project on inventory management with a large energy company. The company provides service on, among other things, meters and transformers for private houses and business buildings in the southeast United States. The meters and transformers are owned by the energy company. The company has several geographically located distribution centers, and one warehouse/service center located in central North Carolina. All failed meters/transformers are shipped to the warehouse/service center, diagnosed on operational conditions, and stored in inventory. Returns with very bad conditions are disposed. The company's daily operation is based on an inventory control control system called “Passport", which manages the replenishment of stock levels. However, the system does not take into consideration the returned products, which account for over a third of the company's total business. The problem the company faces is, how to make remanufacturing decisions of the various types of returns jointly with replenishment decision to minimize total cost?

We formulate an inventory control model with multiple types of returns and investigate the structural properties of the cost functions. These properties enable us to characterize the optimal inventory polices for the system. We find that the structure of the optimal policy for the system with a single type return, that was established by Simpson (1978), cannot be extended to the system with multiple types of returns in general. We show that, in some scenarios, the optimal policies are simple and can be completely determined by a sequence of state-independent parameters. But in others, the optimal policies can be quite complex and the control parameters depend on the system states, i.e., both the inventory levels of serviceable and returned products. We present a partial characterization on the optimal control policy for the general case when there are only two types of returns.

The rest of this paper is organized as follows. In Section 2, we introduce the model and give the mathematical formulation. In Section 3, we identify the parameter range and characterize the state-independent optimal inventory and remanufacturing policies, while in Section 4, we show that outside of the parameter range, the optimal control policies are state-dependent. Section 5 concludes the paper.

Throughout the paper, we use terms “remanufacturable products” and “returns”, “remanufacture" and “repair", and “production” and “manufacturing” interchangeably. Boldface notation indicates row vector and a function is increasing in a vector means that it is increasing in every component of the vector.“Increasing" and “decreasing" are used in a non-strict sense, that is, they represent “nondecreasing" and “nonincreasing" respectively.

2 The Model

Consider a periodic review inventory/production system with a planning horizon of N periods, indexed by $1,2,\ldots,N$. The manufacturer produces a single product to satisfy market demand. In each period, the manufacturer receives random customer demand for serviceable products as well as random customer returns of remanufacturable products. The returned products can be brought back to their working condition by remanufacturing that may involve operations such as disassembly, overhaul, repair, and replacement. The returned products are classified into K different types based on their physical conditions. The manufacturer keeps inventories for serviceable products and for each type of returned products. A serviceable product can be either manufactured from raw materials or remanufactured from each type of returned product. Excess demand is fully backlogged.

Let D_n be the demand for period n, and $R_{n,k}$ the number of type k returns in period n, $k = 1,\dots,K$. We assume that the demands $D_1, D_2, \dots$ as well as the returns $R_{1,k}, R_{2,k}, \dots$ are i.i.d. continuous random variables, but D_n and $(R_{n,1},\dots,R_{n,K})$ can have an arbitrary joint probability distribution for each period n. The unit production cost is p and the unit remanufacturing cost for a type k return is r_k, $k = 1,\dots,K$. Note that in practice, remanufacturing process often incurs more costs than just remanufacturing cost, for example, costs of collecting returned products, diagnosing, sorting, etc. which are actually happening before remanufacturing operations. Here, we simply use one single variable cost r_k to partially reflect this nature. In addition, it costs s_k, $k = 1,\dots,K$ to keep one unit of type k returned product per period in stock and the storage cost for returned products can be either higher or lower than the holding cost rate of serviceable products. Here s_k can either represent direct physical storage costs or any indirect cost stemming from holding the unit. The following two arguments can be used to justify different storage costs for different kinds of returns. First, the storage cost often is interpreted as a proxy of money invested on the returned unit. As the remanufacturing costs of different returns are not identical while the selling prices of the serviceable products are the same, the financial loss due to tied-up capital (the selling price minus the remanufacturing cost) is naturally different. Second, our model also resembles the inventory systems with recycling or product recovery operations that recover materials and parts from old or outdated products (including reuse of parts and products). In such case, the different types of returns may be different products but share the same component For example, PC components in Taiwan are mandated to be recycled, and mostly, to extract for metals or other components. There are different types of returns faced by the recycling operator: Notebook computer, main board, hard disk, etc. The Environmental Protection Administration (EPA) of Taiwan provides typical physical storage costs for these different returns(NT/unit, 35 NT=1 USD): Notebook computer 31, main board 38, PC hard disk 55, power supplier 13, etc.(Lee et al. 2000)

We use $G(\cdot)$ to denote the expected one-period convex serviceable inventory holding and customer backlog cost. A typical example is $G(x) = h\mathsf{E}[\max\{x-D,0\}] + b\mathsf{E}[\max\{D-x,0\}]$, where x is the inventory level of serviceable product at the beginning of the period, D is the one-period demand, and h and b are the holding and shortage cost rate. But $G(\cdot)$ can be more general.

The unit production cost is higher than the remanufacturing cost of any returned products, i.e., $p > r_k$ for all $1 \le k \le K$. If this is not true, then remanufacturing would never be economical. Without loss of generality, we index the returned products according to

$$(1-\alpha)r_1 - s_1 \le \cdots \le (1-\alpha)r_K - s_K,$$

where α is the discount factor, i.e., $0 \le \alpha \le 1$. This means, loosely speaking, when taking

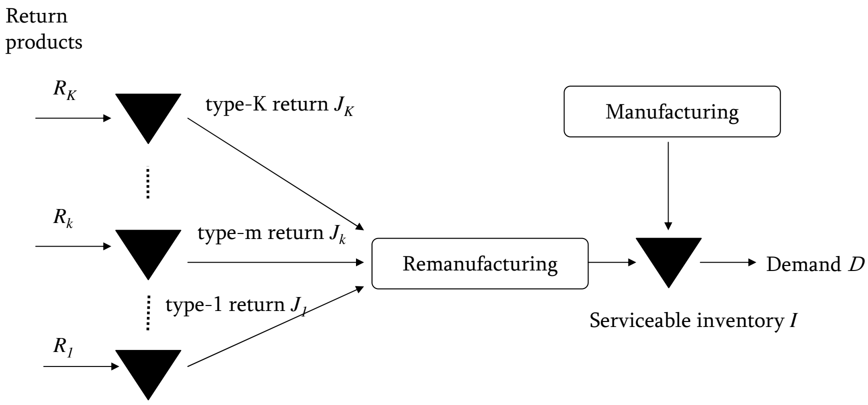

Figure 1: An inventory system with K types of returns

both remanufacturing and holding cost into consideration, type 1 returns have the highest priority to be remanufactured while type K returns have the lowest.

The returned products are not allowed to be disposed. Such systems have their practical meaning. For instance, over 99% of sources from the end product by Fuji-Xerox are reused rather than disposed in the Asia-Pacific region, which is 70% of Fuji-Xerox's spare parts need (Fuji Xerox 2007). We can also consider that the returned products in inventory have been processed with some costs. Therefore, they are "filtered" and not supposed to be disposed.

We assume remanufacturing leadtimes for different types of returns are identical and the same as production leadtime. We note that if the remanufacturing leadtime is different from the production leadtime, the optimal policy will become very complicated and the control parameters are state-dependent even for the single-type return model (Inderfurth 1997). Under the identical leadtime assumption, the system can be transformed into an equivalent model with zero leadtimes. Hence, for ease of exposition, we focus on the problem with zero leadtimes, i.e., the quantities manufactured and remanufactured at the beginning of a period can be used to satisfy demand in the same period. The objective is to find the optimal production, remanufacturing and inventory control policies to minimize the expected total discounted cost over a finite planning horizon.

Additional notation that will be used in this paper is summarized in the following. We let the subscript n denote the period n.

$I_n =$ the starting inventory level of serviceable product;
$J_{n,k} =$ the starting inventory level of type k returned product;
$J_n = (J_{n,1}, \dots, J_{n,K})$;
$i_n =$ the inventory level of serviceable product after manufacturing and remanufacturing decisions but before demand is realized;
$j_{n,k} =$ the inventory level of type k returned product after remanufacturing and disposal decisions but before return occurs;

$j_n = (j_{n,1}, \dots, j_{n,K})$;
$w_{n,k}$ = the remanufacturing quantity of type k return;
$w_n = (w_{n,1}, \dots, w_{n,K})$;
$R_n(k) = \sum_{\ell=1}^{k} R_{n,\ell}$, the total one period return of type 1 to type k.
$R_{n,k} = (R_{n,1}, \dots, R_{n,k})$, $k = 1, \dots, K$.

The time sequence of events is as follows. At the beginning of each period, the firm first decides remanufacturing quantities from each type of returned products; second, the firm decides how much to manufacture from raw material if needed; third, customer demand and product returns are realized; fourth, all costs are calculated. Figure 1 visualizes such an inventory system with some of the above notation.

Given the starting inventory levels of serviceable products I_n and returns J_n, let $V_n(I_n, J_n)$ be the minimum total discounted cost from period n to the end of the planning horizon. The dynamic programming formulation of the problem is

$$\begin{aligned} V_n(I_n, J_n) \quad = \quad & \min_{w_n, j_n, i_n} \left\{ \sum_{k=1}^{K} r_k w_{n,k} + p\left(i_n - I_n - \sum_{k=1}^{K} w_{n,k} \right) + \sum_{k=1}^{K} s_k j_{n,k} \right. \\ & \left. + G(i_n) + \alpha \mathsf{E}[V_{n+1}(i_n - D_n, j_n + R_{n,K})] \right\}, \qquad (1) \end{aligned}$$

subject to $0 \le w_{n,k} = J_{n,k} - j_{n,k}$, $j_{n,k} \ge 0$ for $k = 1, \dots, K$, and $\sum_{k=1}^{K} w_{n,k} \le i_n - I_n$. As in Simpson (1978), we assume $V_{N+1}(i, j) = 0$ for any i, j.

