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A New T-F Function Theory and Algorithm for Nonlinear Integer Programming*

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Abstract We introduce a new T-F function for solving discrete general minimization problems with a general function over box-constrained domain. A T-F function is constructed at a local minimizer of the objective function such that it achieves local maximum at the current solution. Moreover, a local minimizer of the T-F function leads to a new solution to the original problem with lower objective function value. Iteration follows in this manner to reach a global minimizer. Promising computational results are included and show the efficiency of the T-F function method.

Keywords Nonlinear integer programming; global optimization; T-F function method; local minimizer; global minimizer.

1. Introduction

Consider the following discrete global minimization problem:

$$(P) \quad \min_{x\in X} \ f(x),$$

where $f : Z^n \to X \subset Z^n$, and Z^n is the set of integer points in R^n. This problem is important since we can find a variety of practical problems in which discrete global minimizer has to be obtained. The difficulties for searching a global optimum present at two sides: how to leave from a local minimizer to a smaller one and how to judge the current minimizer is a global one.

During the past three or four decades, many theories and algorithms in discrete global optimization have been developed(see [1, 2, 4, 5, 6, 7]). Among these methods, a well-known and practical methods for discrete global optimization is discrete filled function method introduced by [3] and farther developed by [8],[9],[10],[11],[12] to overcome the first difficulty.

In this paper, we propose a new T-F function method for solving discrete minimization problems with a general function over box-constrained domain. The method

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iterates from one local minimum to a better one. In each iteration, we construct a T-F function that attains strict local maximum at the current solution. A local minimizer of the T-F function leads to a new solution of reduced objective function value. Iteration follows in this manner to reach a global minimizer. Some promising computational are reported.

We present some basic knowledge of discrete global optimization in Section 2 and we introduce the new T-F function and study its properties for discrete global minimization problems in Section 3. Then we propose a solution method that uses the T-F function to find a discrete global minimizer in Section 4. Following that we report computational results on the proposed method in Section 5. Concluding remarks are given in the last section.

2. Preliminaries

Considering problem P, in order to introduce the concept of T-F function method, we first recall some definitions involved in nonlinear integer programming problem and then define the discrete tunnel function and T-F function.

Definition 2..1.
The set of all ***axial directions*** *in Z^n is defined by $D = \{\pm e_i : i = 1,2,\ldots,n\}$, where e_i is the ith unit vector (the n dimensional vector with the ith component equal to one and all other components equal to zero).*

Definition 2..2.
The set of all ***feasible directions*** *at $x \in X$ is defined by $D_x = \{d \in D \ : \ x+d \in X\}$, where D is the set of axial directions.*

Definition 2..3.
For any $x \in Z^n$, the ***discrete neighborhood*** *of x is defined by $N(x) = \{x, x \pm e_i, i = 1,2,\ldots,n\}$.*

Definition 2..4.
A point $x^* \in X$ is called a ***discrete local minimizer*** *of $f(x)$ over X if $f(x^*) \leq f(x)$, for all $x \in X \cap N(x^*)$. Furthermore, if $f(x^*) \leq f(x)$, for all $x \in X$, then x^* is called a* ***discrete global minimizer*** *of $f(x)$ over X. If, in addition, $f(x^*) < f(x)$, for all $x \in X \cap N(x^*) \setminus \{x^*\}$ ($x \in X$), then x^* is called a* ***strict discrete local (global) minimizer*** *of $f(x)$ over X.*

Definition 2..5.
For any $x \in X, d \in D$ is said to be a ***discrete descent direction*** *of $f(x)$ at x over X if $x+d \in X$ and $f(x+d) < f(x)$.*

Algorithm 2..1.
(Discrete local minimization method)
1. *Start from an initial point $x \in X$.*
2. *If x is a local minimizer of f over X, then stop; Otherwise, let*

$$d^* := arg \min_{d_i \in D_x} \{f(x+d_i) : f(x+d_i) < f(x)\}.$$

3. Let $x := x + d^*$, go to Step 2.

Definition 2..6.
$P(x,x^*)$ is called a discrete filled function of $f(x)$ at a discrete local minimizer x^* if $P(x,x^*)$ has the following properties:

1. *x^* is a strict discrete local maximizer of $P(x,x^*)$ over X.*
2. *$P(x,x^*)$ has no discrete local minimizers in the region $S_1 = \{x \mid f(x) \geq f(x^*),\, x \in X \backslash x^*\}$.*
3. *If x^* is not a discrete global minimizer of $f(x)$, then $P(x,x^*)$ does have a discrete minimizer in the region $S_2 = \{x \mid f(x) < f(x^*),\, x \in X\}$.*

Definition 2..7.
$P(x,x^*)$ is called a discrete tunnel function of $f(x)$ at a discrete local minimizer x^* if, for any $x^0 \in X$ with $r > 0$, $P(x,x^*) = 0$ if and only if $f(x^0) - f(x^*) + r \leq 0$.

Definition 2..8.
$P(x,x^*)$ is called a T-F function of $f(x)$ at a discrete local minimizer x^* if it is both a discrete tunnel function and a discrete filled function.

3. A new Discrete T-F function

Throughout this paper, we suppose the following assumption holds:

Assumption 3..1.
For any $x \in Z^n \backslash X$, $f(x) = +\infty$. This implies: $\min_{x \in Z^n} f(x) = \min_{x \in X} f(x)$.

Assumption 3..2.
$f(x)$ satisfies the following Lipschitz condition for every $x,\, y \in X$: $\mid f(x) - f(y) \mid \leq L\|x - y\|$, where $0 < L < +\infty$ is a constant.

Before defining the T-F function, for any $r > 0$, we first introduce a function $h_r(t)$ as follows:

$$h_r(t) = \begin{cases} 1, & t \geq 1 + r \\ \phi_r(t), & 1 \leq t \leq 1 + r \\ 0, & t \leq 1 \end{cases} \tag{1}$$

where $\phi_r(t)$ satisfies the following conditions:

For $1 < t < 1 + r$, $1 \geq \phi_r(t) \geq 0$. $\phi_r(1) = 0$, $\phi_r(r+1) = 1$.

For problem P, we define a function $T(x,x^*,r,q)$ of f at given $x^* \in X$ with $r > 0$ and $q > 0$ as follows

$$T(x,x^*,r,q) = \frac{1}{1 + \|x - x^*\|} h_r(f(x) - f(x^*) + r + 1) + q \max(0, f(x) - f(x^*)), \tag{2}$$

where $0 < r < \min_{f(x_1) \neq f(x_2),\, x_1,x_2 \in X} |f(x_1) - f(x_2)|$.

It is easy to see that $T(x,x^*,r,q)$ is tunnel function. Moreover, the following theorems show that it is also a filled function, so this function is a T-F function.

Theorem 3..1.
If x^* is a local minimizer of $f(x)$ and $0 < q < \frac{1}{2L}$, then x^* is a strictly local maximizer of $F(x, x^*, r, q)$.

Proof. Since x^* is a local minimizer of $f(x)$, for any $x^* + d \in N(x^*) \cap X$, we have $f(x^* + d) \geq f(x^*)$ and $f(x^* + d) - f(x^*) + r + 1 \geq r + 1$, where $d \in D$. Therefore, $T(x^* + d, x^*, r, q) = \frac{1}{1+\|x^* - x^* + d\|} + q[f(x^* + d) - f(x^*)] \leq 0.5 + qL < 1 = T(x^*, x^*, r, q)$. This show x^* is a strictly local maximizer of $T(x^* + d, x^*, r, q)$. ■

Lemma 3..1.
For every $x' \in X$, there exists $d \in D$ such that $\|x' + d - x^*\| > \|x' - x^*\|$.

Proof: If there is an $i \in \{1, 2, \ldots, n\}$ such that $[x']_i \geq [x^*]_i$, where $[x]_i$ is the ith component of any vector $x \in X$, then $d = e_i$. On the other hand, if there is an $i \in \{1, 2, \ldots, n\}$ such that $[x']_i \leq [x^*]_i$, then $d = -e_i$. ■

Theorem 3..2.
Given that x^* is a discrete local minimizer of $f(x)$, for any $x \in S_1$, if $q < \frac{1}{L(K+1)(K+2)(2K+1)}$, where $K = \max_{x_1, x_2 \in X} \|x_1 - x_2\|$, then x is not a discrete local minimizer of $F(x, x^*, r, q)$.

