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Four Examples of Phase Synchronization in Oscillator Networks*

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Abstract This paper presents four kinds of phase synchronization in oscillator networks by constructing the matrix of coupling strength. Numerical simulations are implemented to verify the results.

1 Introduction:

Considerable attention has been devoted to the different synchronization phenomena [1] in biology. Such as complete synchronization, cluster synchronization, anti-synchronization, and so on[2, 3, 4, 5, 6]. During recent twenty years, phase synchronization has attracted much interest. Many dynamical behaviors in biological systems can be explained by phase synchronization finally[8]. This synchronization is different from the classical concept of synchronization. In the research of phase synchronization, one of the simplest model is Kuramoto model[7].

The Kuramoto model describe the coupling dynamics of a set of n oscillators. The Kuramoto model is given by:

$$\dot{\phi}_i = \omega_i + \sum_{j=1}^{n} c_{ij} sin(\phi_j - \phi_i),\ i = 1, 2, ..., n. \tag{1}$$

where c_{ij} is the coupling strength, a key parameter in the problem and ω_i is natural frequency.

In this article we accommodate the coupling strength c_{ij} to simulate the different phase synchronization including frequency synchronization, complete phase synchronization, cluster phase synchronization and anti-phase synchronization. Finally we approve the feasibility of different phase synchronization using Kuramoto model.

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2 Four kinds of phase synchronization

2.1 Frequency synchronization

First, we study frequency synchronization. For the oscillator network (1), frequency synchronization[4] means that all oscillators rotate with the same frequency i.e.:

$$\lim_{t\to\infty} |\dot{\phi}_i(t) - \dot{\phi}_j(t)| = 0.\ i, j = 1, 2, ..., n.$$

where $\dot{\phi}_i(t)$ is the frequency of ith oscillator.

Here the following example shows that the frequency synchronization can be achieved. Let $n = 8$ in network (1) and the matrix of coupling strength $C = (c_{ij})$ is:

$$C = \begin{pmatrix} 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & \frac{4}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & 0 & \frac{4}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{\sqrt{3}} & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}} & 0 & \frac{4}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{\sqrt{3}} & 0 & \frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 & \frac{4}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{\sqrt{3}} & 0 & \frac{2}{\sqrt{3}} \\ \frac{4}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 \end{pmatrix},$$

and the frequency of the eight oscillators: $w_1 = w_3 = w_5 = w_7 = 3\sqrt{3}, w_2 = w_4 = w_6 = w_8 = \sqrt{3}$,

Then the classical numerical results are carried out using MATLAB in the following figures:

The above figures present that oscillators with different nature frequency can achieve frequency synchronization by constructing the matrix of coupling strength in Kuramoto model.

2.2 Complete phase synchronization

Next we consider complete phase synchronization. Complete phase synchronization means the phase of all oscillators converge to the same. i.e. the following equation holds:

$$\lim_{t\to\infty} |\phi_i(t) - \phi_j(t)| = 0.\ i, j = 1, 2, ..., n.$$

where $\phi_i(t)$ is the phase of ith oscillator.

Here we give the following example to illustrate the existence of complete phase synchronization. Let $n = 8$ in network (1) and the matrix of coupling strength $C =$

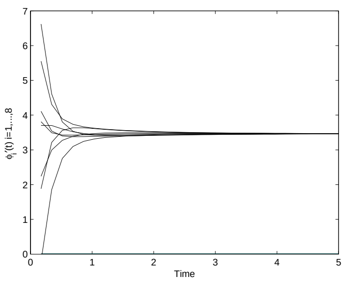

Figure 1: The eight oscillators become frequency synchronization.