In (1), the first term inside the brackets is the total remanufacturing cost of returned products; the second term is the production cost, the third term is the total storage cost of all returned products; the fourth term is the inventory holding and shortage costs of serviceable products; and the last term is the minimum discounted expected total cost from period $n+1$ to period N. For the constraints, the first set requires that the remanufactured quantity for type k return be nonnegative and equal to the difference between its starting inventory level and ending inventory level before demand and return occur since disposal is not allowed (hereafter we omit "before demand and return occur" unless confusion may arise); the second set ensures that the ending inventory of type k return is nonnegative while the last constraint states that the total remanufactured quantity must be less than or equal to the increment of serviceable inventory level because there may be some units produced from raw materials.

Depending on the system cost parameters, the control parameters of the optimal policies can be either state-independent or state-dependent, which will be discussed in the two subsequent sections separately.

3 State-Independent Optimal Policies

For notational convenience, we will suppress the time index n unless confusion may arise. Since $w_k = J_k - j_k, k = 1, \dots, K$, the number of units that is produced from raw material is $i+\sum_{k=1}^K j_k - (I+\sum_{k=1}^K J_k)$. Consequently, the formulation (1) is simplified to,

$$\begin{aligned} V_n(I,J) &= \min_{i,j}\left\{\sum_{k=1}^K r_k(J_k-j_k)+p\left(i+\sum_{k=1}^K j_k - I - \sum_{k=1}^K J_k\right)\right. \\ &\qquad \left. +\sum_{k=1}^K s_k j_k + G(i) + \alpha \mathsf{E}[V_{n+1}(i-D, j+R_K)]\right\}, \end{aligned} \tag{2}$$

subject to $i+\sum_{k=1}^K j_k \geq I+\sum_{k=1}^K J_k$ and $0 \leq j_k \leq J_k$, for $k=1,\dots,K$. To make the system more amenable to analyze, we define

$$\begin{aligned} y_0 &= i, \\ y_k &= i+\sum_{\ell=1}^k j_\ell, \quad k=1,\dots,K, \end{aligned} \tag{3}$$

and

$$\begin{aligned} x_0 &= I, \\ x_k &= I+\sum_{\ell=1}^k J_\ell, \quad k=1,\dots,K. \end{aligned} \tag{4}$$

This transformation is critical and it sets the stage for the remaining analysis. We can interpret x_0 (resp. y_0) as the starting (resp., ending) serviceable inventory level, x_k (resp. y_k) as the starting (resp., ending) aggregate inventory level of serviceable and type 1 to k returned products, $k=1,\dots,K$. By definition, we have $x_0 \leq x_1 \leq \cdots \leq x_K$ and $y_0 \leq y_1 \leq \cdots \leq y_K$.

After some algebra and with a slight abuse of notation, we let $\mathbf{x}=(x_0,\dots,x_K)$, $\mathbf{y}=(y_0,\dots,y_K)$, and rewrite (2) as

$$V_n(\mathbf{x}) = \min_{\mathbf{y}}\{H_n(\mathbf{y})\} - r_1x_0 + \sum_{k=1}^{K-1}(r_k-r_{k+1})x_k + (r_K-p)x_K \tag{5}$$

$$\begin{aligned} \text{s.t.} \quad & x_0 \leq y_0 \leq y_1 \leq \cdots \leq y_K, \\ & x_K \leq y_K, \\ & y_{k+1}-y_k \leq x_{k+1}-x_k, \quad k=0,\dots,K-1, \end{aligned}$$

where

$$\begin{aligned} H_n(\mathbf{y}) &= (r_1-s_1)y_0 + G(y_0) + \sum_{k=1}^{K-1}(r_{k+1}-r_k+s_k-s_{k+1})y_k + (p-r_K+s_K)y_K \\ &\quad +\alpha\mathsf{E}[V_{n+1}(y_0-D, y_1+R_1-D, y_2+R(2)-D,\dots,y_K+R(K)-D)]. \end{aligned}$$

Note that, in the constraints above, $y_k \geq x_k$ is implied by $y_K \geq x_K$ together with $y_{k+1}-y_k \leq x_{k+1}-x_k$, for $k=0,\dots,K-1$.

The problem (5) is to minimize a multi-dimensional convex function subject to a set of linear state-dependent constraints. The optimal solution is, in general, complicated and depends on the state $\mathbf{x}$ of the system, and the structure of the optimal policy does not have a simple form. In what follows, we show that when the system parameters satisfy

$$r_1 - s_1 \leq r_2 - s_2 \leq \cdots \leq r_K - s_K, \tag{6}$$

then the optimal strategy has an exceedingly simple structure. This condition will hold, for instance, when the reparing costs of types 1 to K return are ascending while their storage costs are descending in the type index. This happens in practice, when type 1 has the best quality (lowest repairing cost) and the storage cost mainly due to the capital tied-up (interest rate$\times$ $(p - r_k)$).

The following result shows the convexity of the value function, which can be easily proved by induction hence its proof is not given.

Lemma 1.
$V_n(\mathbf{x})$ and $H_n(\mathbf{y})$ are jointly convex functions for all n.

The key for having a simple structure of the optimal control policy is the following proposition.

Proposition 1. *If the system parameters satisfy (6), then, for $n = 1,\ldots,N+1$, $V_n(\mathbf{x})$ can be decomposed as $V_n(\mathbf{x}) = \sum_{k=0}^{K} Q_{n,k}(x_k)$, in which $Q_{n,k}(\cdot)$, $k = 0,\ldots,K$, is a univariate convex function.*

The following is the main result of this section.

Theorem 2.
For the inventory system with K types of product returns and no disposal, suppose the current state is $\mathbf{x}$ (define $x_{K+1} = \infty$ and $\xi_{n,-1} = \infty$). If the system parameters satisfy (6), then there exists a sequence of constants $\xi_{n,K} \leq \xi_{n,K-1} \leq \cdots \leq \xi_{n,1} \leq \xi_{n,0}$, such that the optimal inventory levels after decisions, $y_0^*,\ldots,y_K^*$, are given by the following:

For $k = 0,\ldots,K$, if $\xi_{n,k} \leq x_k < \xi_{n,k-1}$, then $y_0^* = y_1^* = \cdots = y_k^* = x_k$, $y_{k+1}^* = x_{k+1},\ldots,$ $y_K^* = x_K$; if $x_k < \xi_{n,k} \leq x_{k+1}$, then $y_0^* = y_1^* = \cdots = y_k^* = \xi_{n,k}$, $y_{k+1}^* = x_{k+1},\ldots,y_K^* = x_K$.

Proof. The following important properties of $H_n(\mathbf{y})$ are essential for the proof of Theorem 2.

Lemma 3.
For $n = 1,\ldots,N$,

(a) *$H_n(\mathbf{y})$ is increasing in $y_1, y_2,\ldots,y_K$.*
(b) *$\partial V_n(\mathbf{x})/\partial x_k \geq r_k - r_{k+1}$ for $k = 1,\ldots,K-1$, and $\partial V_n(\mathbf{x})/\partial x_K \geq r_K - p$.*

Proofs of Proposition 1, Theorem 2, and Lemma 3. In the following, we prove Lemma 3, Proposition 1 and Theorem 2 together by induction on n. Because the proof is relatively long, we first summarize the flow of the proof as follows. We start with showing Proposition 1 is true for period $N+1$ and Part (a) of Lemma 3 is true for period $n = N$. Then assume the proposition is true for some $n = t+1$ and part (a) of the lemma is true for $n = t$ for $t \leq N$, we show that this implies the theorem for the period $n = t$ (in fact, we just need to have inductive assumption on the lemma and proposition). After that, based on the theorem, we show part (b) of the lemma is true for period $n = N$ as well as period

$n = t < N$. Finally, we prove that the proposition is true for $n = t$ and, based on part (b), part (a) is true for $n = t-1$ so that the proof is complete.

Notice that for period N, as $V_{N+1}(\mathbf{x}) = 0$, the proposition is trivially true. For part (a) of the lemma, it is not hard to show that $H_N(\mathbf{y})$ is increasing in y_k, $k \geq 1$, from the condition (6). Part (b) will be shown after we prove the theorem. Suppose Proposition 1 is true for period $n = t+1$ and part(a) of Lemma 3 is true at period $n = t$ for $t \leq N$. We next show the policy presented in the theorem is optimal for period $n = t$. Define the sequence of constants $(\xi_{t,0}, \ldots, \xi_{t,K})$ as follows.

$$\begin{aligned}
\xi_{t,k} &= \arg\min_x H_t(x, x, \ldots, x, y_{k+1}, \ldots, y_K), \qquad \text{for } k = 0, \ldots, K-1, \\
&= \arg\min_x \left\{ (r_{k+1} - s_{k+1})x + G(x) + \sum_{l=0}^{k} \alpha \mathsf{E}\left[Q_{t+1,l}\Big(x - D + R(l)\Big) \right] \right\}, \\
\xi_{t,K} &= \arg\min_x H_t(x, x, \ldots, x) = \arg\min_x \left\{ px + G(x) + \sum_{l=0}^{K} \alpha \mathsf{E}\left[Q_{t+1,l}\Big(x - D + R(l)\Big) \right] \right\}.
\end{aligned}$$

