The Sixth International Symposium on Operations Research and Its Applications (ISORA'06)
Xinjiang, China, August 8–12, 2006

Some Results on Fractional Covered Graphs*

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Abstract Let $G = (V(G), E(G))$ be a graph. and let g, f be two integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for all $x \in V(G)$. G is called fractional (g, f)-covered if each edge e of G belongs to a fractional (g, f)-factor G_h such that $h(e) = 1$, where h is the indicator function of G_h. In this paper, sufficient conditions related to toughness and isolated toughness for a graph to be fractional 1-covered, 2-covered are given. The results are proved to be best possible in some sense. In particular, a necessary and sufficient condition for a (g, f)-factor covering a given k-matching is obtained.

Keywords fractional (g, f)-factor; fractional perfect matching; fractional covered

1 Introduction

In this paper, we consider fractional covered graphs, a class of graphs which consists of some special fractional factors. Fractional factor theory has extensive applications in some areas such as network design, combinatorial topology, decision lists and so on. For example, in the communication networks, if we permit that large date packages can be partitioned into some parts to send to some different destinations by different channels, then the running efficiency of networks will be greatly improved. Feasible and efficient assignment for date package can be viewed as a problem to find a fractional factors satisfying some special conditions. Some results for the existence of fractional factors have been obtained [7, 10-12].

Our terminology and notation are standard. Readers are referred to [2] and [9] for undefined terms. All graphs considered in this paper will be finite simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, The degree of x in G is denoted by $d_G(x)$. $\lambda(G)$ and $\kappa(G)$ denote the edge connectivity and connectivity of G, respectively. $\delta(G)$ denotes the minimal degree of G. If S is a subset of $V(G)$, $G[S]$ denotes the induced subgraph by S. The set of isolated vertices of $G \setminus S$ is denoted by $I(G \setminus S)$ and $|I(G \setminus S)| = i(G \setminus S)$. For two disjoint subsets S, T of $V(G)$, $E_G(S, T)$ denotes the set of edges with one vertex in S and another in T and $|E_G(S, T)| = e_G(S, T)$.

*The work is supported by NNSF (10471078) of China and RFDP (20040422004) of Higher Education, Promotional Foundation (2005BS01016) for Excellent Middle-aged or Young Scientists of Shandong Province and DRF of QFNU.

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Let g and f be two integer-valued functions such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. A (g,f)-factor F of G is a spanning subgraph of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. A fractional (g,f)-indicator function is a function h that assigns to each edge of graph G a number in $[0,1]$, so that for each vertex $x \in V(G)$ we have $g(x) \leq d_{G_h}(x) \leq f(x)$, where $d_{G_h}(x) = \sum_{e\in E_x} h(e)$ is the fractional degree of $x \in V(G)$, with $E_x = \{e : e = xy \in E(G)\}$. Let h be a fractional (g,f)-indicator function of a graph G. Set $E_h = \{e : e \in E(G) \text{ and } h(e) \neq 0\}$. If G_h is a spanning subgraph of G such that $E(G_h) = E_h$, then G_h is called a fractional (g,f)-factor of G. If $g(x) = f(x) = k$ (k is a non-negative integer) for all $x \in V(G)$, the a fractional (g,f)-factor is called a fractional k-factor. In particular, a fractional 1-factor is also called a fractional perfect matching.

(g,f)-covered graph was introduced in [6]. A graph is called (g,f)-covered if for each edge of G there is a (g,f)-factor containing it. Li,Yan and Zhang introduced fractional (g,f)-covered graph in [4] and obtained some basic results. A graph is fractional (g,f)-covered if for each edge e of G there is a fractional (g,f)-factor with the indicator function h, such that $h(e) = 1$.

Anstee[1] obtained a necessary and sufficient condition for a graph to have a fractional (g,f)-factor.

Theorem 1. [1] *Let G be a graph, then G has a fractional (g,f)-factor if and only if for every subset S of $V(G)$*

$$g(T) - d_{G\setminus S}(T) \leq f(S),$$

where $T = \{x : x \in V(G)\setminus S \text{ and } d_{G\setminus S}(x) \leq g(x)\}$.

