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Nonlinear Optimization Subject to a System of Fuzzy Relational Equations with Max-min Composition*

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Abstract In this paper we show that the problem of minimizing a nonlinear objective function subject to a system of fuzzy relational equations with max-min composition can be reduced to a 0-1 mixed integer programming problem. The reduction method can be extended to the case of fuzzy relational equations with max-T composition as well as those with more general composition.

Keywords Fuzzy relational equations; fuzzy optimization; mixed integer programming

1 Introduction

A system of fuzzy relational equations with max-min composition, briefly max-min equations, is of the form

$$\bigvee_{j\in N}(a_{ij}\wedge x_j)=b_i, \quad i\in M, \tag{1.1}$$

where $M=\{1,2,\cdots,m\}$ and $N=\{1,2,\cdots,n\}$ are two index sets, $A=(a_{ij})_{m\times n}\in[0,1]^{mn}$, $x=(x_1,x_2,\cdots,x_n)^T\in[0,1]^n$, $b=(b_1,b_2,\cdots,b_m)^T\in[0,1]^m$ and the notations $\vee$ and $\wedge$ denote the *maximum* and *minimum* operators, respectively. In the matrix form, a system of max-min equations can be represented by $A\circ x=b$ where "$\circ$" denotes the max-min composite operation. The resolution of a system of max-min equations $A\circ x=b$, with given A and b, is to determine the solution set $S(A,b)=\{x\in[0,1]^n \mid A\circ x=b\}$. The resolution problem was first investigated by Sanchez [14, 15] and then widely studied by many researchers. It is well-known that the consistency of $A\circ x=b$ can be verified in polynomial time by constructing and checking a potential maximum solution. Moreover, its solution set $S(A,b)$, when it is nonempty, can be characterized by one maximum solution and a finite number of minimal solutions. However, as shown in Chen and Wang [1],

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Markovskii [13] and Li and Fang [9], the detection of all minimal solutions is closely related to the *set covering problem* and hence an NP-hard problem.

Instead of obtaining all minimal solutions of $A \circ x = b$, a specific solution that minimizes a user's criterion function $f(x)$ is of more interests in some circumstances. Consequently, we are interested in solving the following nonlinear optimization problem subject to a system of max-min equations:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & \\ & A \circ x = b, \\ & x \in [0,1]^n. \end{aligned} \tag{1.2}$$

Since $S(A,b)$ is non-convex in general, conventional optimization methods may not be directly employed to solve this problem.

The problem of minimizing a linear objective function subject to a system of max-min equations was first investigated by Fang and Li [4] and later by Wu *et al.* [18] and Wu and Guu [17]. It was shown by Li and Fang [9] that this type of optimization problems can be reduced to a 0-1 integer programming problem in polynomial time and hence is NP-hard in general.

When the problem of minimizing a nonlinear objective function is concerned, the situation could be very complicated. Lu and Fang [12] designed a genetic algorithm to solve nonlinear optimization problems subject to a system of max-min equations. So far, except for some particular scenarios, see e.g., Li and Fang [11] and references therein, there is no efficient method to deal with this type of nonlinear optimization problems.

In this paper, we show that the problem of minimizing a nonlinear objective function subject to a system of max-min equations can be in general reduced to a 0-1 mixed integer programming problem and hence can be handled by taking the advantage of some well developed techniques in integer programming and combinatorial optimization. The rest of the paper is organized as follows. The solution methods of a system of max-min equations are summarized in Section 2. The relation between the max-min equation constrained optimization and 0-1 mixed integer programming is illustrated in Section 3. Some generalizations and related issues are discussed in Section 4.

2 Resolution of Fuzzy Relational Equations

In this section, we recall some basic concepts and important results associated with the resolution of a system of max-min equations $A \circ x = b$ where $A = (a_{ij})_{m \times n} \in [0,1]^{mn}$ is the coefficient matrix, $b = (b_i)_{m \times 1} \in [0,1]^m$ is the right hand side vector and $x = (x_j)_{n \times 1} \in [0,1]^n$ is an unknown vector. Without loss of generality, we can assume that $b_1 \geq b_2 \geq \cdots \geq b_m > 0$. To make the paper succinct and readable, all proofs are omitted in this section. The reader may refer to Li and Fang [9] and references therein for the detailed discussion on this issue.