Because of Proposition 1, $\mathbf{y}^*$ is the solution of the following optimization problem,

$$\begin{aligned}
\min_{\mathbf{y}} \{H_t(\mathbf{y})\} &= \min_{\mathbf{y}} \left\{ (r_1 - s_1)y_0 + G(y_0) + \sum_{k=1}^{K-1} (s_k - s_{k+1} + r_{k+1} - r_k) y_k \right. \\
&\qquad \left. + (p - r_K + s_K) y_K + \sum_{k=0}^{K} \alpha \mathsf{E}\left[Q_{t+1,k}\Big(y_k - D + R(k)\Big) \right] \right\} \\
\text{s.t.} \quad & y_K \geq x_K; \\
& x_0 \leq y_0 \leq y_1 \leq \cdots \leq y_K; \\
& y_{k+1} - y_k \leq x_{k+1} - x_k, \quad k = 0, \ldots, K-1.
\end{aligned}$$

By Part(a) of the lemma, $H_t(\mathbf{y})$ is increasing in $y_1, \ldots, y_K$. This and the convexity of $H_t(\mathbf{y})$ imply $\xi_{t,K} \leq \xi_{t,K-1} \leq \cdots \leq \xi_{t,1} \leq \xi_{t,0}$. Next, we provide the detailed proof for each case in the theorem for period $n = t$.

i) If $x_K \leq \xi_{t,K} < x_{K+1} = \infty$, then $x_0 \leq x_1 \leq \cdots \leq x_K \leq \xi_{t,K}$. Because $H_t(x, \mathbf{y})$ is a convex function and increasing in $y_1, \ldots, y_K$, $\xi_{t,K}$ is the global minimizer under the constraint $y_0 \leq y_1 \leq \cdots \leq y_K$. As long as the policy can achieve $y_0 = y_1 = \cdots y_K = \xi_{t,K}$ without violating any other constraints, then it must be optimal. It can be easily verified that, in this case, all the constraints are satisfied when $y_0^* = y_1^* = \cdots = y_K^* = \xi_{t,K}$.
We show that if $\xi_{t,K} < x_K$, then $y_K^* = x_K$. We prove this by contradiction. Because no disposal is allowed, $y_K^* \geq x_K^*$. Suppose $y_K^* > x_K$, which implies that we produce some items. We argue that, this also implies $y_{K-1}^* = y_K^*$. Because, otherwise, we can set y_K^* to be $y_K^* - \varepsilon$ (produce less) without violating any constraint, and keep $y_k = y_k^*$ for $k = 0, \ldots, K-1$, which will result in a lower cost as $H_t(\mathbf{y})$ is increasing in y_K. Contradiction! Thereby, $y_{K-1}^* = y_K^* > x_K$. Moreover, because $H_t(y_0, \ldots, y_{K-2}, y, y)$ is also increasing in y, applying the previous argument, we can show that $y_{K-2}^* = y_{K-1}^* = y_K^* > x_K$. Repeating the same argument, we can finally reach $y_0^* = y_1^* =$

$\cdots = y_K^* > x_K$. However, since $H_t(\mathbf{x})$ is increasing in $\mathbf{x}$ for $x > \xi_{t,K}$, then $y_0^* = y_1^* = \cdots = y_K^* - \varepsilon$ (produce less) will result in a lower total cost, which contradicts the optimality of the original policy. Hence, $y_K^* = x_K$.

ii) If $\xi_{t,k} < x_k \leq \xi_{t,k-1}$ for $k \leq K-1$, for $k \leq K$, then it implies $x_0 \leq x_1 \leq \cdots \leq x_{k-1} \leq \xi_{t,k-1}$. Because $\xi_{t,k-1} \leq \xi_{t,0}$, it is optimal to increase x as much as possible since $H_t(x,\cdot,\ldots,\cdot)$ is decreasing, which implies $y_0^* = y_1^*$. Furthermore, as $\xi_{t,k-1} \leq \xi_{t,1}$, $H_t(y,y,\cdot,\ldots,\cdot)$ is decreasing in y, thus $y_0^* = y_1^* = y_2^*$. Repeat this argument, $y_0^* = y_1^* = y_2^* = \cdots = y_{k-1}^*$. By applying the similar argument as that in i), we can show $y_k^* = x_k, \ldots, y_K^* = x_K$. Because $H_t(x,x,\ldots,x,x_k,\ldots,x_K)$ is decreasing for $x < \xi_{t,k-1}$. Therefore, $y_0^* = y_1^* = \cdots = y_{k-1}^* = x_k$ and $y_k^* = x_k, \ldots, y_K^* = x_K$.

If $x_k \leq \xi_{t,k} < x_{k+1}$, then $y_{k+1}^* = x_{k+1}, \ldots, y_K^* = x_K$ from above analysis. Following from the same argument as in the previous case, we have $y_0^* = y_1^* = \cdots = y_k^*$. But as $x_k \leq \xi_{t,k} \leq x_{k+1}$, the objective function $H_t(x,\ldots,x,x_{k+1},\ldots,x_K)$ can achieve its minimum $\xi_{t,k}$ without violating any constraint, so $y_0^* = y_1^* = \cdots = y_k^* = \xi_{t,k}$.

iii) If $\xi_{t,1} < x_1 \leq \xi_{t,0}$, then $y_1^* = x_1, \ldots, y_K^* = x_K$. Because $x_0 \leq x_1 \leq \xi_{t,0}$, $H_t(x,x_1,\ldots,x_K)$ is decreasing in x. So $y_0^* = x_1$.

iv) If $x_1 > \xi_{t,0}$, then still $y_1^* = x_1, y_2^* = x_2, \ldots, y_K^* = x_K$. However, depending on x_0, $y_0^* = \max\{x_0, \xi_{t,0}\}$ because $H_t(x,x_1,\ldots,x_K)$ is convex in x.

Now we are ready to verify Proposition 1 for period $n = t$. It is sufficient to show that $\partial V_t(\mathbf{x})/\partial x_k$ only depends on x_k, $k = 0,\ldots,K$. Depending on the value of $\mathbf{x}$, the optimal policy presented above divides the $\mathbb{R}_+^{K+1}$ into $2(K+1)$ regions. Within the interior of each region, it is not hard to see $\partial V_t(\mathbf{x})/\partial x_k$ only depends on x_k, $k = 0,\ldots,K$, since $H_t(\mathbf{y})$ can be decomposed by the inductive assumption. So we only need to check the boundaries between different regions, which may not be differentiable. We use one case to illustrate that, even at the boundary, $V_t(\mathbf{x})$ can still be decomposed. Other cases can be similarly proved so we omit them here. Note that at the boundary of $x_K = \xi_{t,K}$, the right-sided derivative is

$$\begin{aligned}
\left.\frac{\partial V_t(\mathbf{x})}{\partial x_K}\right|_{x_K=\xi_{t,K}+} &= \lim_{\varepsilon\to 0} \frac{V_t(x_0,x_1,\ldots,x_{K-1},\xi_{t,K}+\varepsilon) - V_t(x_0,x_1,\ldots,x_{K-1},\xi_{t,K})}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{H_t(\xi_{t,K}+\varepsilon,\ldots,\xi_{t,K}+\varepsilon) - H_t(\xi_{t,K},\ldots,\xi_{t,K}) + (r_K - p)\varepsilon}{\varepsilon} \\
&= r_K + G'(\xi_{t,K}) + \sum_{k=0}^{K} \alpha \mathsf{E}\left[\Big(Q_{t+1,k}\Big(\xi_{t,K} - D + R(k)\Big)\Big)'\right],
\end{aligned}$$

and the left-sided derivative is

$$\begin{aligned}
\left.\frac{\partial V_t(\mathbf{x})}{\partial x_K}\right|_{x_K=\xi_{t,K}-} &= \lim_{\varepsilon\to 0} \frac{V_t(x_0,x_1,\ldots,x_{K-1},\xi_{t,K}) - V_t(x_0,x_1,\ldots,x_{K-1},\xi_{t,K}-\varepsilon)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{H_t(\xi_{t,K},\xi_{t,K},\ldots,\xi_{t,K}) - H_k(\xi_{t,K},\xi_{t,K},\ldots,\xi_{t,K}) + (r_K - p)\varepsilon}{\varepsilon} \\
&= r_K - p,
\end{aligned}$$

which is independent of $x_0, x_1, \ldots,$ and x_{K-1}.

Recall that we have not shown part (b) of the lemma yet. From the preceding proof, it should be noted that the proof of optimal policy only relies on part (a) and the Proposition. We now verify part (b). Based on the optimal policy presented in the theorem for period N (not rely on the above lemma), the results can be easily shown by taking derivative of $V_N(\mathbf{x})$ as $V_{N+1} = 0$. We leave the detailed steps to the reader.

Suppose part (b) is true for $n = t+1$, we next prove $n = t$ based on the optimal policy of the period t proved above. We need to discuss different regions of $\mathbf{x}$ since they result in different $V_t(\mathbf{x})$. To avoid a lengthy proof, we only show this for one region, i.e., $\xi_{t,1} < x_1 \le \xi_{t,0}$. In this case,

$$\begin{aligned} V_t(\mathbf{x}) &= (r_1 - s_1)x_1 + G(x_1) + \sum_{k=1}^{K-1}(s_k - s_{k+1} - r_k + r_{k+1})x_k + (s_K + p - r_K)x_K \\ &\quad + \sum_{k=0}^{K}\alpha\mathsf{E}[Q_{t+1,k}(x_k - D + R(k))] - r_1 x_0 + \sum_{k=1}^{K-1}(r_k - r_{k+1})x_k. \end{aligned}$$

Thereby, for $k = 1$,

$$\begin{aligned} &\frac{\partial V_t(\mathbf{x})}{\partial x_1} \\ &= r_1 - s_1 + G'(x_1) + \alpha\mathsf{E}[(Q_{t+1,0}(x_1 - D))'] + (-r_1 + s_1 - s_2 + r_2) \\ &\quad + \alpha\mathsf{E}[(Q_{t+1,1}(x_1 - D + R_1))'] + r_1 - r_2 \\ &\ge r_1 - r_2 \end{aligned}$$

where the inequality follows from that $x_1 > \xi_{t,1}$.

