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Extra Resource Allocation Problem with the Most Compromise Common Weights Based on DEA*

Xiao-Ya Li   Jin-Chuan Cui

Institute of Applied Mathematics, Academy of Mathematics and Systems Science
Chinese Academy of Sciences, Beijing 100080, China

Abstract In this paper, we propose a method for the extra resource allocation problem based on data envelopment analysis (DEA). Suppose there are some extra resource such as bonus or material benefits, which can be given to all or only a part of decision making units (DMUs), and if we want the allocation to be most beneficial to the whole system, how the extra resource should be distributed. Since the selection of DMUs to receive the extra resource should depend on not only its efficiency, but also its scale, this extra resource allocation problem is complicated to be solved. We construct an improved Common Weights Analysis (CWA) model to obtain common weights for the full ranking. Adopting the common weights, we propose an algorithm for the extra resource allocation problem. The extra resource allocation we proposed is regarded as the most compromise solution.

Keywords Date envelopment analysis (DEA); common weights analysis (CWA); extra resource allocation problem (ERAP); decision making unit (DMU), compromising solution; efficiency score; scale.

1 Introduction

Data envelopment analysis method, firstly proposed by Charnes, Cooper and Rhodes in 1978 [1], has become an increasingly important tool not only for assessing the relative efficiency of homogeneous operating decision-making units (DMUs), but also for performance forecasting and resource estimation. Instead of the original efficiency evaluation function of DEA, the resource allocation problem based on DEA method has become a hot topic in DEA field. We propose an interesting resource allocation problem which is termed as extra resource allocation problem (ERAP) in this paper. Let us consider a decision-making environment in which a set of DMUs is operating. Each unit produces multiple outputs by costing multiple inputs. Suppose the central decision maker wants to allocate some extra resource such as bonus or material benefits to some or all units, if the decision maker wants the allocation to be most beneficial to the whole system, how much should every unit get. The extra resource allocation try to allocate extra resource to reach the stimulation target.

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2 Literature Review

2.1 DEA Framework

Charnes, Cooper and Rhodes [1] introduced the CCR ratio definition which generalized the single-output to single-input classical engineering-science ratio definition to multiple outputs and inputs without requiring preassigned weights. This is done via the extremal principle incorporated in the following model:

$$
\begin{aligned}
(CCR_{FP}) \quad \mathbf{max} \quad & h_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\
\mathbf{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leqslant 1, \quad j = 1, 2, \ldots, n \\
& u = (u_1, u_2, \ldots, u_s)^T \geqslant 0 \\
& v = (v_1, v_2, \ldots, v_m)^T \geqslant 0
\end{aligned} \tag{1}
$$

where the $y_{rj}, x_{ij} > 0$ represent output and input data for DMU$_j$, an optimal $h_o^* = max$ h_o will always satisfy $0 \leq h_o^* \leq 1$ with optimal solution values $u_r^*, v_i^* > 0$.

The above fractional program (1) can be replaced by a linear program. The dual model is always used in realistic applications, which is expressed with a real variable θ and a nonnegative vector $\lambda = (\lambda_1, \ldots, \lambda_n)^{\mathbf{T}}$ of variables as follows [2]:

$$
\begin{aligned}
(CCR_{DLP}) \quad \mathbf{min} \quad & \theta \\
\mathbf{s.t.} \quad & X\lambda \leqslant \theta x_o \\
& Y\lambda \geqslant y_o \\
& \lambda \geqslant 0
\end{aligned} \tag{2}
$$

2.2 Common Weights Analysis (CWA)

From CCR ratio model, different inputs and outputs weights are allowed for evaluating different DMUs. But efficiency assessments in real applications often demand a general view of the relative importance of inputs and outputs by using same inputs and outputs weights. By using DEA method, we can get a category of efficient DMUs (eDMUs), but DEA can not provide enough information to rank the eDMUs. So if one further wants to understand which DMU the best is, he/she needs another indicator to discriminate among the eDMUs, the common weights is just such indicator.

Using DEA method to solve the common weights, the idea is first proposed by cook et al.(1990) [3], Andersen and Petersen (1993) [4] developed procedures for ranking only the efficient units in the DEA, Liu and Peng (2006) [5] proposed common weights analysis (CWA) model to rank DMUs in the category of efficient. CWA determines an implicit datum under the assumption that the maximum efficiency is equal to 1 among the eDMUs. The CWA model is as follows:

$$\begin{aligned}
\textbf{min} \quad & \sum_{i\in E}(|\Delta_I^i|+|\Delta_O^i|) \\
\textbf{s.t.} \quad & \frac{\sum_{j=1}^{s} u_j y_{ij}+\Delta_O^i}{\sum_{j=1}^{m} v_j x_{ij}+\Delta_I^i}=1, \quad i\in E \\
& u_i>0, \quad i=1,2,\ldots,s \\
& v_i>0, \quad i=1,2,\ldots,m \\
& \Delta_O^i,\Delta_I^i \quad free
\end{aligned} \tag{3}$$

The optimal solution $weights=(v_1,v_2,\cdots,v_m,u_1,u_2,\cdots,u_s)$ is regarded as the most compromised common weights for all the eDMUs.