The most basic equation involved in the resolution of a system of max-min equations is the equation $a \wedge x = b$. It is clear that for any $a, b \in [0,1]$, $a \wedge x \leq b$ if and only if

$x \leq a @ b$ where

$$a @ b = \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{otherwise.} \end{cases}$$

Therefore, $a \wedge x = b$ has a solution if and only if $b \leq a$, in which case the solution set of $a \wedge x = b$ is the closed interval $[b,\ a@b]$. Actually, $a \wedge x = b$ has multiple solutions only when $a = b < 1$.

A system of max-min equations $A \circ x = b$ is called consistent if $S(A,b) \neq \emptyset$, otherwise, it is inconsistent. A partial order can be defined on $S(A,b)$ by extending the natural order such that for any $x^1, x^2 \in S(A,b)$, $x^1 \leq x^2$ if and only if $x^1_j \leq x^2_j$ for all $j \in N$.

Due to the monotonicity property of the *minimum* operator involved in the composition, the solution set $S(A,b)$, when it is nonempty, is "order convex", i.e., if $x^1,\ x^2 \in S(A,b)$, any x satisfying $x^1 \leq x \leq x^2$ is also in $S(A,b)$. See, for instance, Di Nola *et al.* [3] and De Baets [2]. We now focus on the so called extremal solutions.

Definition 2.1.
A solution $\check{x} \in S(A,b)$ is called a minimal solution if $x \leq \check{x}$ implies $x = \check{x}$ for any $x \in S(A,b)$. A solution $\hat{x} \in S(A,b)$ is called a maximum solution if $x \leq \hat{x}$, $\forall\, x \in S(A,b)$.

Lemma 2.2.
Let $A \circ x = b$ be a system of max-min equations. A vector $x \in [0,1]^n$ is a solution of $A \circ x = b$ if and only if, for each $i \in M$, there exists an index $j_i \in N$ such that $a_{ij_i} \wedge x_{j_i} = b_i$ and $a_{ij} \wedge x_j \leq b_i$, $i \in M,\ j \in N$.

Theorem 2.3.
A system of max-min equations $A \circ x = b$ is consistent if and only if the vector $A^T @ b$ with its components being defined by

$$(A^T @ b)_j = \min\{a_{ij} @ b_i \mid i \in M\}, \quad j \in N, \tag{2.1}$$

is a solution of $A \circ x = b$. Moreover, if the system is consistent, the solution set $S(A,b)$ can be fully determined by one maximum solution and a finite number of minimal solutions, i.e.,

$$S(A,b) = \bigcup_{\check{x} \in \check{S}(A,b)} \left\{x \in [0,1]^n \mid \check{x} \leq x \leq \hat{x}\right\}, \tag{2.2}$$

where $\check{S}(A,b)$ is the set of all minimal solutions of $A \circ x = b$ and $\hat{x} = A^T @ b$.

With the potential maximum solution $\hat{x}$, the characteristic matrix $\tilde{Q} = (\tilde{q}_{ij})_{m \times n}$ of $A \circ x = b$ is defined by

$$\tilde{q}_{ij} = \begin{cases} [b_i,\ \hat{x}_j], & \text{if } a_{ij} \wedge \hat{x}_j = b_i, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{2.3}$$

Note that $\tilde{q}_{ij}$ indicates all the possible values for variable x_j to satisfy the ith equation without violating other equations from the upper side. Consequently, a system $A \circ x = b$ is consistent if and only if each row of $\tilde{Q}$ contains at least one nonempty element. Note that all nonempty elements in each column of $\tilde{Q}$ share a common right endpoint. An equivalent form of Lemma 2.2 can be stated via the characteristic matrix $\tilde{Q}$.