For $1 < k < K$,

$$\begin{aligned} \frac{\partial V_t(\mathbf{x})}{\partial x_k} &= (s_k - s_{k+1} - r_k + r_{k+1}) + (r_k - r_{k+1}) + \alpha\mathsf{E}\left[\Big(Q_{t+1,k}(x_k - D + R(k))\Big)'\right] \\ &\ge s_k - s_{k+1} + \alpha(r_k - r_{k-1}) \\ &\ge (r_k - r_{k+1}), \end{aligned}$$

in which the first inequality is due to the inductive assumption and the second one follows from $(1-\alpha)r_k - s_k \le (1-\alpha)r_{k+1} - s_{k+1}$.

For $k = K$,

$$\begin{aligned} \frac{\partial V_t(\mathbf{x})}{\partial x_K} &= (p - r_K + s_K) + \alpha\mathsf{E}\left[\Big(Q_{t+1,K}(x_K - D + R(K))\Big)'\right] + r_K - p \\ &\ge \alpha(r_K - p) + s_K \\ &\ge r_K - p, \end{aligned}$$

in which the first inequality follows from the inductive assumption and the second inequality follows from $r_K < p$ and $s_K \ge 0$.

So part (b) is proved. Finally, we prove part (a) of lemma for period $n = t$, which is in fact implied by part (b) for period $t+1$. Because from Equation (5), for $k = 1, \ldots, K-1$,

$$\frac{\partial H_t(\mathbf{y})}{\partial y_k} \ge (s_k - s_{k+1} - r_k + r_{k+1}) + \alpha(r_k - r_{k+1}) \ge 0$$

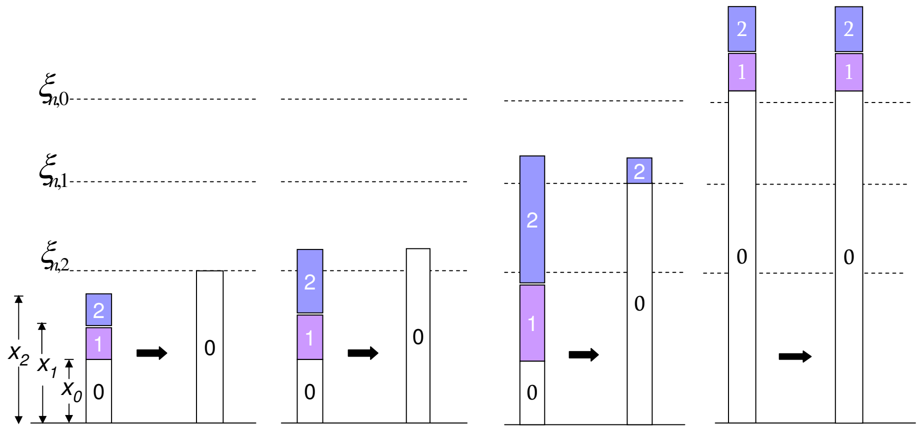

Figure 2: Illustration of the Optimal Policy: $K = 2$

where the first inequality follows from part (b) and the second inequality follows from $(1-\alpha)r_1 - s_1 \leq \ldots \leq (1-\alpha)r_K - s_K$. Similarly, $\partial H_t(\mathbf{y})/\partial y_K \geq (p - r_K + s_K) + \alpha(r_K - p) = (1-\alpha)(p - r_K) + s_K \geq 0$. Therefore, part (a) is also valid and the proofs of Lemma 3, Proposition 1 and Theorem 2 are thus completed. □

We illustrate the optimal policy for a system with two-type returns in Figure 3.

This optimal policy works as follows. In period n, if the total inventory (both the serviceable products and all returns) level x_K is less than $\xi_{n,K}$, then all of the returns should be remanufactured and some new products should be manufactured to bring the serviceable inventory level to $\xi_{n,K}$. As long as the total inventory level x_K is higher than $\xi_{n,K}$, it is never optimal to manufacture and the firm only needs to consider remanufacturing. In general, for $k = 1, \ldots, K$, if the aggregate inventory level of the serviceable and types 1 to k (resp., $k+1$) returns is less (resp., greater) than $\xi_{n,k}$, then it should remanufacture all type 1 to k and some type $k+1$ returns to bring the serviceable inventory to $\xi_{n,k}$; if the aggregate inventory level of the serviceable and type 1 to k returns x_k is higher than $\xi_{n,k}$ but less than $\xi_{n,k-1}$, then the firm should remanufacture all types 1 to k returns. Finally, if the aggregate inventory level of the serviceable and type 1 return x_1 is higher than $\xi_{n,0}$, then it should bring the serviceable inventory level to $\xi_{n,0}$ by remanufacturing some type 1 returns if possible. Note that, the optimal policy suggests the firm first remanufacture type 1 returns, then type 2, type 3, ..., and finally, type K returns.

We end this section by the following proposition.

Proposition 2. *(a) For $k = 0, \ldots, K-1$, $\xi_{n,k}$ is decreasing in r_{k+1}, increasing in s_{k+1}, increasing in r_j and decreasing in s_j, for $j = 1, \ldots, k$.*

(b) $\xi_{n,K}$ is decreasing in p, increasing in r_j and decreasing in s_j for $j = 1, \ldots, K$.

Proof. In the following proof, in order to show the dependence on r_j or s_j, we include r_j or s_j in V_n and H_n whenever needed.

We first show that, for any given k, $(V_n(\mathbf{x}, r_{k+1}))''_{x_\ell, r_{k+1}} = (Q_{n,\ell}(x, r_{k+1}))''_{x, r_{k+1}} \geq -1$ for

$\ell \le k$, which we prove by induction. This is trivially true for $N+1$. Suppose this is true for $n+1$, then for n,

$$V_n(\mathbf{x}, r_{k+1}) = \min_{\mathbf{y}} \Bigg\{ (r_1 - s_1)y_0 + G(y_0) + \sum_{\ell=1}^{k-1} (r_{\ell+1} - r_\ell + s_\ell - s_{\ell+1})y_\ell + (r_{k+1} - r_k + s_k$$
$$- s_{k+1})y_k + (r_{k+2} - r_{k+1} + s_{k+1} - s_{k+2})y_{k+1} + \sum_{\ell=k+2}^{K} (r_{\ell+1} - r_\ell + s_\ell - s_{\ell+1})y_\ell$$
$$+ \alpha \mathsf{E}[V_{n+1}(y_0 - D, \ldots, y_k + R(k) - D, y_{k+1} + R(k+1) - D, \ldots, y_K + R(K) - D, r_{k+1})] \Bigg\}$$
$$- r_1 x_0 + \sum_{\ell=1}^{k-1} (r_\ell - r_{\ell+1})x_\ell + (r_k - r_{k+1})x_k + (r_{k+1} - r_{k+2})x_{k+1} + \sum_{\ell=k+2}^{K} (r_\ell - r_{\ell+1})x_\ell.$$

From the optimal policy presented in Theorem 2, there are several cases. First, if $x_\ell < \xi_{n,\ell} \le x_{\ell+1}$ for any $\ell \ge k$ or $\xi_{n,\ell} \le x_\ell < \xi_{n,\ell-1}$ for any $\ell \ge k+1$, the objective function in the brackets is independent of x_j, for $j \le k$, then the result is immediate, i.e., $V_n(\mathbf{x}, r_{k+1}))''_{x_\ell, r_{k+1}} = 0$ for $\ell < k$ and $V_n(\mathbf{x}, r_{k+1}))''_{x_\ell, r_{k+1}} = -1$ for $\ell = k$.

Second, if $\xi_{n,\ell} \le x_\ell < \xi_{n,\ell-1}$ for any $\ell < k+1$, then the resulting optimal policy is $y_0^* = y_1^* \cdots, y_\ell^* = x_\ell$ and $y_{\ell+1}^* = x_{\ell+1}, \ldots, y_K^* = x_K$. As a result, for $j < \ell$, $V_n(\mathbf{x}, r_{k+1}))''_{x_j, r_{k+1}} = 0$. For $k > j \ge \ell$,

$$\begin{aligned}
& (V_n(\mathbf{x}, r_{k+1}))''_{x_j, r_{k+1}} \\
= \quad & \alpha \Bigg\{ \mathsf{E}[V_{n+1}(x_\ell - D, \ldots, x_\ell - D + R(\ell), x_{\ell+1} - D + R(\ell+1), \ldots, x_k + R(k) - D, \\
& x_{k+1} + R(k+1) - D, \ldots, x_K + R(K) - D, r_{k+1})] \Bigg\}''_{x_j, r_{k+1}} \\
\ge \quad & -\alpha \ge -1,
\end{aligned}$$

where the first inequality follows from the inductive assumption. For $j = k$,

$$\begin{aligned}
& (V_n(\mathbf{x}, r_{k+1}))''_{x_k, r_{k+1}} \\
= \quad & 1 + \alpha \Bigg\{ \mathsf{E}[V_{n+1}(x_\ell - D, \ldots, x_\ell - D + R(\ell), x_{\ell+1} - D + R(\ell+1), \ldots, x_k + R(k) - D, \\
& x_{k+1} + R(k+1) - D, \ldots, x_K + R(K) - D, r_{k+1})] \Bigg\}''_{x_k, r_{k+1}} - 1 \\
\ge \quad & -\alpha \ge -1.
\end{aligned}$$

Third, if $x_\ell < \xi_\ell \le x_{\ell+1}$ for $\ell < k$, then the optimal policy is $y_0^* = y_1^* = \cdots = y_\ell^* = \xi_\ell$, $\ell < k$ and $y_{\ell+1}^* = x_{\ell+1}, \ldots, y_K^* = x_K$. Hence, for $j \le \ell$, $V_n(\mathbf{x}, r_{k+1}))''_{x_j, r_{k+1}} = 0$; for $k > j > \ell$, $V_n(\mathbf{x}, r_{k+1}))''_{x_j, r_{k+1}} = \alpha \mathsf{E}[Q_{n+1,j}(x - D + R(j), r_{k+1})]''_{x, r_{k+1}} \ge -1$ by the inductive assumption; finally, for $j = k$, similar to the previous argument, $(V_n(\mathbf{x}, r_{k+1}))''_{x_k, r_{k+1}} \geq -\alpha \geq -1$. So we complete the induction.

