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The Incidence Chromatic Number of 2-connected 1-trees

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Abstract In this paper, the structural properties of 1-trees are discussed in details firstly. Based on the properties of 1-trees, the incidence chromatic number of 2-connected 1-trees can be determined.

Keywords 1-tree; Incidence set; Incidence chromatic number

1 Introduction

The incidence coloring of graph is introduced by Brualdi and Massey[1] for solving Erdős' strong edge coloring's Conjecture in 1993.

Let $G=(V,E)$ be a graph of order n and of size m. Let $I=\{(v,e): v\in V, e\in E, v \textit{ is incident with } e\}$ be the set of incidences of G. We say that two incidences (v,e) and (w,f) are neighborly iff one of the following conditions satisfies:

(1) $v=w$; (2) $e=f$; (3) the edge $\{v,w\}$ equals e or f.

We define an incidence coloring of G to be a coloring of its incidences in which neighborly incidences are assigned different colors. The incidence chromatic number of G denoted by $\chi_i(G)$ is the smallest number of colors in an incidence coloring.

The incidence chromatic number and the strong chromatic number of graphs have close relations. A strong edge coloring[2] of G is a coloring of the edges of G in which the edges with the same colors form an induced matching. The strong chromatic number $sq(G)$ equals the smallest number of colors in a strong edge coloring. Let H be the bipartite multigraph of order $n+m$ with bipartition V,E in which v_i is adjacent to e_j iff v_i is incident with e_j in G. An incidence coloring of G corresponds to a partition of the edges of H into induced matchings. Thus $\chi_i(G)=sq(H)$.

Therefore, the determination of the incidence chromatic number of a graph is a fascinating question. The incidence coloring number of complete graphs, complete bipartite graphs and trees are determined in [1]. Putting forward the following Conjecture:

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Conjecture: Every graph can be incident colored with $\Delta+2$ colors, namely $\chi_i(G) \leq \Delta+2$ where Δ is the maximum degree of G.

In [3], it is proved that the above conjecture is not true using Paley graphs. [3] presented that the upper bound for the incidence chromatic number of a graph is $\Delta + O(log\Delta)$. [5] determined the incidence coloring number of paths, cycles, fans, wheels, adding-edge wheels and complete multipartite graphs. [6] determined the incidence coloring number of $P_n \times P_m$ and generalized complete graphs. [7] determined the incidence chromatic number of Halin graphs and outerplane graphs($\Delta \geq 4$). [8] determined the incidence chromatic number of cubic graphs.

Let G be a plane graph. If $\exists v \in V(G)$ such that $G-v$ is a forest, G is called 1-tree, and v is called a root vertex. A vertex with degree k in a graph G is called a k-vertex. For a incidence coloring f of G, $v \in V(G)$, $N(v) = \{u|uv \in E(G), u \in V(G)\}$. We call $f[v) = \{f(v,vu), u \in N(v)\}$ is the near incidence set of v, and $f(v] = \{f(u,uv), u \in N(v)\}$ is the far incidence set of v. Let $f(vu)$ denote $\{f(v,vu), f(u,uv)\}$, and $f[v]$ denote $\{f(vu)|u \in N_G(v)\}$.

In addition, other terms and notations not stated can be found in [9].

2 Structural properties of 1-trees

Lemma 2.1[4] If F is a forest, then $|V_1(F)| \geq \Delta(F)$, where $V_1(F)$ denotes all 1-vertices of F.

Lemma 2.2[4] If T is tree with $\Delta(T) \geq 2$, then $|S(T)| \geq 1$ and $|S(T)| = 1$ iff $T \cong K_{1,p-1}$, where $(S(T) = \cup_{0 \leq i \leq 1} V_i(T - V_1(T)))$.

Lemma 2.3[4] If G is a 2-connected 1-tree, then $\delta(G) \leq 2$.

In this paper, we discuss 2-connected 1-trees G. Hence, $\delta(G) = 2$ and G is cycle when $\Delta(G) = 2$. The incidence chromatic number of cycles has been determined. Therefore, we discuss $\Delta(G) \geq 3$ in the following.