For any $S \subseteq V(G)$, Let $T = \{x : x \in V(G)\setminus S \text{ and } d_{G\setminus S}(x) \leq g(x)\}$, we define $\varepsilon(S)$ as follows

$\varepsilon(S) = 2$, if S is not independent;

$\varepsilon(S) = 1$, if S is independent and there is an edge joining S and $V(G)\setminus(S\cup T)$, or there is an edge $e = uv$ joining S and T such that $v \in T$, $d_{G\setminus S}(v) = g(v)$;

$\varepsilon(S) = 0$ if neither (1) nor (2) holds.

Theorem 2. [4] *Let G be a graph, g and f be two integer-value functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then G is fractional (g,f)-covered if and only if for all $S \subseteq V(G)$ and $T = \{x : x \in V(G)\setminus S \text{ and } d_{G\setminus S}(x) \leq g(x)\}$*

$$\delta(S,T) \geq \varepsilon(S),$$

where $\varepsilon(S)$ is as defined earlier.

By Theorem 2, it is easily to obtain the following

Theorem 3. [5] *Let G be a graph, and $k > 0$ be an integer. Then G is fractional k-covered if and only if for all $S \subseteq V(G)$, $T = \{x : x \in V(G)\setminus S \text{ and } d_{G\setminus S}(x) \leq k\}$*

$$\sum_{j=0}^{k-1}(k-j)P_j(G\setminus S) \leq k|S| - \varepsilon(S),$$

where $P_j(G\setminus S)$ denote the number of vertices in $G\setminus S$ with degree j.

In [3], Chv*á*tal introduced the definition of toughness $t(G)$ of graph G. When G is not complete graph

$$t(G) = \min\{\frac{|S|}{\omega(G\setminus S)} : S \subseteq V(G), \omega(G\setminus S) \geq 2\}.$$

Theorem 4. [3] *If G is not a complete graph. Then $\kappa(G) \geq 2t(G)$.*

Theorem 5. [7] *Let G be a graph, and $k \geq 2$ an integer. If $t(G) \geq k - \frac{1}{k}$, and $V(G) \geq k+1$, then G has a fractional k-factor.*

In this paper, we study sufficient conditions related to toughness and isolated toughness for a graph to be fractional 1-covered or 2-covered. In particular, we give a necessary and sufficient condition for a (g, f)-factor covering a given k-matching.

2 Main results and proofs

Theorem 6. *Let G be a graph, when G is an even cycle, or $t(G) > \frac{3}{2}$, then G is fractional 1-covered.*

Proof. Let $k = 1$ in Theorem 3. It suffice to prove that

$$i(G\setminus S) \leq |S| - \varepsilon(S), \tag{1}$$

holds for all $S \subseteq V(G)$.

If G is a complete graph, obviously the theorem holds. In the following we suppose that G is not a complete graph. By Theorem 4, we have $\delta(G) \geq \kappa(G) \geq 2t(G) > 3$.

Case 1. $|S| \leq 3$.

Since $\delta(G) > 3$, we have $i(G\setminus S) = 0$. For $\varepsilon(S) \leq 2$, then $i(G\setminus S) < |S| - \varepsilon(S)$. (1) holds.

Case 2. $|S| \geq 4$.

Since $\frac{3}{2} < t(G) \leq \frac{|S|}{\omega(G\setminus S)}$, we have $|S| > \frac{3}{2}\omega(G\setminus S)$. suppose that G is not fractional 1-covered. By Theorem 3, there exists $\phi \neq S_0 \subseteq V(G)$, $|S_0| \geq 4$, satisfying $i(G\setminus S_0) > |S_0| - \varepsilon(S_0)$. Since $\varepsilon(S_0) \leq 2$, we have

$$i(G\setminus S_0) \geq |S_0| - 1. \tag{2}$$

Thus, $\omega(G\setminus S_0) \geq i(G\setminus S_0) \geq |S_0| - 1 \geq 3$ by (2). Since

$$\frac{3}{2} < t(G) \leq \frac{|S_0|}{\omega(G\setminus S_0)},$$

we have $\frac{3}{2}\omega(G\setminus S_0) < |S_0|$. Thus,

$$\omega(G\setminus S_0) + \frac{3}{2} \leq \frac{3}{2}\omega(G\setminus S_0) \leq |S_0| - 1 \leq i(G\setminus S_0),$$

a contradiction.