Note that $\xi_{n,k}$ is the solution of

$$
\begin{aligned}
& (H_n(y,y,\ldots,y,y_{k+1},\ldots,y_K,r_{k+1}))'_y = \Bigg[G(y) + (r_{k+1} - s_{k+1})y \\
& +(r_{k+2} - r_{k+1} + s_{k+1} - s_{k+2})y_{k+1} + \sum_{\ell=k+2}^{K} (r_{\ell+1} - r_\ell + s_\ell - s_{\ell+1})y_\ell \\
& +\alpha \mathsf{E}[V_{n+1}(y-D,\ldots,y+R(k)-D, y_{k+1}+R(k+1)-D,\ldots,y_K+R(K)-D, r_{k+1})] \Bigg]'_y \\
= \; & G'(y) + r_{k+1} + \alpha \mathsf{E}[\sum_{\ell=0}^{k} Q_{n+1,\ell}(y-D+R(\ell), r_{k+1})]' = 0
\end{aligned}
$$

Based on the result we just showed, the left hand side of the last equality is increasing in r_{k+1} and so $\xi_{n,k}$ is increasing in r_{k+1} by the convexity.

The remaining parts of the proposition can be similarly proved and we leave the details to the reader. □

4 State-Dependent Optimal Policies

In this section, we investigate the optimal policy for systems in which condition (6), i.e., $r_1 - s_1 \leq r_2 - s_2 \leq \cdots \leq r_K - s_K$, is not satisfied. We show that in this case the simple structure of the optimal control policy presented in the previous section is no longer true, and its optimal policy is a very complicated, state-dependent, policy. For ease of exposition, in this section we only consider two types of returns $K = 2$. That is, we consider a system whose parameters satisfy $(1-\alpha)r_1 - s_1 \leq (1-\alpha)r_2 - s_2$ but $r_1 - s_1 > r_2 - s_2$. We partially characterize the structure of the optimal control policy for this system.

With only two types of returns, the optimality equation (5) is reduced to

$$
\begin{aligned}
V_n(\mathbf{x}) \quad &= \quad \min_{\mathbf{y}} \{H_n(\mathbf{y})\} - r_1 x_0 + (r_1 - r_2)x_1 + (r_2 - p)x_2 \qquad (7) \\
\text{s.t.} \quad & \quad y_2 \geq x_2, \\
& \quad y_k - y_{k-1} \leq x_k - x_{k-1}, \text{ for } k = 1,2 \\
& \quad x_0 \leq y_0 \leq y_1 \leq y_2, \\
\text{where} & \\
H_n(\mathbf{y}) \quad &= \quad (r_1 - s_1)y_0 + G(y_0) + (r_2 - s_2 - (r_1 - s_1))y_1 + (p - r_2 + s_2)y_2 \\
& \quad +\alpha \mathsf{E}[V_{n+1}(y_0 - D, y_1 + R_1 - D, y_2 + R(2) - D)]. \qquad (8)
\end{aligned}
$$

Note that $\mathbf{x} = (x_0, x_1, x_2)$ and $\mathbf{y} = (y_0, y_1, y_2)$.

In the following, we first provide an example demonstrating that the optimal policy is not a simple threshold type, and then we present a partial characterization to the optimal control policy.

Example 1.
Consider a system with two types of returns and parameters $r_1 = 4, r_2 = 2, s_1 = 2, s_2 = 1, p = 5, \alpha = 1$, $G(x) = 3\mathrm{E}[\max\{x-D,0\}] + 5\mathrm{E}[\max\{D-x,0\}]$. One period demand is Poisson distributed with rate 10, type 1 and type 2 returns have Poisson distributions with rates 3 and 4, respectively. Let $N = 2$, then at period 1, the systems states and the corresponding optimal strategies are given in Table 1.

Table 1: An Example with State-Dependent Optimal Policy

(x_0,x_1,x_2)	(y_0^*,y_1^*,y_2^*)
(4,14,17)	(12,14,17)
(4,15,16)	(13,15,16)
(4,15,17)	(13,15,17)
(4,15,18)	(12,15,18)
(4,15,19)	(12,15,19)

It can be seen from the table that the optimal inventory level y_0^* of the serviceable product no longer follows a remanufacture-up-to policy, but depends on (x_1,x_2). For example, if the starting state is $(4,15,17)$, the optimal policy is to remanufacture 9 units of type 1 return to bring the serviceable inventory level y_0^* to 13. However, if the starting state becomes $(4,15,18)$, then the optimal policy is to bring the serviceable inventory up to 12, i.e., $y_0^* = 12$, instead of 13.

To characterize the optimal policy for such a system, we need to first define

$$\begin{aligned} \xi_{n,2} &= \arg\min_x H_n(x,x,x), \\ \xi_{n,1}(y_2) &= \arg\min_x H_n(x,x,y_2), \\ \xi_{n,0}(y_1,y_2) &= \arg\min_x H_n(x,y_1,y_2), \\ (\theta_{n,1}(y_2),\delta_{n,1}(y_2)) &= \arg\min_{x,y} H_n(x,y,y_2), \\ \theta_{n,2}(Z-z,y_2) &= \arg\min_x H_n(x,x+Z-z,y_2), \\ \delta_{n,2}(Z-z,y_2) &= \theta_{n,2}(Z-z,y_2)+Z-z. \end{aligned} \tag{9}$$

Here, with a slight abuse of notation we still use $\xi_{n,k}$ to represent the control limits, since they have parallel meaning to those in Theorem 2, though they depend on the system states now.

Lemma 4.
For $n = 1,\ldots,N$,

(a) $\xi_{n,1}(y_2) \geq \xi_{n,2}$, $\theta_{n,1}(y_2) \geq \xi_{n,2}$, *and* $\xi_{n,0}(y_1,y_2) \geq \xi_{n,2}$.
(b) $\partial\xi_{n,0}(y_1,y_2)/\partial y_1 \leq 1$.

The relationship between the control parameters specified in this lemma will help us partially characterize the optimal policy presented in the following.

Theorem 5.
For the inventory system with two types of returns, there exists a parameter $\xi_{n,2}$ and a set of functions

$$\xi_{n,1}(y_1), \xi_{n,0}(y_1,y_2), \delta_{n,1}(y_2), \delta_{n,2}(Z-z,y_2), \theta_{n,1}(y_2), \theta_{n,2}(Z-z,y_2),$$

such that, when the current state is (x_0,x_1,x_2), the optimal production/remanufacturing strategy for period n is determined, according to two possible scenarios, as follows:

i) If $x_2 < \xi_{n,2}$, then $y_0^* = y_1^* = y_2^* = \xi_{n,2}$.
(1) $\theta_{n,1}(x_2) \geq \delta_{n,1}(x_2)$
  ii) If $\xi_{n,2} \leq x_2 < \xi_{n,1}(x_2)$, then $y_0^* = y_1^* = y_2^* = x_2$.
  iii) If $x_1 < \xi_{n,1}(x_2) \leq x_2$, then $y_0^* = y_1^* = \xi_{n,1}(x_2), y_2^* = x_2$.
  iv) If $\xi_{n,1}(x_2) \leq x_1$, $y_0^* = \xi_{n,0}(x_1,x_2) \perp [x_0,x_1]$, i.e., the closest number in the interval $[x_0,x_1]$ with respect to $\xi_{n,0}(x_1,x_2)$, $y_1^* = x_1$, and $y_2^* = x_2$.

(2) $\theta_{n,1}(x_2) < \delta_{n,1}(x_2)$
  ii) If $x_1 - x_0 \geq \delta_{n,1}(x_2) - \theta_{n,1}(x_2)$, then
    (a) if $x_2 < \delta_{n,1}(x_2)$, then $y_0^* = \min\{\xi_{n,0}(x_2,x_2),x_2\}, y_1^* = y_2^* = x_2$.
    (b) if $x_1 < \delta_{n,1}(x_2) \leq x_2$, then $y_0^* = \theta_{n,1}(x_2), y_1^* = \delta_{n,1}(x_2), y_2^* = x_2$.
    (c) if $x_1 \geq \delta_{n,1}(x_2)$, then $y_0^* = \max\{\xi_{n,0}(x_1,x_2),x_0\}, y_1^* = x_1, y_2^* = x_2$.
  iii) If $x_1 - x_0 < \delta_{n,1}(x_2) - \theta_{n,1}(x_2)$, then
    (a) if $x_2 < \delta_{n,2}(x_1-x_0,x_2)$, then $y_0^* = \xi_{n,0}(x_2,x_2) \perp [x_0+x_2-x_1,x_2], y_1^* = y_2^* = x_2$.
    (b) if $x_1 < \delta_{n,2}(x_1-x_0,x_2) \leq x_2$, then $y_0^* = \theta_{n,2}(x_1-x_0,x_2), y_1^* = \delta_{n,2}(x_1-x_0,x_2), y_2^* = x_2$.
    (c) if $x_1 \geq \delta_{n,2}(x_1-x_0,x_2)$, then $y_0^* = x_0, y_1^* = x_1, y_2^* = x_2$.