In the following discussions, we denote a plane graph S_p with p vertices $u, v, x_1, \ldots, x_{p-2}$ and $2p-4$ edges $ux_1, ux_2, \ldots, ux_{p-2}, vx_1, \ldots, vx_{p-2}$. $\overline{S_p}$ has the same vertex set with S_p, and $E(\overline{S_p}) = E(S_p) \cup \{uv\}$.

For $p \geq 6$, $S_p^1, S_p^2, \ldots, S_p^k (k = p-5)$, are a family of plane graphs with the same vertex set with S_p. $E(S_p^1) = E(S_p) \backslash \{ux_1, vx_2\} \cup \{x_1x_2\}$,$E(S_p^2) = E(S_p^1) \backslash \{vx_3\} \cup \{x_1x_3\}$, $E(S_p^3) = E(S_p^2) \backslash \{vx_4\} \cup \{x_1x_4\}$, ... We denote the family of graphs by $\aleph$. The maximum degree of every graph of $\aleph$ is $p-3$.

Lemma 2.4[4] If G is a 2-connected 1-tree with $\Delta(G) \geq 3$, then at least one of the following cases is true:

1) There are two adjacent 2-vertices u and v;

2) There is a 3-face uvw such that $d(u) = 2$ and $d(v) = 3$;

3) There is a 4-cycle $uxvyu$ whose interior contains at most one edge xy and $d(u) = d(v) = 2$, $d(x) \leq \Delta(G) - 1$;

4) $G \cong S_p$ or $G \cong \overline{S_p}$.

Lemma 2.5 Let G be a 2-connected 1-tree and t be the root vertex of G, then $d_G(t) = \Delta(G)$.

Proof: Let $F = G - t$. Since G is 2-connected 1-tree, $\delta(G) = 2$ and $V_1(F) \subseteq N_G(t)$. If G is cycle, the conclusion is true. When G is not cycle, suppose $d_G(t) < \Delta(G)$, then there exists $v \in V(F)$, $d_G(v) = \Delta(G)$. If $vt \notin E(G)$, then $d_G(v) = d_F(v)$. By Lemma 2.1, $\Delta(G) = \Delta(F) \leq |V_1(F)| \leq |d_G(t)| < \Delta(G)$. The contradiction occurs. If $vt \in E(G)$, by Lemma 2.1, $\Delta(G) - 1 \leq \Delta(F) \leq |V_1(F)| \leq |d_G(t)| - 1 < \Delta(G) - 1$. There is also contradiction. Therefore, $d_G(t) = \Delta(G)$.

Lemma 2.6 Let G be a 2-connected 1-tree and t be the root vertex. For any $x, y \in V(G) \backslash \{t\}$, $d_G(x) + d_G(y) \leq \Delta + 2$.

Proof: Let G be a 2-connected 1-tree, $F = G - t$, and F be a tree and $V_1(F) \subseteq N_G(t)$. $\forall x, y \in V(F)$, $d_F(x) - 1 + d_F(y) - 1 \leq |V_1(F)| \leq \Delta(G)$.

1) If $xt, yt \notin E(G)$, then $d_G(x) + d_G(y) \leq \Delta + 2$.

2) If one of $xt \in E(G)$ and $yt \in E(G)$ is correct, without loss of generality, we assume $xt \in E(G)$. If x is a leaf of F, $d_G(x) = 2$, $d_F(y) = d_G(y) \leq |V_1(F)| \leq \Delta(G)$, hence $d_G(x) + d_G(y) \leq \Delta + 2$. If x isn't a leaf of F, $(d_G(x) - 1) - 1 + d_G(y) - 1 \leq |V_1(F)| \leq \Delta(G) - 1$, $d_G(x) + d_G(y) \leq \Delta + 2$

3) If $xt, yt \in E(G)$, x and y are leaves of F. The conclusion is true. When x and y are not leaves of F, $(d_G(x) - 1) - 1 + (d_G(y) - 1) - 1 \leq |V_1(F)| \leq \Delta(G) - 2$, and $d_G(x) + d_G(y) \leq \Delta + 2$. When one of x and y is a leaf of F, the proof similar to the case 2).