The theorem is proved. □

Remark 1. The condition $t(G) > \frac{3}{2}$ in the theorem is best possible. Let $G_1 = \overline{K}_2 \vee P_3$. We have $t(G_1) = \frac{3}{2}$. Let $S = V(P_3)$, $i(G \setminus S) = 2$, $\varepsilon(S) = 2$, then $i(G \setminus S) > |S| - \varepsilon(S)$, G_1 is not fractional 1-covered.

Theorem 7. *Let G be a graph, if $t(G) \geq \frac{3}{2}$, and $V(G) \geq 3$. Then G is fractional 2-covered.*

Proof. Let $k = 2$ in Theorem 3. We only need to prove that

$$2P_0(G \setminus S) + P_1(G \setminus S) \leq 2|S| - \varepsilon(S), \tag{3}$$

for any $S \subseteq V(G)$, $T = \{x : x \in V(G) \setminus S, d_{(G \setminus S)}(x) \leq 2\}$.

If G is a complete graph, as $|V(G)| \geq 3$, obviously G is fractional 2-covered. In the following we suppose G is not a complete graph. By Theorem 4, we have $\delta(G) \geq \kappa(G) \geq 2t(G) \geq 3$.

Case 1. $|S| \leq 1$.

Since $\delta(G) \geq 3$, we have $P_0(G \setminus S) = P_1(G \setminus S) = 0$. If $|S| = 0$, then $\varepsilon(S) = 0$. (3) holds. If $|S| = 1$ $\varepsilon(S) \leq 1$, then $2P_0(G \setminus S) + P_1(G \setminus S) \leq 2|S| - 1 \leq 2|S| - \varepsilon(S)$. Hence (3) holds.

Case 2. $|S| \geq 2$.

Suppose that G is not fractional 2-covered. By Theorem 3, there exists $\phi \neq S_0 \subseteq V(G)$, and $|S_0| \geq 2$, such that $2P_0(G \setminus S_0) + P_1(G \setminus S_0) > 2|S_0| - \varepsilon(S_0)$. Since $\varepsilon(S_0) \leq 2$, we have $2P_0(G \setminus S_0) + P_1(G \setminus S_0) > 2|S_0| - 2$, that is, $P_0(G \setminus S_0) + \frac{1}{2}P_1(G \setminus S_0) > |S_0| - 1$. Therefore,

$$P_0(G \setminus S_0) + \frac{1}{2}P_1(G \setminus S_0) \geq |S_0|. \tag{4}$$

Denote $T_1 = \{x : x \in V(G) \setminus S_0, d_{G \setminus S_0}(x) = 1\}$. Then $N_{G \setminus S_0}(T_1) = \{y : x \in T_1, xy \in E_{G \setminus S_0}(G)\}$ and $E(T_1) = \{e : e = uv, u, v \in T_1\}$. We consider the following subcases:

Subcase 2.1 $P_1(G \setminus S_0) = 0$.

In this case, we have $\omega(G \setminus S_0) \geq P_0(G \setminus S_0) \geq |S_0| \geq 2$ by the hypothesis.

$$\frac{3}{2} \leq \frac{|S_0|}{\omega(G \setminus S_0)} \leq \frac{|S_0|}{P_0(G \setminus S_0)} \leq 1,$$

a contradiction.

Subcase 2.2 $P_1(G \setminus S_0) \geq 1$.

In this case $\omega(G \setminus (S_0 \cup V_1)) \geq 2$. Let $|V_1| = |N_{G \setminus S_0}(T_1)|$. Then

$$\begin{aligned}
\frac{3}{2} &\leq \frac{|S_0 \cup V_1|}{\omega(G \setminus (S_0 \cup V_1))} \\
&\leq \frac{P_0(G \setminus S_0) + \frac{1}{2}P_1(G \setminus S_0) + |V_1|}{\omega(G \setminus S_0) + P_1(G \setminus S_0) - |E(T_1)|} \\
&\leq \frac{\omega(G \setminus S_0) - |E(T_1)| + \frac{1}{2}P_1(G \setminus S_0) + |V_1|}{\omega(G \setminus S_0) - P_1(G \setminus S_0) - |E(T_1)|} \\
&\leq \frac{\omega(G \setminus S_0) - |E(T_1)| + \frac{1}{2}P_1(G \setminus S_0) + P_1(G \setminus S_0) - |E(T_1)|}{\omega(G \setminus S_0) - |E(T_1)| + P_1(G \setminus S_0)} \\
&< \frac{3}{2}.
\end{aligned}$$

A final contradiction. The theorem is proved. □

Remark 2. Note that Theorem 5 is best possible in [7], i.e, the condition $t(G) \geq \frac{3}{2}$ for graph G has a fractional 2-factor is best possible. Therefore the condition $t(G) \geq \frac{3}{2}$, G is fractional 2-covered is also the best possible in this sense.