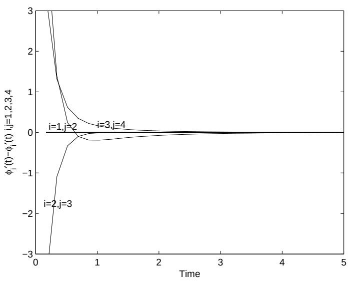

Figure 2: The time evolution of $\dot{\varphi}_1 - \dot{\varphi}_2$, $\dot{\varphi}_2 - \dot{\varphi}_3$,$\dot{\varphi}_3 - \dot{\varphi}_4$

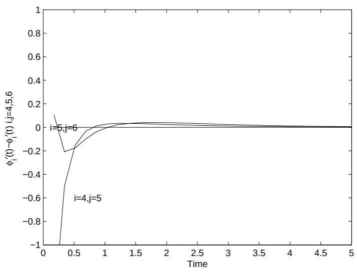

Figure 3: The time evolution of $\dot{\varphi}_4 - \dot{\varphi}_5$, $\dot{\varphi}_5 - \dot{\varphi}_6$,

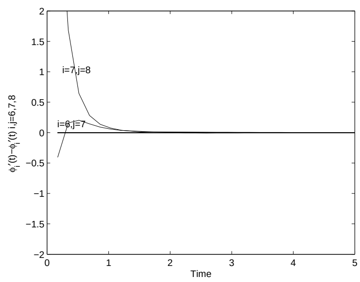

Figure 4: The time evolution of $\dot{\varphi}_6 - \dot{\varphi}_7$, $\dot{\varphi}_7 - \dot{\varphi}_8$,

(c_{ij}) is:

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

here all the oscillators have the same frequency $w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = w_7 = w_8 = 1$.And initial phase array is $[0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6]$

Then we get the following numerical results as shown in Fig5,6,7,8:

We get the similar results for various initial values. So complete phase synchronization can be achieved by constructing the matrix of coupling strength in Kuramoto model.

2.3 Cluster phase synchronization

Further we study cluster phase synchronization[10]. Cluster phase synchronization is defined as a spatiotemporal pattern[9]. In this pattern, all the oscillators are

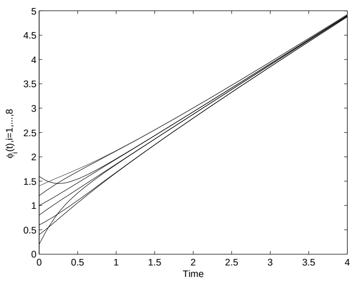

Figure 5: The phase of the eight oscillators being the same

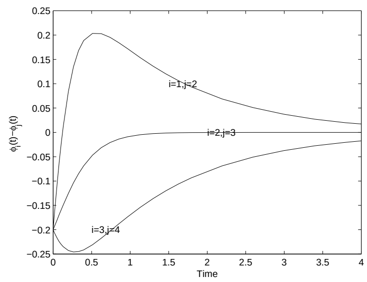

Figure 6: The time evolution of $\varphi_1 - \varphi_2$, $\varphi_2 - \varphi_3, \varphi_3 - \varphi_4$

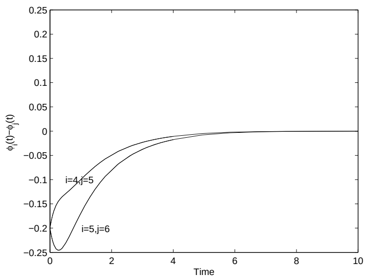

Figure 7: The time evolution of $\varphi_4 - \varphi_5$, $\varphi_5 - \varphi_6$

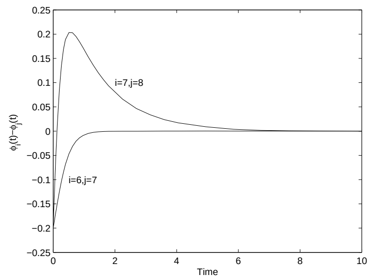

Figure 8: The time evolution of $\varphi_6 - \varphi_7$, $\varphi_7 - \varphi_8$

divided into m clusters where the difference of the phase of the oscillators in the same cluster converge to zero but not in different cluster, i.e,

$$\lim_{t\to\infty} |\phi_i(t) - \phi_j(t)| = 0.\ i \in n_l, j \in n_l, l = 1, \cdots, m,$$

$$\lim_{t\to\infty} |\phi_i(t) - \phi_j(t)| \neq 0.\ i \in n_l, j \in n_k, l, k = 1, \cdots, m, l \neq k,$$

$$n = n_1 + n_2 + \cdots + n_m,$$

where $\phi_i(t)$ is the phase of ith oscillator.