Therefore, The conclusion is true.

Lemma 2.7 Let G be a 2-connected 1-tree and t be a root vertex. If $\exists x, y \in V(G) \backslash \{t\}$ such that $d_G(x) + d_G(y) = \Delta + 2$, then $\forall z \in V(G) \backslash \{x, y, t\}$, $d_G(z) = 2$.

Proof(disproof): Let $F = G - t$. $\forall z \in V(G) \backslash \{x, y, t\}$. Suppose that $d_G(z) \geq 3$. Since G is 2-connected, F is a tree and $|V_1(F)| \leq \Delta(G)$.

Case 1 $zt \in E(G)$, then $|V_1(F)| \leq \Delta(G) - 1$. For $\forall x, y \in V(G) \backslash \{t\}$, $d_F(x) - 1 + d_F(y) - 1 \leq |V_1(F)|$.

Subcase 1.1 $xt, yt \notin E(G)$, then $d_G(x) = d_F(x)$ and $d_G(y) = d_F(y)$. Therefore, $d_G(x) - 1 + d_G(y) - 1 \leq |V_1(F)|$, that is $d_G(x) + d_G(y) \leq \Delta + 1$. There is contradiction with the proposition.

Subcase 1.2 One of $xt \in E(G)$ and $yt \in E(G)$ is true, without loss of generality, we assume $xt \notin E(G)$ and $yt \in E(G)$. Thus, $d_G(x) = d_F(x)$, $d_G(y) = d_F(y) + 1$, and $d_G(x) - 1 + (d_G(y) - 1) - 1 \leq |V_1(F)|$. If y is a leaf of F, then $d_G(x) + d_G(y) \leq \Delta + 1$. If y isn't a leaf of F, then $|V_1(F)| \leq \Delta - 2$, and $d_G(x) + d_G(y) \leq \Delta + 1$. There is contradiction with the proposition.

Subcase 1.3 $xt, yt \in E(G)$, there are three cases as the following:

1) x is a leaf and y isn't a leaf, then $d_G(y) = d_F(y) + 1 \leq |V_1(F)| + 1 \leq (\Delta - 2) + 1 = \Delta - 1$, and $d_G(x) + d_G(y) \leq \Delta + 1$.

2) x and y are all leaves, then $d_G(x) + d_G(y) = 4$.

3) x and y are all not leaves, then $(d_G(x) - 1) - 1 + (d_G(y) - 1) - 1 \leq |V_1(F)| \leq \Delta - 2$. That is $d_G(x) + d_G(y) \leq \Delta + 1$. There is contradiction with the proposition.

Case 2 $zt \notin E(G)$, then $\forall x, y \in V(G)\backslash\{t\}$, $d_F(x) - 1 + d_F(y) - 1 + d_F(z) - 1 \leq |V_1(F)| \leq \Delta(G)$. There are also three subcases, and the proof is similar to Case1.

Lemma 2.8 If G is a 2-connected 1- tree and G is not cycle, then the number of the maximum degree vertex of G is at most 2. When there are two maximum degree vertices, other vertices are 2-degree vertices.

Proof: Suppose that there are three maximum degree vertices x, y, z. Let x be the root vertex. By Lemma 2.6, $d_G(y) + d_G(z) \leq \Delta + 2$, namely $\Delta \leq 2$. Since G is 2-connected, $\delta(G) = 2$. It is contradict with that G isn't cycle. Hence, the number of the maximum degree vertex of G is at most 2.

Let x and y be two maximum degree vertices of G, and x be a root vertex. $\forall z \in V(G)\backslash\{x, y\}$, by Lemma 2.6, $d_G(y) + d_G(z) \leq \Delta + 2$, $d(z) \leq 2$, and $\delta(G) = 2$. Therefore, $d(z) = 2$. The conclusion is true.

3 Incidence chromatic number of 2-connected 1-trees

Lemma 3.1[1,5] For any graphs G with the maximum degree Δ, $\chi_i(G) \geq \Delta + 1$.

Lemma 3.2[7] Let the maximum degree of G be Δ and there exist a $(\Delta + 1)$-incidence coloring. The far incidence of the maximum degree vertex is colored the same color.