Isolated toughness $I(G)$ of graph G was introduced in [10]. When G is not complete,

$$I(G) = min\{\frac{|S|}{i(G \setminus S)} : S \subseteq V(G), i(G \setminus S) \geq 2\},$$

when G is complete, $I(G) = +\infty$.

Theorem 8. *Let G be a graph, and $\delta(G) \geq 3$, If $I(G) > \frac{3}{2}$ for all $S \subseteq V(G)$, then G is fractional 1-covered.*

Proof. Let $k = 1$ in Theorem 3. It suffice to prove that

$$i(G \setminus S) \leq |S| - \varepsilon(S), \tag{5}$$

holds for all $S \subseteq V(G)$.

We consider two cases.

Case 1. $|S| < 2$.

Since $\delta(G) \geq 3$, $T = \{x : x \in V(G) \setminus S \text{ and } d_{G \setminus S}(x) \leq 1\} = \emptyset$, and $i(G \setminus S) = 0$. If $S = \emptyset$, in this case $\varepsilon(S) = 0$, then $i(G \setminus S) = 0 \leq |S| - \varepsilon(S)$. Thus (1) holds. Else $|S| = 1$, in this case $\varepsilon(S) \leq 1$, then $i(G \setminus S) = 0 \leq |S| - \varepsilon(S)$. Thus (5) holds.

Case 2. $|S| \geq 2$.

Subcase 2.1 $i(G \setminus S) \leq 1$.

Since $\varepsilon(S) \leq 2$, $i(G \setminus S) \leq 1 = 3 - 2 \leq |S| - 2 \leq |S| - \varepsilon(S)$, hence (5) holds.

Subcase 2.2 $i(G \setminus S) \geq 2$.

Since $I(G) > \frac{3}{2}$, we have $\frac{|S|}{i(G \setminus S)} > \frac{3}{2}$, that is, $2|S| > 3i(G \setminus S)$. Thus, $2|S| - 1 \geq 3i(G \setminus S)$. Since $i(G \setminus S) \geq 2$, we have $2|S| - 1 \geq 2i(G \setminus S) + 2$. Hence $i(G \setminus S) \leq |S| - \frac{3}{2}$. Moreover, $i(G \setminus S) \leq |S| - \lceil \frac{3}{2} \rceil$ since $i(G \setminus S)$ is an integer. Thus $i(G \setminus S) \leq |S| - \varepsilon(S)$ since $\varepsilon(S) \leq 2$.

The theorem is proved. □

Remark 3. (1) The condition $\delta(G) \geq 3$ in Theorem 8 can not be reduced to $\delta(G) \geq 2$. Let G_1 be a graph which consists of two cycles $C_1 = v_1v_2v_3$ and $C_2 = v_1v_4v_5v_6$ with one common vertex v_1 and v_4v_6 is a chord of C_2. Clearly $\delta(G_1) = 2$, $I(G_1) = 2 \geq \frac{3}{2}$. Let $S = \{v_1, v_2\}$, $\varepsilon(S) = 2$, $i(G \setminus S) = 1 > |S| - \varepsilon(S)$. Therefore G_1 is not fractional 1-covered.

(2) The condition $I(G) \geq \frac{3}{2}$ is best possible. Let G_2 be a cycle $C_5 = v_1v_2v_3v_4v_5$ with chords v_1v_4, v_1v_5 and v_2v_3. Clearly $\delta(G_2) = 3$, $I(G_2) = \frac{3}{2}$. Let $S = \{v_1, v_3, v_4\}$, $\varepsilon(S) = 2$, $i(G \setminus S) = 2 > |S| - \varepsilon(S)$. Therefore G_2 is not fractional 1-covered.