In this paper, we present an example with $n = n_1 + n_2 + n_3 = 8$, where n_1, n_2, n_3 correspondingly represent the size of the each cluster. Let $n_1 = 3, n_2 = 3, n_3 = 2$. The matrix of coupling strength $C = (c_{ij})$ is:

$$C = \begin{pmatrix} 0 & 3 & 0 & \varepsilon_1 & \varepsilon_1 & \varepsilon_1 & \varepsilon_3 & \varepsilon_3 \\ 3 & 0 & 2 & \varepsilon_1 & \varepsilon_1 & \varepsilon_1 & \varepsilon_3 & \varepsilon_3 \\ 0 & 2 & 0 & \varepsilon_1 & \varepsilon_1 & \varepsilon_1 & \varepsilon_3 & \varepsilon_3 \\ \varepsilon_1 & \varepsilon_1 & \varepsilon_1 & 0 & \sqrt{3} & 0 & \varepsilon_2 & \varepsilon_2 \\ \varepsilon_1 & \varepsilon_1 & \varepsilon_1 & \sqrt{3} & 0 & 1 & \varepsilon_2 & \varepsilon_2 \\ \varepsilon_1 & \varepsilon_1 & \varepsilon_1 & 0 & 1 & 0 & \varepsilon_2 & \varepsilon_2 \\ \varepsilon_3 & \varepsilon_3 & \varepsilon_3 & \varepsilon_2 & \varepsilon_2 & \varepsilon_2 & 0 & 1 \\ \varepsilon_3 & \varepsilon_3 & \varepsilon_3 & \varepsilon_2 & \varepsilon_2 & \varepsilon_2 & 1 & 0 \end{pmatrix},$$

$\varepsilon_1 = \frac{1}{32}, \varepsilon_2 = \frac{1}{34}, \varepsilon_3 = \frac{1}{35}$, the frequency: $w_1 = w_2 = w_3 = \frac{3}{32} + \frac{3\sqrt{3}}{35}, w_4 = w_5 = w_6 = \frac{1}{34} + \frac{2\sqrt{3}}{35}, w_7 = w_8 = \frac{\sqrt{3}}{70}$. The initial phase array is $\phi_0 = [0.5, 1, 3, 4, 2, 1.5, 3, 3.5]$

The clustering phase synchronization can be shown clearly, as presented in Fig.9,10,11,12.:

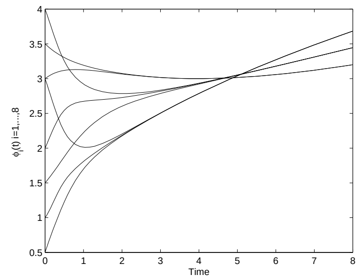

Figure 9: The eight oscillators become cluster synchronization

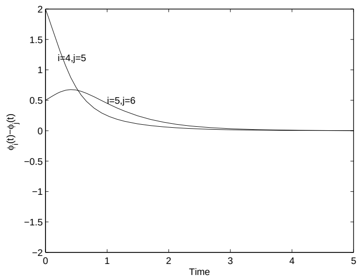

Figure 10: The time evolution of $\varphi_1 - \varphi_2$, $\varphi_2 - \varphi_3$. They belong to the first cluster.

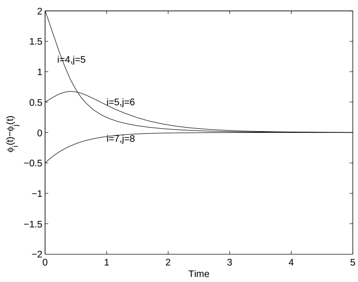

Figure 11: The time evolution of $\varphi_4 - \varphi_5$, $\varphi_5 - \varphi_6$in the second cluster, $\varphi_7 - \varphi_8$ in the third cluster

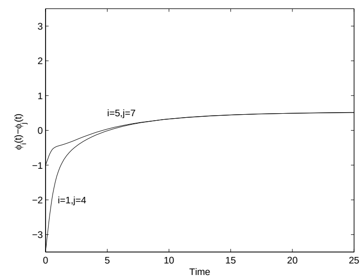

Figure 12: The time evolution of $\varphi_1 - \varphi_4, \varphi_5 - \varphi_7$

We get the similar results for various initial values. Clustering phase synchronization can be achieved by constructing the matrix of coupling strength in Kuramoto model.