Proof: For any $x \in S_1$, by Lemma 3..1, there exists a direction $d \in D$ such that $x + d \in X$ and $\|x + d - x^*\| > \|x - x^*\|$. For the above d, consider the following four cases:

Case i: If $f(x + d) \geq f(x) \geq f(x^*)$.

Since $\|x - x^*\| \leq K, \|x - x^* + d\| \leq K + 1, \|x - x^* + d\|^2 - \|x - x^*\|^2 \geq 1$, we have

$$
\begin{aligned}
& \frac{\|x - x^* + d\| - \|x - x^*\|}{(1 + \|x - x^* + d\|)(1 + \|x - x^*\|)} \\
= & \frac{\|x - x^* + d\|^2 - \|x - x^*\|^2}{(1 + \|x - x^* + d\|)(1 + \|x - x^*\|)(\|x - x^* + d\| + \|x - x^*\|)} \\
\geq & \frac{1}{(K+1)(K+2)(2K+1)},
\end{aligned}
$$

therefore,

$$
\begin{aligned}
& T(x + d, x^*, r, q) - T(x, x^*, r, q) \\
= & \frac{1}{1 + \|x - x^* + d\|} - \frac{1}{1 + \|x - x^*\|} + q[f(x + d) - f(x)] \\
\leq & -\frac{\|x - x^* + d\| - \|x - x^*\|}{(1 + \|x - x^* + d\|)(1 + \|x - x^*\|)} + q[f(x + d) - f(x)] \\
\leq & -\frac{1}{(K+1)(K+2)(2K+1)} + qL < 0.
\end{aligned}
$$

that is $T(x + d, x^*, r, q) < T(x, x^*, r, q)$.

Case ii: If $f(x) > f(x+d) \geq f(x^*)$.

$$\begin{aligned} & T(x+d,x^*,r,q) - T(x,x^*,r,q) \\ = & \frac{1}{1+\|x-x^*+d\|} - \frac{1}{1+\|x-x^*\|} + q[f(x+d)-f(x)] < 0. \end{aligned}$$

that is $T(x+d,x^*,r,q) < T(x,x^*,r,q)$.

Case iii: If $f(x) \geq f(x^*) > f(x+d) > f(x^*) - r$.

Since $f(x+d) - f(x^*) + r + 1 > 1$, we have

$$\begin{aligned} T(x+d,x^*r,q) &= \frac{1}{1+\|x-x^*+d\|} h_r(f(x+d)-f(x^*)+r+1) \\ &\leq \frac{1}{1+\|x-x^*+d\|} \\ &< \frac{1}{1+\|x-x^*\|} + q[f(x)-f(x^*)] = T(x,x^*,r,q). \end{aligned}$$

that is $T(x+d,x^*,r,q) < T(x,x^*,r,q)$.

Case iv: If $f(x) \geq f(x^*) > f(x+d)$ and $f(x+d) - f(x^*) + r \leq 0$.

In this case $T(x+d,x^*r,q) = 0 < \frac{1}{1+\|x-x^*\|} \leq T(x,x^*r,q)$.

The above four cases imply that any $x \in S_1$ is not a discrete local minimizer of $F(x,x^*,r,q)$ when $q > 0$ is small enough. ■

Theorem 3..3.
If x^* is a discrete local minimizer of $f(x)$ over X but not a discrete global minimizer of $f(x)$ in X, then there exists a discrete minimizer x_1^* of $F(x,x^*,r,q)$ in the region $S_2 = \{x|f(x) < f(x^*), x \in X\}$.

Proof. Let x' be a global minimizer of (P), by the condition, we have $f(x') - f(x^*) + r + 1 < 1$, thus $F(x',x^*,r,q) = 0$.

On the other hand, by the definition of $h_r(t)$, for any $x \in X$,we have $T(x,x^*,r,q) \geq 0$.