To prove Lemma 4 and this theorem, we need to have the following lemma. For convenience, let $U_n(y_0,y_1,y_2) = \mathsf{E}[V_{n+1}(y_0-D, y_1-D+R(1), y_2-D+R(2))]$.

Lemma 6.
For $n = 1,\ldots,N$,

(a) $H_n(y_0,y_1,y_2)$ is increasing in y_2 and $H_n(x,y,y)$ is increasing in y.
(b) $\partial V_n(x_0,x_1,x_2)/\partial x_1 + \partial V_n(x_0,x_1,x_2)/\partial x_2 \geq r_1 - p$ and $\partial V_n(x_0,x_1,x_2)/\partial x_2 \geq r_2 - p$.
(c) $\partial^2 V_n(x_0,y_1,y_2)/\partial x_2 + \partial^2 V_n(x_0,x_1,x_2)/\partial x_0 \partial x_1 \geq 0$.

We first prove Lemma 4 by assuming Lemma 6 holds true. Then we verify Lemma 6 and the theorem together by induction.

Proofs of Lemma 4, Lemma 6, and Theorem 5.

Suppose Lemma 6 is true. Then, part (a) follows from the convexity of H_n and the definitions of $\xi_{n,1}(y_2)$, $\theta_{n,1}(y_2)$ and $\xi_{n,0}(y_1,y_2)$.

For part (b), by definition, $\xi_{n,0}(y_1,y_2)$ is the solution of

$$\frac{\partial H_n(y_0,y_1,y_2)}{\partial y_0} = 0.$$

Thereby,

$$\left(\frac{\partial^2 H_n(y_0,y_1,y_2)}{\partial y_0^2}\frac{\partial \xi n,0(y_1,y_2)}{\partial y_1}+\frac{\partial^2 H_n(y_0,y_1,y_2)}{\partial y_0 \partial y_1}\right)\Bigg|_{y_0=\xi_{n,0}(y_1,y_2)}=0,$$

and

$$\begin{aligned}\frac{\partial \xi n,0(y_1,y_2)}{\partial y_1} &= \left.\frac{-\partial^2 H_n(y_0,y_1,y_2)/\partial y_0 \partial y_1}{\partial^2 H_n(y_0,y_1,y_2)/\partial y_0^2}\right|_{y_0=\xi_{n,0}(y_1,y_2)} \\ &= \left.\frac{-\partial^2 U_n(y_0,y_1,y_2)/\partial y_0 \partial y_1}{\partial^2 U_n(y_0,y_1,y_2)/\partial y_0^2}\right|_{y_0=\xi_{n,0}(y_1,y_2)} \le 1,\end{aligned}$$

where the last inequality follows from the inductive assumption of Lemma 6 part (b). □

Now we prove Lemma 6 and the theorem together by induction on n. Firstly, part (a) of Lemma 6 is true based on and $V_{N+1}=0$. Note that, because condition (6) is not satisfied, $H_n(y_0,y_1,y_2)$ is not increasing in y_1 any more. Suppose part (a) is valid at period $n=t$, then we prove the theorem is true for $n=t$ by the following three steps.

Step 1. We first show that, if $x_2<\xi_{t,2}$, $y_0^*=y_1^*=y_2^*=\xi_{t,2}$. Again, because of the inductive assumption of part (a) Lemma 6, i.e., $H_t(y_0,y_1,y_2)$ is increasing in y_2 and $H_t(y_0,y,y)$ is increasing in y, $\xi_{t,2}$ is the global minimizer of $H_t(y_0,y_1,y_2)$ under constraint $y_0\le y_1\le y_2$. Hence, if $y_0=y_1=y_2=\xi_{t,2}$ is attainable without violating any other constraint, it must be the optimal solution. It is not hard to check it is indeed the case when $x_2<\xi_{t,2}$. Thus, i) is proved.

Step 2. Similar to the proof of Theorem 2, we can show that, if $x_2\ge\xi_{t,2}$, then $y_2^*=x_2$.

Step 3. For $x_2\ge\xi_{t,2}$, we fix $y_2^*=x_2$ and the problem becomes

$$\begin{aligned}\min_{y_0,\,y_1}\quad & H_t(y_0,y_1,x_2)\\ \text{s.t.}\quad & y_0\le y_1\\ & y_1-y_0\le x_1-x_0,\\ & x_1\le y_1\le x_2.\end{aligned}\tag{10}$$

Note that $H_t(y_0,y_1,x_2)$ is joint convex in (y_0,y_1) but may not be increasing in y_1. The following argument is used throughout the remaining proof of this step. By definition, if $(\theta_{t,1}(x_2),\delta_{k,1}(x_2))$ can be attained without violating the constraints in (10), it must be the minimizer; otherwise, the optimal solution must be attained at some boundary of the regions defined by the violated constraints (some violated constraint must be binding).

Case (1): If $\theta_{t,1}(x_2)>\delta_{t,1}(x_2)$, then under the constraint $y_0\le y_1$, the minimizer of $H_t(y_0,y_1,x_2)$ is $y_0=y_1=\xi_{t,1}(x_2)=\arg\min_y H_t(y,y,x_2)$ instead of $y_0=\theta_{t,1}(x_2)$ and $y_1=\delta_{t,1}(x_2)$.

ii) If $\xi_{t,2}\le x_2<\xi_{t,1}(x_2)$, then $x_0\le x_1\le\xi_{t,1}(x_2)$ and $(\xi_{t,1}(x_2),\xi_{t,1}(x_2))$ is not attainable. For (10), if first ignore the constraint $y_1\ge x_1$ and $y_1-y_0\le x_1-x_0$, then at optimum of the relaxed problem, $y_0=y_1$ or $y_1=x_2$ or both (otherwise, the optimum of the relaxed problem is $(\xi_{t,1}(x_2),\xi_{t,1}(x_2))$). Suppose $y_0=y_1$, then by the definition of $\xi_{t,1}(x_2)$, $H_t(x,x,x_2)$ is decreasing for $x \leq \xi_{t,1}(x_2)$, so $y_0 = y_1 = x_2$. Hence $y_1 = x_2$ always holds at optimum. Thus, by the definition of $\xi_{t,0}(y_1,y_2)$, $y_0^* = \xi_{t,0}(x_2,x_2) \perp [x_0+x_2-x_1,x_2]$. Finally, as $y_1 = x_2 \geq x_1$, both of the constraints we relaxed before are not violated, so $y_{0*} = \xi_{t,0}(x_2,x_2) \perp [x_0+x_2-x_1,x_2]$ and $y_{1*} = y_{2*} = x_2$. Next, we further show that $\xi_{t,0}(x_2,x_2) \geq x_2$ based on Lemma 4 by discussing the following two scenarios.

If $x_2 \leq \delta_{t,1}(x_2)$, then following from the inductive assumption of Lemma 6 and Lemma 4,

$$\int_{x_2}^{\delta_{t,1}(x_2)} \frac{\partial \xi_{t,0}(y,x_2)}{\partial y} dy \leq \int_{x_2}^{\delta_{t,1}(x_2)} dy.$$

Hence, $\xi_{t,0}(\delta_{t,1}(x_2),x_2) - \xi_{t,0}(x_2,x_2) \leq \delta_{t,1}(x_2) - x_2$, which implies $\xi_{t,0}(x_2,x_2) \geq x_2 + \theta_{t,1}(x_2) - \delta_{t,1}(x_2) \geq x_2$ since $\xi_{t,0}(\delta_{t,1}(x_2),x_2) = \theta_{t,1}(x_2)$.

If $\delta_{t,1}(x_2) < x_2 < \xi_{t,1}(x_2)$, then consider the relaxed problem of (10) with constraints:

$$0 \leq y_1 - y_0 \leq x_1 - x_0, x_2 \geq y_1.$$

Because $x_2 > \delta_{t,1}(x_2)$, the optimal solution $(\theta_{t,1}(x_2), \delta_{t,1}(x_2))$ only violates the constraint $y_1 - y_0 \geq 0$. Thus, at optimum, $y_0 = y_1$. Moreover, because $y_1 \leq x_2 < \xi_{t,1}(x_2)$, so at optimum $y_0 = y_1 = x_2$. Notice this optimal solution of the relaxed problem does not violate the relaxed constraint $y_1 \geq x_1$, so it is optimal for (10).