Theorem 2.4.
Let $A \circ x = b$ be a system of max-min equations with a potential maximum solution $\hat{x}$ and a characteristic matrix $\tilde{Q}$. A vector $x \in [0,1]^n$ is a solution of $A \circ x = b$ if and only if $x \leq \hat{x}$ and the induced matrix $Q_x = (q'_{ij})_{m\times n}$ has no zero rows where

$$q'_{ij} = \begin{cases} 1, & \text{if } x_j \in \tilde{q}_{ij}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Note that if all of the nonempty elements of $\tilde{Q}$ are singletons, we can define a 0-1 matrix $Q = (q_{ij})_{m\times n}$ with

$$q_{ij} = \begin{cases} 1, & \text{if } \tilde{q}_{ij} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

In this case, if Q has no zero rows, then $A \circ x = b$ is consistent and each equation can be satisfied by a variable, say x_j, at a unique value $\hat{x}_j$. Hence we say a system $A \circ x = b$ is "simple" if all nonempty elements of its characteristic matrix are singletons. It is clear that $A \circ x = b$ is simple if for each $i \in M$, $b_i \neq a_{ij}$ holds for all $j \in N$.

The consistency of $A \circ x = b$ can be verified by constructing and checking the potential maximum solution in a time complexity of $O(mn)$. Once the maximum solution is obtained, the characteristic matrix $\tilde{Q}$ can be constructed in a time complexity of $O(mn)$. However, the detection of all minimal solutions is a complicated and challenging issue for investigation.

3 Max-min Equation Constrained Nonlinear Optimization Problems

Now we consider the following nonlinear optimization problem subject to a system of max-min equations

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & \\ & A \circ x = b, \\ & x \in [0,1]^n, \end{aligned} \tag{3.1}$$

provided that the feasible domain is nonempty, i.e., $S(A,b) \neq \emptyset$.

Let $\tilde{Q} = (\tilde{q}_{ij})_{m\times n}$ be the characteristic matrix of $A \circ x = b$. Denote r_j the number of different values in $\{b_i \mid \tilde{q}_{ij} \neq \emptyset,\ i \in M\}$ and $K_j = \{1, 2, \cdots, r_j\}$ for each $j \in N$ and $r = \sum_{j\in N} r_j$. Let $\check{v}_{jk}$, $k \in K_j$, be the different values in $\{b_i \mid \tilde{q}_{ij} \neq \emptyset,\ i \in M\}$ and $\check{v}^j = (\check{v}_{j1}, \check{v}_{j2}, \cdots, \check{v}_{jr_j})^T$ for each $j \in N$. It is clear that $\check{v}^j$ contains all possible values that x_j may assume in a minimal solution of $A \circ x = b$ such that $a_{ij} \wedge x_j = b_i$ for some $i \in M$.

To represent $S(A,b)$ in an ordinary manner other than a system of max-min equations, we need to define two binary matrices $G = (g_{jk})_{n\times r}$ and $Q = (q_{ik})_{m\times r}$, respectively, by

$$g_{jk} = \begin{cases} 1, & \text{if } \sum_{s=1}^{j-1} r_s < k \leq \sum_{s=1}^{j} r_s, \\ 0, & \text{otherwise,} \end{cases} \qquad \forall j \in N \tag{3.2}$$

and

$$q_{ik} = \begin{cases} 1, & \text{if } \sum_{s=1}^{j-1} r_s < k \le \sum_{s=1}^{j} r_s,\ \check{v}_{jk'} \in \tilde{q}_{ij} (\text{with } k' = k - \sum_{s=1}^{j-1} r_s), \text{for some } j \in N, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

In Li and Fang [9], G and Q are called the coefficient matrix of inner-variable incompatibility constraints and the augmented characteristic matrix, respectively.