This shows x' is a minimizer of $T(x,x^*,r,q)$. ■

4. Numerical implementation and solution algorithm

Assume $x^* \in X$ is a local minimizer of $f(x)$. Based on the properties of the T-F function proposed in the previous section, we propose the following algorithm to find a global minimizer of problem P:

Algorithm 2 (Discrete T-F function method)

1. Input the lower bound of r, namely $r_L = 1e-8$. Input an initial point $x_0^{(0)} \in X$. Let $D = \{\pm e_i, i = 1, 2, \cdots, n\}$.
2. Starting from an initial point $x_0^{(0)} \in X$, minimize $f(x)$ and obtain the first local minimizer x_0^* of $f(x)$. Set $k := 0$, $q = 1$, and $r = 1$.

3. Set $x_k^{(0)i} = x_k^* + d_i$, $d_i \in D$,$i = 1,2,\cdots,2n$, $J = [1,2,\cdots,2n]$ and $j = 1$.
4. Set $i = J_j$ and $x = x_k^{(0)i}$.
5. If $f(x) < f(x_k^*)$, then use x as initial point for discrete local minimization method to find x_{k+1}^* such that $f(x_{k+1}^*) < f(x_k^*)$. Set $k = k+1$, go to step3.
6. Let $D_0 = \{d \in D : x+d \in X\}$. If there exists $d \in D_0$ such that $f(x+d) < f(x_k^*)$, then use $x+d^*$, where $d^* = arg\min_{d\in D_0}\{f(x+d)\}$, as an initial point for a discrete local minimization method to find x_{k+1}^* such that $f(x_{k+1}^*) < f(x_k^*)$. set $k := k+1$ and go to 3.
7. Let $D_1 = \{d \in D_0 : \|x+d-x^*\| > \|x-x^*\|\}$. If $D_1 = \emptyset$ then go to step 10.
8. If there exists $d \in D_1$ such that $T(x+d,x_k^*,r,q) \geq T(x,x_k^*,r,q)$, then set $q = 0.1q$, $J = [J_j,\cdots,J_{2n},J_1,\cdots,Jj-1]$, $j = 1$, go to step 4.
9. Let $D_2 := \{d \in D_1 : f(x+d) < f(x), T(x+d,x_k^*,r,q) < T(x,x_k^*,r,q).\}$ If $D_2 \neq \emptyset$, then set $d^* := arg\min_{d\in D_2}\{f(x+d)+T(x+d,x_k^*,r,q);\}$ Otherwise set $d^* := arg\min_{d\in D_1}\{T(x+d,x_k^*,r,q)\}$. After that, set $x = x+d^*$ and go to step 6.
10. If $i < 2n$, then set $i = i+1$ and go to step 4
11. Set $r = 0.1r$. If $r \geq r_L$, go to step 3; Otherwise, the algorithm is incapable of finding a better minimizer starting from the initial points, $\{x_k^{(0)i} : i = 1,2,\cdots,2n\}$. The algorithm stops and x_k^* is taken as a global minimizer. ■

5. Numerical Experiment

The algorithm in Fortran 95 is successfully used to find the global minimizers of some test problems. Through out the tests, we use the discrete local minimization method as shown in Algorithm 1 to perform local searches. In the following part, several test problems are given and results of the algorithm in solving these problem are reported.

The main iterative results are summarized in tables for each function. The symbols used are shown as follows:

x_k^0 or y_k^0	: The k-th initial point.
k	: The iteration number in finding the k-th local minimizer.
x_k^* or y_k^*	: The k-th local minimizer.
$f(x_k^*)$ or $f(y_k^*)$	: The function value of the k-th local minimizer.
time	: The CPU time in seconds for the algorithm to stop.