For $S_p, \overline{S_p}$ and the graphs of $\aleph$, there are the following three Lemmas.

Lemma 3.3 For $p \geq 5$, $inc(S_p) = \Delta(S_p) + 2 = p$ and $inc(\overline{S_p}) = \Delta(\overline{S_p}) + 1 = p$.

Proof: Firstly, we construct a p-incidence coloring f of S_p, and $f : I(S_p) \to C = \{1, 2, \ldots, p\}$.

$$f(u, ux_i) = f(v, vx_i) = i, i = 1, 2, \ldots, p-2,$$
$$f(x_i, x_i u) = p-1, f(x_i, x_i v) = p, i = 1, 2, \ldots, p-2$$

Hence, $\chi_i(S_p) \leq p$. Now we prove $\chi_i(S_p) \geq p$

If S_p isn't satisfied, there is $\chi_i(S_p) \leq p - 1 = \Delta(S_p) + 1$. u and v are the maximum degree vertices. If f is a $(\Delta(S_p) + 1)$-incidence coloring of S_p, then the near incidence of u and v need $\Delta(S_p)$ colors. The other two colors far incidence coloring of u and v is needed at least one color same to $f[u] \cup f[v]$. This is contradict with the definition of incidence coloring. Hence, $\chi_i(S_p) \geq p$, and $\chi_i(S_p) = p$.

For $\overline{S_p}$, we may get by the incidence coloring of S_p. If the colors of the far incidence of u and v color the same color, we get a p-incidence coloring of $\overline{S_p}$. Hence $inc(\overline{S_p}) = p$.

Lemma 3.4 For $p \geq 6$, the incidence coloring number of graphs of graph family $\aleph$ is $p - 2$.

Proof: By Lemma 3.1, if only we give a $(p-2)$-incidence coloring of $\aleph$. Now we construct an incidence coloring f of S_P^k, f:$I(S_p^k) \to C = \{1, 2, \ldots, p-2\}$. Let the vertex set be $\{u, v, x_1, \ldots, x_{p-2}\}$.

$f(u, ux_i) = i$, $f(x_i, x_i u) = f(x, xx_1) = 1$, $i = 2, \ldots, p-2$.

$f(v,vx_i) = f(u,ux_i), f(x_1,x_1v) = f(x_i,x_iv) = 2,\ i = 2+k,\dots,p-2,\ 1 \le k \le p-5$.

$f(x_1,x_1x_2) = f(u,ux_{p-2}) = p-2$, $f(x_1,x_1x_i) = \cdots = f(u,ux_i)$, $i = 3,\dots,1+k$.

$f(x_i,x_ix_1) = f(u,ux_{2+k})$, $i = 2,\dots,1+k$.

Hence, the conclusion is true.

Theorem 3.5 If G is a 2-connected 1-tree with the maximum degree $\Delta \ge 3$ and two maximum degree vertices, and $G \ne S_p^1$, then there exists a $(\Delta+2)$-incidence coloring of G, such that the far incidence of every vertex has the same color.

Proof: If G is S_p or $\overline{S_p}$, the conclusion is true by Lemma 3.3. For other graphs, suppose u and v are two maximum degree vertices. Other vertices of G are 2-vertices by Lemma 2.6. There exists Δ uv-paths with different length. We construct a $(\Delta+2)$-incidence coloring f of G, $f: I(G) \to C = \{1,2,\dots,\Delta+2\}$.

Let $\sum_{i=1}^{r} k_i = \Delta$, where k_i denotes the number of the uv-paths whose length is i and $k_1 = 1$. We color every path in increasing order. Let $f(u] = \Delta+1$ and $f(v] = \Delta+2$. Thus, the far incidences of u and v are colored.

1) For $i = 2$, we color the near incidences of u and v with the same color and the color is $1,2,\dots,k_2$, respectively.

2) For $i = 3$, let $uu_1^iu_2^iv(i = 1,2,\dots,k_3)$ be k_3 uv-paths, and let $f(u,uu_1^i) = f(u_2^i,u_2^iu_1^i) = k_2+i$, $f(v,vu_2^i) = f(u_1^i,u_1^iu_2^i) = \Delta - i$, and $i = 1,2,\dots,k_3$.