Combine above two scenarios, the optimal solution of this case is $y_0^* = y_1^* = y_2^* = x_2$.

iii) If $x_1 < \xi_{t,1}(x_2) \leq x_2$, because $(\xi_{t,1}(x_2), \xi_{t,1}(x_2))$ is the minimizer under the constraint $y_0 \leq y_1$ and it can be attained in this case, then $y_0^* = y_1^* = \xi_{t,1}(x_2), y_2^* = x_2$.

iv) If $\xi_{t,1}(x_2) \leq x_1$, then $(\xi_{t,1}(x_2), \xi_{t,1}(x_2))$ is not attainable. We first relax the constraint $y_1 \leq x_2$. We note that, at optimum of the relaxed problem, it must be true that either $y_0^* = y_1^*$ or $y_1^* = x_1$ (otherwise, the relaxed problem can attain $(\xi_{t,1}(x_2),\xi_{t,1}(x_2))$). Suppose $y_0^* = y_1^*$. Because $H_t(x,x,x_2)$ is increasing in x for $y_0 \geq x_1 \geq \theta_{t,1}(x_2)$, we must have $y_1^* = x_1$. Thus, $y_1^* = x_1$ must hold at optimum. Therefore, by the definition of $\xi_{t,0}(x_1,x_2)$, $y_0^* = \xi_{t,0}(x_1,x_2) \perp [x_0,x_1]$. Again, as $y_1^* = x_1 \leq x_2$, the constraint we relaxed is not violated. In such a case, $y_0^* = \xi_{t,0}(x_1,x_2) \perp [x_0,x_1]$, $y_1^* = x_1$, and $y_2^* = x_2$ is optimal.

Case (2): If $\theta_{t,1}(x_2) \leq \delta_{t,1}(x_2)$, then if both of them are attainable, at optimum, $y_0^* = \theta_{t,1}(x_2)$ and $y_1^* = \delta_{t,1}(x_2)$.

ii).a If $x_1 - x_0 \geq \delta_{t,1}(x_2) - \theta_{t,1}(x_2)$ and $x_2 < \delta_{t,1}(x_2)$, then $x_1 \leq x_2 < \delta_{t,1}(x_2)$ and $x_0 < \theta_{t,1}(x_2)$. Hence $y_1^* = x_2$ since $H_t(y_0,y_1,y_2)$ is decreasing in y_1 when $y_1 < \delta_{t,1}(x_2)$. Then by the definition of $\xi_{t,0}(y_1,y_2)$, $y_0^* = \xi_{t,0}(x_2,x_2) \perp [x_0+x_2-x_1,x_2]$. Moreover, note that, from Lemma 4,

$$\int_{x_2}^{\delta_{t,1}(x_2)} \frac{\partial \xi_{t,0}(y,x_2)}{\partial y} dy \leq \int_{x_2}^{\delta_{t,1}(x_2)} dy.$$

Hence, $\xi_{t,0}(\delta_{t,1}(x_2),x_2) - \xi_{t,0}(x_2,x_2) \leq \delta_{t,1}(x_2) - x_2$, or $\xi_{t,0}(x_2,x_2) \geq x_2 - (\delta_{t,1}(x_2) - \xi_{t,0}(x_2)) \geq x_2 - (x_1 - x_0)$. Hence, $y_0^* = \min\{\xi_{t,0}(x_2,x_2), x_2\}$.

ii).b If $x_1 - x_0 \geq \delta_{t,1}(x_2) - \theta_{t,1}(x_2)$ and $x_1 < \delta_{t,1}(x_2) \leq x_2$, then it implies $x_0 < \theta_{t,1}(x_2)$. So $y_0^* = \theta_{t,1}(x_2), y_1^* = \delta_{t,1}(x_2)$ as they are attainable.

ii).c If $x_1 - x_0 \geq \delta_{t,1}(x_2) - \theta_{t,1}(x_2)$ and $x_1 \geq \delta_{t,1}(x_2)$, then $y_1^* = x_1$, and by definition, $y_0^* = \xi_{t,0}(x_1,x_2) \perp [x_0,x_1]$. Moreover, since

$$\int_{\delta_{t,1}(x_2)}^{x_1} \frac{\partial \xi_{t,0}(y,x_2)}{\partial y} dy \leq \int_{\delta_{t,1}(x_2)}^{x_1} dy$$

$\xi_{t,0}(x_1,x_2) - \theta_{t,1}(x_2) \leq x_1 - \delta_{t,1}(x_2)$ or $\xi_{t,0}(y,x_2) \leq x_1 - (\delta_{t,1}(x_2) - \theta_{t,1}(x_2)) \leq x_1$, so $y_0^* = \max\{\xi_{t,0}(x_1,x_2), x_0\}$.

iii).a If $x_1 - x_0 \leq \delta_{t,1}(x_2) - \theta_{t,1}(x_2)$, then it implies that $(\delta_{t,1}(x_2), \theta_{t,1}(x_2))$ cannot be attained. We first relax the constraint that $y_1 \geq x_1$. So at optimum, the relationship $y_1 - y_0 = x_1 - x_0$ or $y_1 = x_2$ or both holds. Suppose $y_1 - y_0 = x_1 - x_0$, then the unconstrained minimizer is $\theta_{t,2}(x_1 - x_0, x_2)$. If $x_2 \leq \delta_{t,2}(x_1 - x_0, x_2)$, then $x_1 \leq \theta_{t,2}(x_1 - x_0, x_2)$, so $y_1^* = y_0^* + x_1 - x_0 = x_2$ because $H_t(x, x + x_1 - x_0, x_2)$ is decreasing in x. Hence, $y_1^* = x_2$ is always valid at optimum. By definition of $\xi_{t,0}(x_2,x_2)$, $y_0^* = \xi_{t,0}(x_2,x_2) \perp [x_0 + x_2 - x_1, x_2]$.

iii).b If $x_1 - x_0 < \delta_{t,1}(x_2) - \theta_{t,1}(x_2)$ and $x_1 \leq \delta_{t,2}(x_1 - x_0, x_2) \leq x_2$, since $(\theta_{t,2}(x_1 - x_0, x_2), \delta_{t,2}(x_1 - x_0, x_2))$ is the optimal point under the constraint $y_1 - y_0 \leq x_1 - x_0$, and can be attained, then $y_0^* = \theta_{t,2}(x_1 - x_0, x_2), y_1^* = \delta_{t,2}(x_1 - x_0, x_2)$.

iii).c If $x_1 - x_0 < \delta_{t,1}(x_2) - \theta_{t,1}(x_2)$ and $x_1 \geq \delta_{t,2}(x_1 - x_0, x_2)$, then $x_0 \geq \theta_{t,2}(x_1 - x_0, x_2)$ and if we first relax $y_1 \leq x_2$, then at optimum, either $y_1^* - y_0^* = x_1 - x_0$ or $y_1^* = x_1$. Suppose $y_1^* - y_0^* = x_1 - x_0$. Because $x_1 \geq \delta_{t,2}(x_1 - x_0, x_2)$ implies that $H_t(x, x + x_1 - x_0, x_2)$ is increasing in x. So $y_1^* = x_1$ must hold at optimum. By definition, $y_0^* = \xi_{t,0}(x_1,x_2) \perp [x_0,x_1]$. We next further refine y_0^*.
If $x_1 \geq \delta_{t,1}(x_2)$, then

$$\int_{\delta_{t,1}(x_2)}^{x_1} \frac{\partial \xi_{t,0}(y,x_2)}{\partial y} dy \leq \int_{\delta_{t,1}(x_2)}^{x_1} dy,$$

which implies $\xi_{t,0}(x_1,x_2) - \xi_{t,0}(\delta_{t,1}(x_2),x_2) \leq x_1 - \delta_{t,1}(x_2)$, or $\xi_{t,0}(x_1,x_2) \leq x_1 + \theta_{t,1}(x_2) - \delta_{t,1}(x_2) \leq x_1 + x_0 - x_1 = x_0$. So $y_0^* = x_0$.
If $\delta_{t,2}(x_1 - x_0, x_2) \leq x_1 < \delta_{t,1}(x_2)$, then consider the relaxed problem of (10) with only constraints $y_1 - y_0 \leq x_1 - x_0$ and $y_1 \geq x_1$. Then, at optimum, because $(\theta_{t,1}(x_2), \delta_{t,1}(x_2)$ is not attainable, either $y_1^* = x_1$ or $y_1^* - y_0^* = x_1 - x_0$ (or both hold). Suppose $y_1^* - y_0^* < x_1 - x_0$, then $y_1^* > x_1$ because $x_1 < \delta_{t,1}(x_2)$. so at optimum, $y_1^* - y_0^* = x_1 - x_0$ As we show $y_1^* = x_1$ Hence, $y_0^* = x_0$ as it does not violate the constraint $y_1 \leq x_2$.
Overall, by the preceeding argument, the optimal solution in this case is $y_0^* = x_0, y_1^* = x_1$.

We next prove part (b) and (c) given the theorem is valid for $n = t \leq N$. When $n = N$, following from the optimal policy and $V_{N+1} = 0$, it is not hard to show part (b) and (c) are valid. Suppose they are true for $t+1$. For $n = t$, to show part (b), it is sufficient to show that

$$\frac{V_t(x_0,x_1,x_2)}{\partial x_1} + \frac{\partial V_t(x_0,x_1,x_2)}{\partial x_2} \geq r_1 - p.$$

To simplify the proof, we sketch the idea for (x_0,x_1,x_2) within one specific region among the regions divided by the optimal policy, i.e., $x_1 - x_0 \le \delta_{t,1}(x_2) - \theta_{t,1}(x_2)$ and $x_1 \ge \delta_{t,2}(x_1 - x_0, x_2)$. All other regions can be similarly proved. Substitute the optimal solution (y_0^*, y_1^*, y_2^*) into (7),

$$
\begin{aligned}
V_t(x_0,x_1,x_2) &= H_t(y_0^*,y_1^*,y_2^*) - r_1x_0 + (r_1-r_2)x_1 + (r_2-p)x_2 \\
&= H_t(\xi_{t,0}(x_1,x_2) \perp [x_0,x_1], x_1, x_2) - r_1x_0 + (r_1-r_2)x_1 + (r_2-p)x_2.
\end{aligned}
$$