Let S be a subset of $V(G)$, and M be a matching of G. Let
$T = \{x : x \in V(G) \setminus S \text{ and } d_{G\setminus S}(x) \leq g(x)\}$,
$D = V(G) \setminus (S \cup T)$,
$E_G(S) = \{e : e = xy \in E(G), x, y \in S\}$,
$E' = M \cap E_G(S)$,
$E'' = M \cap E_G(S, D)$,
$H = G[E' \cup E'']$,
$\beta_G(S, M) = 2|E'| + |E''| = \sum_{x \in S} d_H(x)$.

Nextly, we give a necessary and sufficient condition for a graph to have a fractional (g, f)-factor covering a given k-matching.

Theorem 9. *Let G be a graph, g and f be two integer-valued functions defined on $V(G)$, such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$, and let M be a matching of G. Then G has a fractional (g, f)-factor G_h such that $h(e) = 1$ for every $e \in M$, where h is the indicator function of G_h, if and only if*

$$\delta_G(S, T) = d_{G\setminus S}(T) - g(T) + f(S) \geq \beta_G(S, M),$$

for any subset S of $V(G)$, and $T = \{x : x \in V(G) \setminus S \text{ and } d_{G\setminus S}(x) \leq g(x)\}$.

Proof. For all edges of M, define functions f^e and g^e on $V(G)$ such that

$$f^e = \begin{cases} f(x) - 1, & \text{for } x \in V(M); \\ f(x), & \text{otherwise.} \end{cases}$$

and

$$g^e = \begin{cases} g(x) - 1, & \text{for } x \in V(M); \\ g(x), & \text{otherwise.} \end{cases}$$

Clearly, there is a fractional (g, f)-factor of G containing M such that $h(e) = 1$ for ever $e \in M$, if and only if there is a fractional (g^e, f^e)-factor of $G \setminus M$, and by Theorem 1, if and only if

$$\delta_{G\setminus M}(S, T') = d_{((G\setminus M)\setminus S)}(T') - g^e(T') + f^e(S) \geq 0 \ \text{ for every } S \subseteq V(G \setminus M),$$

where $T' = \{x : x \in V((G \setminus M) \setminus S) \text{ and } d_{((G\setminus M)\setminus S)}(x) \leq g^e(x)\}$. Thus G has a fractional (g, f)-factor covering M, if and only if for all $S \subseteq V(G)$

$$\min_{M \subseteq E(G)} \{\delta_{G\setminus M}(S, T')\} \geq 0. \tag{6}$$

Set $T^* = \{x : x \in V(G) \setminus S \text{ and } d_{G\setminus S}(x) < g(x)\}$. We have

$$\delta_G(S, T) = \delta_G(S, T^*),$$

and

$$\delta_{G\setminus M}(S,T') = \delta_{G\setminus M}(S,T^*).$$

Moreover, for $M \subseteq E(G)$.

$$\begin{aligned}
&\delta_G(S,T) - \delta_{G\setminus M}(S,T')\\
&= \delta_G(S,T^*) - \delta_{G\setminus M}(S,T^*)\\
&= (d_{G\setminus S}(T^*) - d_{((G\setminus M)\setminus S)}(T^*)) - (g(T^*) - g^e(T^*)) + (f(S) - f^e(S))\\
&= 2|E'| + |E''|\\
&= \beta(S,M).
\end{aligned}$$

That is,

$$\delta_{G\setminus M}(S,T') = \delta_G(S,T) - \beta(S,M).$$

Thus, (6) holds, if and only if for all $S \subseteq V(G)$,

$$\delta_G(S,T) \geq \beta(S,M).$$

The theorem is proved. □

Let $k \geq 0$ be an integer, and G be a graph with at least $2k+2$ vertices having k-matching. A graph G is called fractional k-extendable if every k-matching M of G contained in a fractional 1-factor G_h of G such that $h(e) = 1$ for all $e \in M$. Let $g(x) = f(x) = 1$ for all $x \in V(G)$. By Theorem 9, we have the following corollary.

Corollary 10. [8] *Let G be a graph with a k-matching and $k \neq 0$, Then G is fractional k-extendable if and only if*

$$i(G \setminus S) \leq |S| - 2k,$$

holds for any $S \subseteq V(G)$, such that $G[S]$ contains a k-matching.

Note that the sharpness of Theorem 5 and Theorem 7 and to obtain better results. We propose the following conjecture.

Conjecture 1. Let G be a graph, $k \geq 2$ be an integer. If $t(G) \geq k - \frac{1}{k}$, and $V(G) \geq k+1$, then G is fractional k-covered.

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