3) For $i = 4$, let $uu_1^iu_2^iu_3^iv(i = 1,2,\dots,k_4)$ be k_4 uv-paths, and $f(u,uu_1^i) = f(u_2^i,u_2^iu_1^i) = k_2+k_3+i$, $f(v,vu_3^i) = f(u_2^i,u_2^iu_3^i) = \Delta - k_3 - i$, and $f(u_1^i,u_1^iu_2^i) = f(u_3^i,u_3^iu_2^i) = \alpha \in C\backslash\{f(uu_1^i), f(vv_3^i)\}$, $i = 1,2,\dots,k_4$.

4) For $i = r$, let $uu_1^iu_2^i\dots u_{r-1}^iv(i = 1,2,\dots,k_r)$ are k_r uv-paths, and $f(u,uu_1^i) = f(u_2^i,u_2^iu_1^i) = k_2+\cdots+k_{r-1}+i$, $f(v,vu_{r-1}^i) = f(u_{r-2}^i,u_{r-2}^iu_{r-1}^i) = \Delta - (k_3+\cdots k_{r-1}) - 1$, $i = 1,2,\dots,k_r$. For any $u_j^i(j = 2,3,\dots,r-2)$, it's the far incidences with at most 4 limits, so there exists colors and $|f(u_j^i]| = 1$.

Therefore, we prove the conclusion.

Theorem 3.6 For $p \ge 7$, the graphs of graph family $\aleph$ do not have $(p-1)$-incidence colors, therefore the far incidence of every vertex is colored the same color.

Proof: $\forall S_p^k \in \aleph$, k is positive integer and $k \le p-5$. Suppose $V(S_p^k) = \{u,v,x_1,\dots,x_{p-2}\}$ and u is the root vertex. We need $p-2$ colors to color the far incidences and the near incidences of u. By the definition of incidence coloring and the demand, the far incidences of v and x_1 need the other two different colors.

So, the conclusion is true.

Theorem 3.7 If G is a 2-connected 1-tree with $\Delta \ge 3$ and only one maximum degree vertex, and $G \notin \aleph$, there exists a $(\Delta+2)$-incidence coloring of G, such that the far incidence of every vertex is colored with the same color.

Proof: We will proceed by induction on the order p of G. There is $p \ge 4$ by the proposition $\Delta \ge 3$. When $p = 4$, it is clear that G is fan F_4, so the result is true. We suppose that the conclusion is correct for graph G with order less than $p(p \ge 5)$.

Now, for any graph G with order p, by Lemma 2.4, we can divide the proof into four cases. Let $C = \{1, 2, \ldots, \Delta+2\}$ be the color set.

Case 1 There are two adjacent 2-vertices u and v. Let $N(u) = \{x, v\}$, $N(v) = \{u, y\}$, $H = G - u + xv$, and $d_G(y) < \Delta$ (by proposition only one maximum degree vertex). The order of H is less than p and $\Delta(H) = \Delta(G) = \Delta$. By induction hypothesis, H has a $(\Delta+2)$-incidence coloring $f^* : I(H) \to C$ such that the far incidence of every vertex is colored the same color. Now we extend f^* to a $(\Delta+2)$-incidence coloring f of G as follows: $f(xu) = f^*(xv)$, $f(v, vu) = f^*(x, xv)$, $f(u, uv) = f(y, yv) = \alpha \in C \backslash \{f^*[y] \cup f^*(x])\}$. The incidence coloring of other elements is same to f^*.

Case 2 There is a 3-face $uvwu$ such that $d_G(u) = 2$ and $d_G(v) = 3$. By Lemma 2.5 and Lemma 2.8, $d_G(w) = \Delta(G)$. Let $H = G - u$, and $\Delta(H) = \Delta(G) - 1$. If $H \notin \aleph$, by induction hypothesis, H has a $(\Delta+2)$-incidence coloring $f^* : I(H) \to C$ such that the far incidence of every vertex is colored with the same color. Now we extend f^* to a $(\Delta+2)$-incidence coloring f of G as follows: $f(u, uw) = f^*(v, vw)$, $f(u, uv) = f^*(w, wv)$, $f(w, wu) = f(v, vu) = \alpha \in C \backslash \{f^*[w] \cup f^*[v)\}$. The incidence coloring of other elements the same to f^*.