Theorem 3.1.
Let $A \circ x = b$ be a consistent system of max-min equations with a maximum solution $\hat{x}$. A vector $x \in [0,1]^n$ with $x \le \hat{x}$ is a solution of $A \circ x = b$ if and only if there exists a binary vector $u \in \{0,1\}^r$ such that $Qu \ge e^m$, $Gu \le e^n$ and $\check{V}u \le x$ where $e^m = (1,1,\cdots,1)^T \in \{0,1\}^m$, $e^n = (1,1,\cdots,1)^T \in \{0,1\}^n$ and $\check{V} = \mathrm{diag}(\check{v}^1, \check{v}^2, \cdots, \check{v}^n)^T \in [0,1]^{nr}$.

Proof: If $x \in S(A,b)$, denote $u = (u_{11}, \cdots, u_{1r_1}, \cdots, u_{n1}, \cdots, u_{nr_n})^T \in \{0,1\}^r$ with

$$u_{jk} = \begin{cases} 1, & \text{if } k = \mathrm{argmax}\{\check{v}_{jk} \mid \check{v}_{jk} \le x_j\}, \\ 0, & \text{otherwise,} \end{cases} \quad j \in N. \tag{3.4}$$

Since $\check{v}_{jk}$, $k \in K_j$, are different from each other for each $j \in N$, it is clear that $\sum_{k \in K_j} u_{jk} \le 1$ and $\sum_{k \in K_j} \check{v}_{jk} u_{jk} \le x_j$. Hence we have $Gu \le e^n$ and $\check{V}u \le x$. Moreover, for each $i \in M$, there exists $j_i \in N$ such that $a_{ij} \wedge x_{j_i} = b_i$, i.e., there exists $k \in K_{j_i}$ such that $u_{j_i k} = 1$ and $q_{ik'} = 1$ with $k' = k + \sum_{s=1}^{j_i - 1} r_s$. Hence, we have $Qu \ge e^m$.

Conversely, denote $u = (u_{11}, \cdots, u_{1r_1}, \cdots, u_{n1}, \cdots, u_{nr_n})^T \in \{0,1\}^r$ and $x_u = \check{V}u$. It is clear that $x_u \le x \le \hat{x}$. Since $Qu \ge e^m$, for each $i \in M$, there exists $j_i \in N$ and $k \in K_{j_i}$ such that $u_{j_i k} = 1$ and $q_{ik'} = 1$ with $k' = k + \sum_{s=1}^{j_i - 1} r_s$. Moreover, we have $(x_u)_{j_i} \in \tilde{q}_{i j_i}$ for each $i \in M$ since $Gu \le e^n$. Hence, we have $x_u \in S(A,b)$ and consequently $x \in S(A,b)$. □

According to Theorem 3.1, the problem of minimizing a nonlinear function $f(x)$ subject to a consistent system of max-min equations $A \circ x = b$ can be reduced to the following 0-1 mixed integer programming problem:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & \\ & Qu \ge e^m, \\ & Gu \le e^n, \\ & \check{V}u \le x \le \hat{x}, \\ & u \in \{0,1\}^r. \end{aligned} \tag{3.5}$$

Notice that this new problem may introduce additional r (up to mn) binary variables. This could dramatically increase the size of the problem compared to the original problem. However, it may be still worthwhile in general to perform this reduction, with which some well developed techniques could be applied. On the other hand, the constraint $\sum_{k \in K_j} q_{jk} u_{jk} \le 1$ is redundant whenever $r_j = 1$ and hence can be removed. Therefore, in case the system $A \circ x = b$ is "simple", the constraint $Gu \leq e^n$ vanishes and the new problem contains only n binary variables and n continuous variables.

Note that the reduction does not involve the objective function $f(x)$. As a consequence, the binary vector u serves merely as a control vector and does not appear in the objective function $f(x)$. Li and Fang [9, 10] showed that for some particular scenarios, for instance, $f(x)$ is linear, linear fractional or, more generally, monotone in each variable separately, the above 0-1 mixed integer optimization problem can be further reduced into a linear/nonlinear 0-1 integer programming problem. Notice that in case $f(x) = \bigvee_{j \in N} f_j(x_j)$ with $f_j(x)$ being a continuous monotone function for each $j \in N$, the corresponding optimization problem can be solved in polynomial time. The reader may refer to Li and Fang [11] and references therein for a detailed discussion on this issue.