Problem 1

$$\begin{aligned} \min \quad & f(x) = 100(x_2 - x_1^2)^2 + (1-x_1)^2 + 90(x_4 - x_3^2)^2 + (1-x_3)^2, \\ & \quad +10.1[(x_2-1)^2 + (x_4-1)^2] + 19.8(x_2-1)(x_4-1), \\ s.t. \quad & 10 \leq x_i \leq 10, \quad x_i \ \ is\ integer, \ \ i = 1,2,3,4. \end{aligned}$$

This problem has $21^4 \approx 1.94 \times 10^5$ feasible points where 41 of them are discrete local minimizers but only one of those discrete local minimizers is the discrete global minimum solution: $x_{global}^* = (1,1,1,1)$ with $f(x_g^*lobal) = 0$. We used five initial points in our experiment: (9,6,5,6), (10,10,10,10), (-10,-10,-10,-10), (-10,10,-10,10), (10,-10,-10,10). For every experiment, the proposed algorithm succeeded in identifying the discrete global minimum. Let $x_1^0 = (9,6,5,6)$, a summary of the computational results are displayed in the following Table 1 and 3.

Table 1.

k	x_k^0	$f(x_k^0)$	x_k^*	$f(x_k^*)$
1	(9, 6, 5, 6)	596070.0	(2, 4, 2, 3)	342.1000
2	(1, 1, 2, 3)	131.4000	(1, 1, 2, 4)	91.90000
3	(1, 1, 0, 1)	91.00000	(1, 1, 1, 1)	0.000000

time=0.1301872

Problem 2

$$\begin{aligned} \min \quad & f(x) = g(x)h(x), \\ s.t. \quad & x_i = 0.001y_i, \ \ -2000 < y_i < 2000, \quad y_i \ \ is\ integer, \ \ i = 1,2, \end{aligned}$$

where

$$\begin{aligned} g(x) &= 1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2), \\ h(x) &= 30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2). \end{aligned}$$

This problem has $4001^2 \approx 1.60 \times 10^7$ feasible points .More precisely, it has 207 and 2 discrete local minimizers in the interior and the boundary of box $-2.00 \leq x_i \leq 2.00, \ \ i = 1,2$, respectively. Nevertheless, it has only one discrete global minimum solution: $x_{global}^* = (0.000, -1.000)$ with $f(x_g^*lobal) = 3$. We used five initial points in our experiment: $(2000,2000)$, $(-2000,-2000)$, $(1196,1156)$, $(-2000,2000)$, $(2000,-2000)$. For every experiment, the proposed algorithm succeeded in identifying the discrete global minimum. the summary of the computational results are displayed in the following Table 2 and 3.

Table 2.

k	y_k^0	$f(y_k^0)$	y_k^*	$f(y_k^*)$
1	(1196, 1156)	1862.019	(1278, 888)	954.1411
2	(1280, 889)	954.1388	(1279, 889.)	954.1375
3	(388, -25)	953.7714	(-600, -400)	30.00000
4	(-271, -720)	29.98818	(0, -1000)	3.000019

time=2.693873

Problem 3

$$\begin{aligned} \min \quad & f(x) = (x_1 - 1)^2 + 5(x_n - 1)^2 + \sum_{i=1}^{n-1} (n-i)(x_i^2 - x_{i+1})^2, \\ s.t. \quad & -5 \leq x_i \leq 5, \ \ x_i \ is\ integer, \ i = 1, 2, \cdots, n \end{aligned}$$

This problem has 11^n feasible points and many local minimizers, but it has only one global minimum solution: $x_{global}^* = (1, \cdots, 1)$ with $f(x_{global}^*) = 0$.

For all problems with different sizes, we used four initial points in our experiment: $(5,\cdots,5)$, $(-5,\cdots,-5)$, $(-5,\cdots,-5,5,\cdots,5)$, $(5,\cdots,5,-5,\cdots,-5)$. For every experiment, the proposed algorithm succeeded in identifying the discrete global minimum. Let $x_1^0=(5,\cdots,5)$, for $n=25,50,100$, respectively, the summary of the computational results are displayed in the Table 3.

Problem 4(Rosenbrock's function)

$$\begin{aligned}\min \quad & f(x)=\sum_{i=1}^{n-1}[100(x_{i+1}-x_i^2)^2+(1-x_i)^2],\\ s.t. \quad & -5\le x_i\le 5,\ \ x_i \textit{ is integer},\ i=1,2,\cdots,n\end{aligned}$$

This problem has 11^n feasible points and many local minimizers, but it has only one global minimum solution: $x^*_{global}=(1,1,\cdots,1)$ with $f(x^*_{global})=0$.