If $H \in \aleph$, we construct a $(\Delta+2)$-incidence coloring of G. Let $V(G) = \{w, u, v, x, y, v_1, \ldots, v_{\Delta-2}\}$ and $E(G) = \{wu, wv, wv_1, \ldots, wv_{\Delta-2}\} \cup \{uv, vx, xy\} \cup \{xv_1, \ldots, xv_{k-1}\} \cup \{yv_k, \ldots, y\, v_{\Delta-2}\}$, $1 \leq k \leq \Delta-2$, $f(w, wu) = f(v, vu) = 1$, $f(v] = 2$, and $f(w, wv_i) = 2 + i, i = 1, \ldots, \Delta-2$,

$f(w] = \Delta+1, f(x] = \Delta+2, f(y] = 1,$

$f(x, xv_i) = f(w, wv_i) i = 1, \ldots, k-1$, $f(y, yv_i) = f(w, wv_i) i = k, \ldots, \Delta-2$.

Clearly, f is an incidence coloring of G which satisfies the demand.

Case 3 There is a 4-cycle $uxvyu$ whose interior contains at most one edge xy, $d_G(u) = d_G(v) = 2$, $d_G(x) \leq \Delta(G) - 1$.

1) $xy \in E(G)$. Let $H = G - u$, $\Delta(H) = \Delta(G) - 1$, H has a $(\Delta+1)$-incidence coloring $f^* : I(H) \to C' = \{1, 2, \ldots, \Delta+1\}$ such that the far incidence of every vertex is color with the same color. Now we extend f^* to a $(\Delta+2)$-incidence coloring f of G as follows: $f(u, ux) = f^*(y, yx)$, $f(u, uy) = f^*(x, xy)$, $f(x, xu) = f(y, yu) = \Delta+2$. The incidence coloring of other elements is same to f^*.

2) $xy \notin E(G)$. Let $H = G - u$, $\Delta(H) = \Delta(G) - 1$, H has a $(\Delta+1)$-incidence coloring $f^* : I(H) \to C' = \{1, 2, \ldots, \Delta+1\}$ such that the far incidence of every vertex is colored with the same color. Now we extend f^* to a $(\Delta+2)$-incidence coloring f of G as follows: $f(u, uy) = f^*(v, vy)$, $f(u, ux) = f^*(v, vx)$, $f(y, yu) = f(x, xu) = \Delta+2$. The incidence coloring of other elements is same to f^*.

Case 4 $G \cong S_p$ or $G \cong \overline{S_p}$. By Lemma3.3, the conclusion is true.

Theorem 3.8 Let G be a 2-connected 1-tree with $\Delta \geq 3$ and $G \neq S_p$ and $G \notin \aleph$, $\chi_i(G) = \Delta+1$.

Proof: We will proceed by induction on the order p of G. There is $p \geq 4$ by the proposition $\Delta \geq 3$. When $p = 4$, clearly, G is fan F_4, so the result is true. We suppose that the conclusion holds for graph G of order less than $p (p \geq 5)$. Now, for any graph G of order p, by Lemma 2.3, we may divide the proof into four cases. Let $C = \{1,2,\ldots,\Delta+1\}$ be the color set.

Case 1 There are two adjacent 2-vertices u and v. Let $N(u) = \{x,v\}$, $N(v) = \{u,y\}$, and $H = G-u+xv$. The order of H is less than p and $\Delta(H) = \Delta(G) = \Delta$. By induction hypothesis, H has a $(\Delta+1)$-incidence coloring $f^* : I(H) \to C$. Now we extend f^* to a $(\Delta+1)$-incidence coloring f of G as follows: $f(xu) = f^*(xv)$, $f(v,vu) = \alpha \in C\backslash\{f^*(vy)\cup f(u,ux)\}$, $f(u,uv) = \beta \in C\backslash\{f(xu)\cup f^*(v,vy)\cup\alpha\}$. The incidence coloring of other elements is same to f^*.