Consider the following nonlinear optimization problem:

$$\begin{aligned} \min \quad & (2x_1 + x_2)^2 + (x_2 - 2x_3)^2 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 0.8 & 0 & 0.8 \\ 0.6 & 0.6 & 0 \\ 0 & 0.4 & 0.2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \end{pmatrix}, \\ & 0 \leq x_j \leq 1, \quad j = 1, 2, 3. \end{aligned} \tag{3.6}$$

It is clear that the system of max-min equations is consistent with a maximum solution $\hat{x} = (1, 1, 1)^T$. The characteristic matrix is

$$\tilde{Q} = \begin{pmatrix} [0.8, 1] & \emptyset & [0.8, 1] \\ [0.6, 1] & [0.6, 1] & \emptyset \\ \emptyset & [0.4, 1] & \emptyset \end{pmatrix}.$$

Therefore, we have $\check{v}^1 = (0.8, 0.6)^T$, $\check{v}^2 = (0.6, 0.4)^T$, $\check{v}^3 = 0.8$ and

$$\check{V} = \begin{pmatrix} 0.8 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.8 \end{pmatrix}. \tag{3.7}$$

Moreover, the augmented characteristic matrix is

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \tag{3.8}$$

and the coefficient matrix of inner-variable incompatibility constraints is

$$G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.9}$$

Note that the third row of G can be removed since $\check{v}^3$ contains only a single value. Denote $u = (u_{11}, u_{12}, u_{21}, u_{22}, u_{31})^T$. The concerned optimization problem is equivalent to the following 0-1 mixed integer programming problem:

$$\begin{aligned}
\min \quad & (2x_1 + x_2)^2 + (x_2 - 2x_3)^2 \\
\text{s.t.} \quad & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.6 & -0.4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -0.8 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (3.10) \\
& u_{11}, u_{12}, u_{21}, u_{22}, u_{31} \in \{0, 1\},\ x_1, x_2, x_3 \in [0, 1].
\end{aligned}$$

Using a commercial solver, e.g., CPLEX, an optimal solution can be obtained as

$$(u^*; x^*) = (0, 0, 1, 0, 1, 0, 0.8, 0.8)^T$$

with an objective value 1.28. Hence $x^* = (0, 0.8, 0.8)^T$ is an optimal solution to the original nonlinear optimization problem subject to a system of max-min equations.

4 Concluding Remarks

In this paper, we showed that the problem of minimizing a nonlinear objective function subject to a system of fuzzy relational equations with max-min composition can be in general reduced to a 0-1 mixed integer programming problem. With this reduction, some well developed techniques in integer programming and combinatorial optimization can be employed to solve the problem.

It has been shown in the literature that a system of fuzzy relational equations can be well defined with respect to the max-T composition where $T : [0, 1]^2 \to [0, 1]$ is a continuous triangular norm, see, e.g., Di Nola *et al.* [3] and Gottwald [5]. The *minimum* operator is the most frequently used triangular norm. According to Li and Fang [9], a system of fuzzy relational equations with max-T composition, briefly max-T equations, can be handled in a completely analogous manner as that of max-min equations. Hence, the reduction method introduced in this paper remains valid for nonlinear optimization problems subject to a system of max-T equations. Moreover, when T is a continuous Archimedean triangular norm, a system of max-T equations with positive right hand side constants is always “simple”.

The proposed reduction method can be extended as well to nonlinear optimization problems subject to a system of fuzzy relational equations or inequalities with max-O or min-O composition where $O : [0, 1]^2 \to [0, 1]$ is a general continuous binary operator as long as the solution set can be fully determined by a maximum solution and a finite number of minimal solution or dually, by a minimum solution and a finite number of maximal solutions. One such scenario is fuzzy relational equations with max-O_{av} composition discussed in Khorram and Ghodousian [6], Wu [16] and Khorram and Hassanzadeh [7], where $O_{av}(a,b) = \frac{1}{2}(a+b)$ for any $a,b \in [0,1]$. The reader may refer to Li and Fang [8, 9] for details.

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