3 Description and Analysis of Extra Resource Allocation

The extra resource allocation problem is: suppose there are some extra resource which can be given to all or only a part of DMUs, and if we want the allocation to be most beneficial to the whole system, how the extra resource should be distributed.

This extra resource allocation problem can be generally found in practice. For example, a factory wants to distribute some premium to several outstanding staffs at the end of the year; The chief bank wants to distribute a great deal of bonus to all branch banks; The government wants to serve out food aid to different disaster areas. How should the premium, bonus or food aid be distributed to realize the fair principle and meanwhile make the whole beneficial.

It is complicated to solve this kind of problem, because the selection of DMUs to receive the extra resource should depend on not only its efficiency, but also its scale. Li and Cui (2007) in [6] analyzed this problem and offered the resource allocation algorithm according to three base models: single input and single output, single input and multi-output, multi-input and single output model. The following theorem was proposed in paper [6].

Theorem 1: for the single output and single input case, it is fair and logical to allocate the extra resource that the allocation weights is equivalent to the proportion of the real output of each DMU. That means: suppose there are n DMUs with single output and single input to be allocated resource, the output value of the n DMUs is $(Y_1,\cdots,Y_n)$. Then we can determine the allocation weights W as $\left(\frac{Y_1}{\sum_{i=1}^{n}Y_i},\cdots,\frac{Y_n}{\sum_{i=1}^{n}Y_i}\right)$.

Proof: suppose n DMUs with single input and single output are considered. The input vector is $X=(X_1,\cdots,X_n)$, and the output vector is $Y=(Y_1,\cdots,Y_n)$. The efficiency scores of the DMUs calculated from CCR ratio model (2) are $\theta=(\theta_1,\cdots,\theta_n)$. We will first prove that there is a constant $h>0$ exists that satisfies: for all $i\in 1,2,\ldots,n$, $h\theta_i X_i=Y_i$.

For DMU$_j$, $\forall j = 1,2,\ldots,n$, the constraints of its related CCR model (1) are:

$$\frac{u_j Y_i}{v_j X_i} \leq 1, \quad i = 1,2,\ldots,n, \quad v_j > 0, u_j > 0$$

Then we have that $\frac{u_j}{v_j} \leq min\{\frac{X_1}{Y_1},\cdots,\frac{X_n}{Y_n}\}$. Let $\frac{X_l}{Y_l} = min\{\frac{X_1}{Y_1},\cdots,\frac{X_n}{Y_n}\}$, then $\frac{u_j}{v_j} \leq \frac{X_l}{Y_l}$ goes. As the X_l and Y_l are known constants, so we know that $max\ \frac{u_j}{v_j} = \frac{X_l}{Y_l}$, $\forall j = 1,\ldots,n$. The object function of CCR model (1) is $max\frac{u_j Y_j}{v_j X_j}$, as $max\ \frac{u_j}{v_j} = \frac{X_l}{Y_l}$ and the X_j and Y_j are known constants, so $max\frac{u_j Y_j}{v_j X_j} = \frac{X_l Y_j}{Y_l X_j} = \theta_j$, then we get that $Y_j = \frac{Y_l}{X_l}\theta_j X_j$, let $h = \frac{Y_l}{X_l}$, so we get that for all $i \in 1,2,\ldots,n$, $h\theta_i X_i = Y_i$.

For DMU$_j$, since $Y_j = h\theta_i X_i$ and $u_j Y_j = v_j \theta_j X_j$, it is obvious that doing allocation according to Y_j or $u_j Y_j$ both make sense because they both take into consideration the efficiency and the scale of DMU$_j$. But since u_j is a variable, given a random positive number $k > 0$, it also goes that $ku_j Y_j = kv_j \theta_j X_j$. $ku_j Y_j$ also can reflects the efficiency and the scale of DMU$_j$ as same as $u_j Y_j$ and Y_j. So we get the conclusion that $ku_j Y_j$ and $u_j Y_j$ do not reflect the DMU$_j$'s scale in a logical way. Otherwise, the Y_j is a constant and reflects the DMU$_j$'s real scale. So the allocation according to real output of the DMUs is a fair and logical method with concerning both the efficiency and the scale factors. □

We accomplish the proof of Theorem 1 on the assumption that the extra resource allocation is fair and logical when it concerns both the efficiency and scale of all the DMUs.