2.4 Anti-phase synchronization

Finally, we consider anti-phase synchronization[11, 12]. Anti-synchronization is a phenomenon that the state vectors of synchronized systems have the same absolute values but opposite signs. We say that anti-synchronization of two systems S_1 and S_2 is achieved if the following equation holds:

$$\lim_{t\to\infty} ||\phi_1(t) + \phi_2(t)|| = 0.$$

where $\phi_1(t), \phi_2(t)$ are the state vectors of the system S_1, S_2. In this paper, all the oscillators in the network are divided into two clusters. Complete phase synchronization is reached in each cluster and the phase of the oscillators in different clusters fulfil the above equation. Because the oscillators in the two clusters have the same 2π cycle, the above equation equals the following:

$$\lim_{t\to\infty} |\phi_1(t) - \phi_2(t)| = \pi.$$

where $\phi_i(t)$ is the phase of ith oscillator. The following example proves the existence of this anti-phase synchronization. Let $n = 7, n_1 = 3, n_2 = 4$ and the the matrix of coupling strength is:

$$C = \begin{pmatrix} 0 & \sqrt{3}-1 & 0 & -2 & 0 & 0 & 0 \\ \sqrt{3}-1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 2 & 0 & -3 \\ -2 & -2 & -5 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 2 & 0 & 0 & \frac{2}{3} & 6 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & -3 & 0 & 6 & 0 & 0 \end{pmatrix},$$

and the frequency of each oscillator: $w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = w_7 = w_8 = 1$,The initial phase array is $\phi_0 = [1, 2, 3, 2.53, 3.5, 2, 4]$

Then we get the following numerical results:

We get the similar results for various initial values .The above figures presents that anti-phase synchronization can be achieved by constructing the matrix of coupling strength in Kuramoto model.

3 Conclusion

By means of studying the phase synchronization of classical Kuramoto model through some examples, we find that changing the matrix of coupling strength result in different kinds of phase synchronization. We confirm it possible that the existence of different synchronization states in nature.

Our study barely stay on examples. The further theoretical analysis will be provided in addition.

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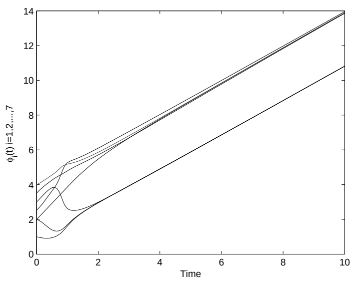

Figure 13: The oscillators become anti-phase synchronization

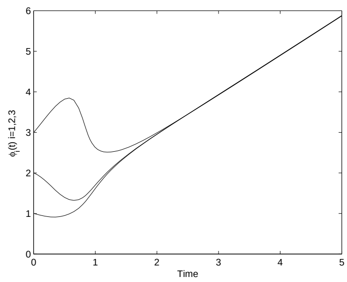

Figure 14: The time evolution of φ_1 , φ_2 and φ_3

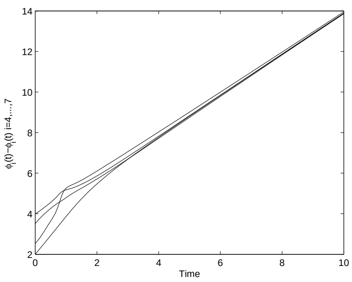

Figure 15: The time evolution of φ_1, φ_2, φ_3 and φ_4 being the same

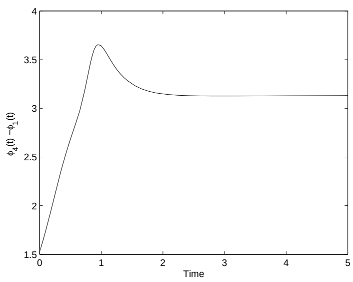

Figure 16: The time evolution of $\varphi_1 - \varphi_4$. They become anti-phase synchronization

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