Hence, if $\xi_{t,0}(x_1,x_2) \in [x_0,x_1]$,

$$
\begin{aligned}
& \frac{\partial V_t(x_0,x_1,x_2)}{\partial x_1} + \frac{\partial V_t(x_0,x_1,x_2)}{\partial x_2} \\
= \; & \frac{\partial H_t(\xi_{t,0}(x_1,x_2),x_1,x_2)}{\partial x_1} + \frac{\partial H_t(\xi_{t,0}(x_1,x_2),x_1,x_2)}{\partial x_2} + r_1 - p \\
= \; & \frac{\partial H_t(y_0,x_1,x_2)}{\partial y_0}\Big|_{y_0=\xi_{t,0}(x_1,x_2)} \frac{\partial \xi_{t,0}(x_1,x_2)}{\partial x_1} + \frac{\partial H_t(y_0,x_1,x_2)}{\partial x_1}\Big|_{y_0=\xi_{t,0}(x_1,x_2)} \\
& + \frac{\partial H_t(y_0,x_1,x_2)}{\partial y_0}\Big|_{y_0=\xi_{t,0}(x_1,x_2)} \frac{\partial \xi_{t,0}(x_1,x_2)}{\partial x_2} + \frac{\partial H_t(y_0,x_1,x_2)}{\partial x_2}\Big|_{y_0=\xi_{t,0}(x_1,x_2)} \\
& + r_1 - p \\
= \; & \frac{\partial H_t(y_0,x_1,x_2)}{\partial x_1}\Big|_{y_0=\xi_{t,0}(x_1,x_2)} + \frac{\partial H_t(y_0,x_1,x_2)}{\partial x_2}\Big|_{y_0=\xi_{t,0}(x_1,x_2)} + r_1 - p \\
\ge \; & (-r_1+s_1+p) + \alpha(r_1-p) + r_1 - p \ge r_1 - p,
\end{aligned}
$$

where the last equality is due to the optimality of $\xi_{t,0}(x_1,x_2)$ and the inequality follows from the inductive assumption.

If $\xi_{t,0}(x_1,x_2) < x_0$, then

$$
\begin{aligned}
\frac{\partial V_t(x_0,x_1,x_2)}{\partial x_1} + \frac{\partial V_t(x_0,x_1,x_2)}{\partial x_2} &= \frac{\partial H_t(x_0,x_1,x_2)}{\partial x_1} + \frac{\partial H_t(x_0,x_1,x_2)}{\partial x_2} + r_1 - p \\
&\ge ((-r_1+s_1+s_2-r_2) + (p-r_2+s_2) \\
&\quad + \alpha(r_1-p)) + r_1 - p \\
&\ge r_1 - p.
\end{aligned}
$$

If $\xi_{t,0}(x_1,x_2) > x_1$, then $y_0^* = y_1^* = x_1, y_2^* = x_2$. Note that for sufficiently small $\varepsilon > 0$, $H_t(x_1+\varepsilon, x_1+\varepsilon, x_2+\varepsilon) \ge H_t(x_1,x_1,x_2)$ because the optimality of $y_0^* = y_1^* = x_1, y_2^* = x_2$. So $\partial H_t(x_1,x_1,x_2)/\partial x_1 + \partial H_t(x_1,x_1,x_2)/\partial x_2 \ge 0$, and

$$
\begin{aligned}
\frac{\partial V_t(x_0,x_1,x_2)}{\partial x_1} + \frac{\partial V_t(x_0,x_1,x_2)}{\partial x_2} &= \frac{\partial H_t(x_1,x_1,x_2)}{\partial x_1} + \frac{\partial H_t(x_1,x_1,x_2)}{\partial x_2} + r_1 - p \\
&\ge r_1 - p.
\end{aligned}
$$

For part (c), we also just show one case, which is (c) of ii) in (2). If $y_0^* = \xi_{t,0}(x_1,x_2)$, then

$$
\frac{\partial^2 V_t(x_0,x_1,x_2)}{\partial x_0^2} + \frac{\partial^2 V_t(x_0,x_1,x_2)}{\partial x_0 \partial x_1} = 0.
$$

If $y_0^* = x_0$, then

$$\begin{aligned} & \frac{\partial^2 V_t(x_0,x_1,x_2)}{\partial x_0^2} + \frac{\partial^2 V_t(x_0,x_1,x_2)}{\partial x_0 \partial x_1} \\ = \quad & G''(x) + \frac{\partial^2 U_t(x_0,x_1,x_2)}{\partial x_0^2} + \frac{\partial^2 U_t(x_0,x_1,x_2)}{\partial x_0 \partial x_1} \geq 0 \end{aligned}$$

by the convexity of G and the inductive assumption.

Finally, based on part (b) and (c) of Lemma 5, we can show part (a) is true for period $n = t-1$. Therefore, we complete the proof. □

We offer an explanation to this policy. If the total inventory x_2 of the serviceable and returned products is lower than $\xi_{n,2}$, then it is optimal to remanufacture all the available returns and produce some units to bring the serviceable inventory level up to $\xi_{n,2}$. If the total inventory level is higher than $\xi_{n,2}$, then it is not optimal to manufacture any units and we only need to consider remanufacturing from returns. In this case there are two possible scenarios:

First, $\theta_{n,1}(x_2) \geq \delta_{n,1}(x_2)$. If $x_2 < \xi_{n,1}(x_2)$, then remanufacture all returned products. If $x_2 \geq \xi_{n,1}(x_2)$ but $x_1 < \xi_{n,1}(x_2)$, then bring the serviceable inventory level up to $\xi_{n,1}(x_2)$ by remanufacturing all type 1 and some type 2 returned products. If $x_1 \geq \xi_{n,1}(x_2)$, then only remanufacture some type 1 returns to raise the serviceable inventory level to $\xi_{n,0}(x_1,x_2)$ if possible.

Second, $\theta_{n,1}(x_2) < \delta_{n,1}(x_2)$. If the inventory of type 1 return $x_1 - x_0$ is more than $\delta_{n,1}(x_2) - \theta_{n,1}(x_2)$, then the optimal policy is: If $x_2 \leq \delta_{n,1}(x_2)$, then bring the serviceable inventory level to $\min\{\xi_{n,0}(x_2,x_2),x_2\}$ by remanufacturing all type 2 and some type 1 returns; if $x_1 < \delta_{n,1}(x_2) \leq x_2$, then first remanufacture some type 2 returns to bring the aggregate inventory level of serviceable and type 1 return to $\delta_{n,1}(x_2)$ and then further repair some type 1 returns to increase the serviceable inventory level to $\theta_{n,1}(x_2)$; if $x_1 \geq \delta_{n,1}(x_2)$, then only remanufacture some type 1 returns to bring the serviceable inventory level to $\max\{\xi_{n,0}(x_1,x_2),x_0\}$. If type 1 return $x_1 - x_0$ is less than $\delta_{n,1}(x_2) - \theta_{n,1}(x_2)$, then the optimal policy is: If $x_2 \leq \delta_{n,2}(x_1 - x_0, x_2)$, then remanufacture all type 2 and some type 1 returns to bring the serviceable inventory level to $\xi_{n,0}(x_2,x_2)$; if $x_1 < \delta_{n,2}(x_1 - x_0, x_2) \leq x_2$, then only remanufacture some type 2 returns so as to raise the serviceable inventory level to $\theta_{n,2}(x_1 - x_0, x_2)$, which also makes the aggregate inventory level of serviceable product and type 1 return $\delta_{n,2}(x_1 - x_0, x_2)$; if $x_1 \geq \delta_{n,2}(x_1 - x_0, x_2)$, then do nothing.

This result is quite different from that of Theorem 2 and it is no longer always optimal to wait to remanufacture type 2 returns until type 1 returns are depleted. Why does the optimal policy become so complicated in this case? Technically, it is mainly due to the loss of some structural properties of $H_n(\mathbf{y})$ in $\mathbf{y}$, specifically, $H_n(\mathbf{y})$ may not be increasing in $y_1, y_2, \ldots, y_K$. Consequently, $V_n(\mathbf{x})$ is not decomposable in contrast to the case $r_1 - s_1 \leq r_2 - s_2$ (see Proposition 1). Intuitively, we can also explain this with the help of the following arguments. Note that $(1-\alpha)r_k - s_k$ is the "actual" unit cost of remanufacturing one unit of type k return to serviceable product if the current period is not period N. In contrast, $r_k - s_k$ is the "actual" type k unit remanufacturing cost when the period is N already. If $(1-\alpha)r_1 - s_1 \leq (1-\alpha)r_2 - s_2$ and $r_1 - s_1 \leq r_2 - s_2$, it implies that it is always better to remanufacture type 1 return first until it is depleted regardless of what will happen in the next period. However, for the case $(1-\alpha)r_1 - s_1 \leq (1-\alpha)r_2 - s_2$ but $r_1 - s_1 > r_2 - s_2$, in some periods close to the end of the planning horizon, it may be better for the firm to remanufacture type 2 returns first while it is better to remanufacture type 1 return first in the periods close to the beginning of the planning horizon. These complicate the decision making for remanufacturing.

It can be expected that as the types of returned product increase, the optimal policy will be even more complicated when condition (6) is not satisfied.

5 Concluding Remarks

Efficient management of inventory systems with returns can reduce costs and increase profitability of firms. In this paper, we develop inventory models with multiple types of returns and characterize the optimal inventory control and remanufacturing policies. We find that simple and state-independent policy is optimal under some scenarios while not in some others. For the latter case, we also analyze the structure of optimal control policy for the case with two types of returns and prove that the optimal policy is quite complicated and state-dependent. We remark that, even though we assume stationary distributions for demand and returns over periods, all the results reported in Sections 3 and 4 can be easily extended to the nonstationary case.

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