4 Counter Example

Section 3 gives a description of the single input and single output DMU case, let us extent it to the multi-output and multi-input DMU case which is more general in practice. Suppose there are n DMUs, for each DMU, say DMU$_j$, the given values on the indices are denoted as $(x_{1j}, x_{2j}, \ldots, x_{mj})$ and $(y_{1j}, y_{2j}, \ldots, y_{sj})$, respectively. We can measure the efficiency of DMU$_j$ by θ_j which is calculated by model (2), and we can get a set of weights vector $w_i = (v_{1j},\cdots,v_{mj}; u_{1j},\cdots,u_{sj})$, under which DMU$_j$ can reach its optimal efficiency score. So the following formulation goes:

$$u_{1j}y_{1j} + u_{2j}y_{2j} + \cdots + u_{sj}y_{sj} = \theta_j(v_{1j}x_{1j} + v_{2j}x_{2j} + \cdots + v_{mj}x_{mj})$$

Let $\overline{Y}_j$ denotes the virtual output of DMU$_j$, where $\overline{Y}_j = u_{1j}y_{1j} + u_{2j}y_{2j} + \cdots + u_{sj}y_{sj}$, and $\overline{X}_j$ denotes the virtual input of DMU$_j$, where $\overline{X}_j = v_{1j}x_{1j} + v_{2j}x_{2j} + \cdots + v_{mj}x_{mj}$. In the same way, other DMU's virtual output and virtual input can be calculated. So we transform the multiple outputs and multiple inputs problem into the single input and single output case. It is natural to get the allocation weights by Theorem 1 as following formulation:

$$\overline{W} = \left(\frac{\overline{Y}_1}{\sum_{i=1}^n \overline{Y}_i},\cdots,\frac{\overline{Y}_n}{\sum_{i=1}^n \overline{Y}_i}\right)$$

.

Let $\overline{w}_j$ denotes the allocation weights of DMU$_j$. Given random positive number $k>1$, it is allowed that we multiply $u_{1j}, u_{2j}, \cdots, u_{sj}$ and $v_{1j}, \cdots, v_{mj}$ with constant k to get $\widehat{u}_{1j}, \cdots, \widehat{u}_{sj}$ and $\widehat{v}_{1j}, \cdots, \widehat{v}_{mj}$. So, the new optimal weights of DMU$_j$ is $\widehat{w}_j = k\overline{w}_j$ and also satisfies $\widehat{u}_{1j}y_{1j} + \cdots + \widehat{u}_{sj}y_{sj} = \theta_j(\widehat{v}_{1j}x_{1j} + \cdots + \widehat{v}_{mj}x_{mj})$, which mean that the new set of weights $\widehat{w}_j$ is also optimal for DMU$_j$. Then we can get $\widehat{Y}_j = k\overline{Y}_j$, and assume that the other DMUs' virtual output is not changed. As the virtual output of DMU$_j$ is greater than before, the allocation weights $\frac{\widehat{Y}_j}{\sum_{i=1}^{n}\widehat{Y}_i}$ of DMU$_j$ will be greater than $\frac{\overline{Y}_j}{\sum_{i=1}^{n}\overline{Y}_i}$.

Because we can randomly set the number k, so the above allocation weights lacks of a uniform solution. To solve this problem, we need find a uniform multiplier to measure all the DMUs' scale on one coordination.

To solve this problem, let us consider the case in which per unit values of the various outputs are known. These values could be income, profit or a weight assigned to each particular output. Let u_r be the value or weight for per unit of output r, and assume $u_r > 0, r = 1, \ldots, s$. We can compute DMU$_j$'s virtual output as $Y_j = u_1y_{1j} + \cdots + u_sy_{sj}$, and let the $W_j = \frac{Y_j}{\sum_{i=1}^{n}Y_i}$ denotes the allocation weight of DMU$_j$. With this common weights as a uniform multiplier, we can get an approach to solve the extra resource allocation problem.

5 θ-CWA — An Improved Model of CWA

As we focus on the allocation process, assume that we have known the DMUs set which will get a part of the extra resource. Without loss of generality, we assume that all the DMUs are going to be allocated some extra resource. From the analysis above, we need first calculate a set of common weights for the whole system to solve this allocation problem.