Case 2 There is a 3-face such that $d_G(u) = 2$ and $d_G(v) = 3$. By Lemma 2.4 and Lemma 2.7, w is a root vertex and $d_G(w) = \Delta(G)$. Let $H = G-u$, then $\Delta(H) = \Delta(G)-1$. If $H \notin \aleph$, by Theorem 3.6 and 3.7, H has a $(\Delta+1)$-incidence coloring f^* : $I(H) \to C$ such that the far incidence of every vertex is colored with the same color. Next, we extend f^* to a $(\Delta+1)$-incidence coloring f of G as follows: $f(u,uw) = f^*(v,vw)$, $f(u,uv) = f^*(w,wv)$, $f(v,vu) = \alpha \in C\backslash f^*[v]$, $f(w,wu) = \beta \in C\backslash f^*[w]$. The incidence coloring of other elements is same to f^*.

If $H \in \aleph$, by Lemma 3.4, H has a (Δ)-incidence coloring $f^* : I(H) \to C' = \{1,2,\ldots,\Delta\}$. Now we extend f^* to a $(\Delta+1)$-incidence coloring f of G as follows: $f(u,uw) = f^*(v,vw)$,$f(w,wu) = f(v,vu) = \Delta+1$, , $f(u,uv) = f^*(w,wv)$. The incidence coloring of other elements is same to f^*.

Case 3 There is a 4-cycle $uxvyu$ whose interior contains at most one edge xy, $d_G(u) = d_G(v) = 2$, $d_G(x) \le \Delta(G)-1$, then $d_G(y) = \Delta(G)$.

1) $xy \in E(G)$. Let $H = G-u$, then $\Delta(H) = \Delta(G)-1$, and $H \notin \aleph$. By Theorem 3.7, H has a $(\Delta+1)$-incidence coloring $f^* : I(H) \to C$ such that the far incidence of every color wi with the same color. Now we extend f^* to a $(\Delta+1)$-incidence coloring f of G as follows: $f(u,uy) = f^*(x,xy)$, $f(u,ux) = f^*(y,yx)$, $f(y,yu) = \alpha \in C\backslash\{f^*[y]\}$, $f(x,xu) = \beta \in C\backslash f^*[x]$.

2) $xy \notin E(G)$. Let $H = G-u$, then $\Delta(H) = \Delta(G)-1$. By Theorem 3.7, H has a $(\Delta+1)$-incidence coloring $f^* : I(H) \to C$ such that the far incidence of every vertex is colored with the same color. Now we extend f^* to a $(\Delta+1)$-incidence coloring f of G as follows: $f(u,uy) = f^*(v,vy)$, $f(y,yu) = \alpha \in C\backslash f^*[y]$. If $\alpha \notin f^*[x)$, $f(x,xu) = \alpha$, $f(u,ux) = \beta \in C \setminus f^*[x) \setminus f(uy)$. If $\alpha \in f^*[x)$, $f(x,xu) = \beta \in C\{\backslash f^*[x] \cup f^*(y]\}$, $f(u,ux) = \gamma \in C \setminus \{f^*[x) \cup f(u,uy) \cup \beta\}$. If $\alpha \in f^*(x]$, when $d(x) < \Delta-1$, $f(x,xu) = \beta \in C \setminus \{f^*[x] \cup f^*(y]\}$, $f(u,ux) = r \in C \setminus \{f^*[x] \cup f^*(y] \cup \beta\}$. When $d(x) = \Delta+1$, by Lemma 2.7 and Lemma 2.8, G has one only one 3-vertex and other vertices are 2-vertices. Let $f(x] = \beta \in C\backslash\{f^*[x] \cup f^*(y]\}$, $f(x,xu) = \alpha = f(y,yu)$, if the near incidence of adjacent vertices of x is colored β in coloring f^*, since there is at most 4 limitations, we may recolor it.

The incidence coloring of other elements is same to f^*.

Case4 $G \cong \overline{S_p}$. By Lemma 2.4, the conclusion is correct.

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