For all problems with different sizes, we used four initial points in our experiment: $(5,\cdots,5)$, $(-5,\cdots,-5)$, $(-5,\cdots,-5,5,\cdots,5)$, $(5,\cdots,5,-5,\cdots,-5)$. For every experiment, the proposed algorithm succeeded in identifying the discrete global minimum. Let $x_1^0=(5,\cdots,5)$, for $n=25,50,100$, respectively,the summary of the computational results are displayed in the Table 3.

Table 3

PN	*DN*	*IN*	*TI*	*FN*	*RA*
1	4	3	0.1301872	85705	0.442
2	2	4	2.693873	2125511	0.133
3	25	2	90.27982	18503950	1.71×10^{-19}
3	50	2	1277.898	148242400	1.27×10^{-44}
3	100	2	6042.128	395363800	2.86×10^{-96}
4	25	2	31.74565	6282030	5.82×10^{-20}
4	50	2	463.3863	49876530	4.26×10^{-45}
4	100	2	7061.204	397503030	2.88×10^{-96}

The symbols used are shown as follows:

PN: The Nth problem.

DN: The dimension of objective function of a problem.

IN: The number of iteration cycles.

TI: The CPU time in seconds for the algorithm to stop.

FN: The number of objective function evaluations for the algorithm to stop.

RA: The ratio of the number of function evaluations for the algorithm to stop to the number of feasible points.

6. Conclusions

In this paper, we have proposed a new T-F function method for solving a class of discrete global minimization problems. The computational results show that this algorithm is quite efficient and reliable. So it may become a new and practical discrete T-F function algorithm for discrete global optimization.

References

[1] O.K. Gupta and A. Ravindran: Branch and bound experiments in convex nonlinear integer programming, Management Science, Vol.31, no.12,pp. 1533-1546, 1985.

[2] F. Körner: A new branching rule for the branch and bound algorithm for solving nonlinear integer programming problems, BIT, Vol. 28, no. 3, pp. 701-708, 1988.

[3] R.P. Ge and C.B.Huang: A Continuous approach to nonlineare integer programming, Applied Mathematics and Computation, Vol. 34, no.1, pp. 39-60, 1989.

[4] D.Li and X.L.Sun: Success grarantee of dual search in integer programming: *p*th power Lagrangian method, Journal of Global Optimization, vol.18, pp,235-254, 2000.

[5] D.Li and D.J.White: *p*th power Lagrangian method for integer programming, Annals of Operations Research, vol. 98, pp.151-170, 2000.

[6] C.Mohan and H.T.Nguyen: A controlled random search technique incorporateing the simulated annealing concept for solving integer and mixed integer global optimization problems, Computational Optimization and Application, vol.14, pp. 103-132, 1999.

[7] X.L.Sun and D.Li: Asymptotic strong duality for bounded integer programming: A logarithmic-exponential dual formulation, Mathematics of Operations Research, vol.25, pp.625-644, 2000.

[8] Y.L.Shang and L.S.Zhang: A filled function method for finding a global minimizer on global integer optimization, J. of Computational and Applied Mathematics, vol.181,pp.200-210, 2005.

[9] W.X.Wang, Y.L.Shang and L.S.Zhang: Two-parameters quasi-filled function algorithm for nonlinear integer programming, Journal of Zhejiang University: Science, (2006), vol.7, pp.2083-2087.2006

[10] Y.J.Yang and Y.M.Liang: A New Discrete Filled Function Algorithm for Discrete Global Optimization, Journal of Computational and Applied Mathematics, vol.202,pp.280-291,2007.

[11] L.S.Zhang and D.Li: Global Search in Nonliear Integer Programming: Filled Function Approach, International Conference on Optimization Techniques and Applications, Perth, pp.446-452,1998.

[12] L.S. Zhang, F. Gao, and W.-X. Zhu: Nonlinear integer programming and global optimization, Journal of Computational Mathematics, vol.2, no.17, pp.179-190, 1999.