Model (3) has provided a method to calculate the common weights, but it has a limitation that it is only used for efficient DMUs (eDMUs). In the conclusion part in paper [5], a future research issue is addressed that how to extent the full ranking to all the DEA-iDMUs. As all the DMUs should be considered, it is necessary to get a full ranking for all the DMUs. Kao and Hung [7] proposed a compromise solution approach in 2005. By combining the idea of compromise solution with CWA, we propose the following θ-CWA model.

$$
\begin{aligned}
(\theta - CWA) \quad \mathbf{min} \quad & \sum_{i=1}^{n}(|\Delta_I^i| + |\Delta_O^i|) \\
\mathbf{s.t.} \quad & \frac{\sum\limits_{j=1}^{s} u_j y_{ij} + \Delta_O^i}{\sum\limits_{j=1}^{m} v_j x_{ij} + \Delta_I^i} = \theta_i, \quad i = 1,2,\ldots,n \\
& u_i > 0, \quad i = 1,2,\ldots,s \\
& v_i > 0, \quad i = 1,2,\ldots,m \\
& \Delta_O^i, \Delta_I^i \quad free
\end{aligned} \tag{4}
$$

The above θ-CWA model can be transformed into the linear programming as follows:

$$
\begin{aligned}
(\theta - CWA_{LP}) \quad \mathbf{min} \quad & \sum_{i=1}^{n}(\Delta_I^{i2} + \Delta_O^{i2}) \\
\mathbf{s.t.} \quad & \sum_{j=1}^{s} u_j y_{ij} + \Delta_O^i - \theta_i \sum_{j=1}^{m} v_j x_{ij} - \theta_i \Delta_I^i = 0, \quad i = 1,2,\ldots,n \\
& u_i > 0, \quad i = 1,2,\ldots,s \\
& v_i > 0, \quad i = 1,2,\ldots,m \\
& \Delta_O^i, \Delta_I^i \quad free
\end{aligned} \tag{5}
$$

By solving the above linear programming formulation, we get a compromise common weights $cw = (v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_s)$ for full ranking. Based on this set of common weights, all DMUs are compared on one scale. This set of weights are different with other evaluation weights. Firstly, because this set of common weights dependent on the efficiency score of all the DMUs. As we all known, the efficiency score computed by CCR model reaches the optimal relative efficiency of the DMU, so the common weights make all the DMUs arrive their optimal efficiency in a compromise way; Secondly, this set of weights are changing with the recently standings' change of all the DMUs, it is not invariable. That means the advantage aspect in last evaluation process may not still increase the DMU's score in new evaluation this time.

6 Extra Resource Allocation Algorithm

Combined the above θ-CWA model with the analyzed method in chapter 3, we present the algorithm for extra resource allocation problem as follows.

Algorithm 1 (Extra resource allocation algorithm):

Given n DMUs and the inputs X and outputs Y, and the $cw = (v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_s)$ computed by θ-CWA model.

Step 1: for j from 1 to n, compute $Y_j = \sum_{i=1}^{s} u_i y_{ij}$;
Step 2: compute $Y_\Sigma = \sum_{i=1}^{n} Y_i$;
Step 3: for j from 1 to n, compute $W_j = \frac{Y_j}{Y_\Sigma}$;
Output $W = (W_1, W_2, \cdots, W_n)$.

The extra resource allocation calculated by algorithm 1 is a fair allocation because it not only considers the efficiency score but also considers the scale aspect.

Corollary 2: The extra resource allocation solution calculated by Algorithm 1 is the most compromise solution.

Proof: Algorithm 1 gives us an approach to solve the extra resource allocation problem. The solution is unique one that reflects the relative efficiency scores of the DMUs and the scales. Since the common weights computed from θ-CWA model (5) allowing a relaxation with the efficiency scores which are regarded as the exact efficiency scores of all the DMUs, and model (5) tries to smallest the errors caused by the relaxation, the solution is called the most compromise common weights [7]. From the analysis in section 4, the exact solution to the extra resource allocation problem is hard to get. As Algorithm 1 utilizes the most compromise common weights, the solution to the extra resource allocation problem is regarded as the most compromise solution. □

7 Conclusion

Although extra resource allocation problem based on DEA method is a new problem proposed, it can be often seen in practice. Adopting common weights analysis method, we propose a scheme and its related algorithm to obtain a most compromise solution to the extra resource allocation problem. Since in multiple inputs and multiple outputs DMU case, the exact solution may not exist, so the solution we proposed in this paper is a feasible, and from the most compromise sense, the best answer can be found to the problem. The principle that the allocation should concerns with both the efficiency and the scale of all the DMUs, is a broad-accepted principle in real management applications. This study extends the research range of the resource allocation problem and the full ranking with common weights based on DEA method.

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