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A Stochastic Inventory Placement Model for a Multi-echelon Seasonal Product Supply Chain with Multiple Retailers*

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Abstract This paper formulates a stochastic inventory model for a single-period seasonal or fashion product inventory system consisting of multiple stocking echelons in series and retailers. By transforming model, an equivalent two-stage stochastic linear program model is developed to solve this stochastic inventory problem, which makes the inventory problem easier to analyze. The optimal placement policy of this inventory system is characterized and its existence is proved. Moreover, the algorithm for finding optimal placement policies is obtained.

Keywords inventory/production; multi-echelon; optimal placement policy; stochastic inventory model; two-stage stochastic program

1 Introduction

In practical situations, many manufacturing processes of seasonal or fashion products can be viewed as serial assembling processes. Here, we consider a production-inventory system of such seasonal or fashion products, which comprises multiple stocking echelons in series and retailers. We assume that the system contains m retailers and its manufacturing process consists of $k-1$ serial assembling/processing facilities, so it includes k different stocking echelons, which hold various kinds of inventory items in the form of raw materials(echelon 1), subassemblies/work-in-processes (echelons 2 to $k-1$) and finished products (echelon k). Through the m retailers, the demand of all customers is satisfied by obtaining the finished products or converting the available raw materials and subassemblies within the selling period. A diagrammatic representation of such a manufacturing system is given in Figure 1

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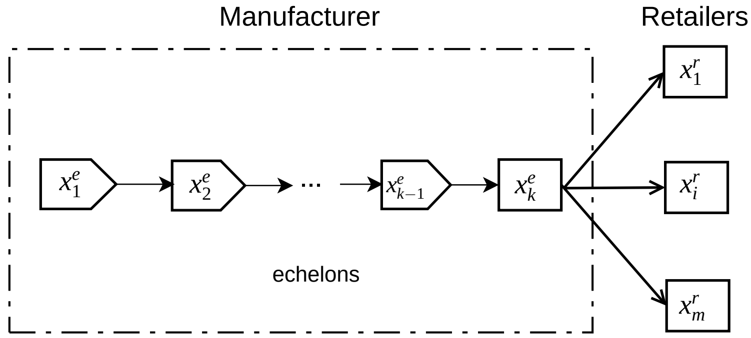

Figure 1: A Production-Inventory System consisting of k echelons in series and m retailers

Because of the short selling period, we consider the case in which there are no opportunities to reorder more raw materials from the suppliers, neither is there time to modify its initial inventory levels at the stocking echelons and the downstream retailers. Therefore, the inventory placement decisions have to be made at the start of the selling period. The major challenge facing manufacturers of the seasonal or fashion products is to plan for the production and inventory levels of their products to meet random demands over a short selling period.

Generally, there are two kinds of inventory policies used for production and inventory planning. One is assemble in advance (AIA) and the other is assemble to order (ATO).In contrast to AIA, which assembles the finished products according to plan (forecast demand) and keeps inventory at the end product level, ATO keeps iinventory at the subassembly/work-in-process level and only assemble a product in response to demand. ATO can postpone the commitment point of subassemblies to end products and thus make it possible to reduce liquidation loss and realize great products variety. However for many seasonal or fashion products with a serial assembly process, because of their short selling period, production capacity limits and delays in a series of converting (assembling) and shipping processes, adopting the ATO inventory strategy will result in the loss of customers who are unwilling to wait. Moreover, as far as production is concerned, the unit cost of a finished product under ATO is higher than that under AIA, because for the latter there is no need for rush orders, and production can be planned and implemented well in advance. On the other hand, when the AIA inventory strategy is used, liquidation loss will be incurred if supply exceeds demand to cause the wasted processing costs and the disposal of unused inventory. Therefore, as shown by Eynan and Rosenblatt [4], a composite of AIA and ATO is the dominating strategy for the serial multi-echelon systems of seasonal or fashion products.

Whitin [10] first presented such a single-period problem with stochastic demand and two echelons. Subsequently, many authors (Brayan et.al. [1], Hanssmann [6], Johson et.al. [8], Gerchak et.al. [5], Eynan and Rosenblatt [4]) analyzed the similar problems with two or four echelons. Recently, Hariga [7] and Chung et al. [3] investigated such a single period stochastic model composed of n echelons, independently. However, these existing models with multiple echelons in series typically assume that only one retailer faces all customer demand. To the best of our knowledge, there are no papers studying the case with multiple retailers. In this paper we consider a single-period mixed ATO and AIA production-inventory system consisting of k echelons and m retailers as illustrated in Figure 1, in which each retailer faces his own stochastic demand and the random demands arriving at the various retailers are allowed to be correlative and their distribution is arbitrary. The profitable stock quantities placed at various stock points of this system have to be made before the start of selling period. In selling period, the customer's demand arriving at a retailer is first satisfied with the retailer's stock. When a shortage occurs at the retailer's inventory, based on the customers' different willingness to wait, the excess demand can be partially satisfied by converting and transporting the available inventory items at the upstream echelons.

The remainder of this paper is organized as follows. In Section 2, we introduce the single-period inventory system with k echelons and m retailers. In Section 3, an equivalent two-stage stochastic linear program model with fixed recourse is developed to analyze this inventory problem. In Section 4, we characterize the optimal solution to this problem , prove the existence of optimal inventory policies and propose an effective algorithms to find optimal inventory placement policies. Finally, we conclude the paper in Section 5.

2 Notation and Model Formulation

As shown in Figure 1, we consider a single-period stochastic production-inventory model composed of m retailers and k stocking echelons which hold various inventory items in the form of raw materials (echelon 1), different grades of subassemblies/work-in-processes (echelons 2 to $k-1$) and finished products (echelon k).

We use the following notation to define this production-inventory system:

$i = 1,2,\cdots,m$ index the retailers.

$j = 1,2,\cdots,k$ index the echelons.

$x_i^r =$ stock quantity placed in the store of retailer i at the beginning of the period, where $x_i^r \geq 0$. We denote the vector $x_r = (x_1^r, x_2^r, \ldots, x_m^r)$.

$x_j^e =$ stock quantity placed in the echelon j at the beginning of the period, where $x_j^e \geq 0$. $x_e = (x_1^e, x_2^e, \ldots, x_k^e)$.

D_i=random demand of the ith retailer, and $D = (D_1, D_2, \cdots, D_m)$ is a random vector with joint distribution function $F(x_1, x_2, \cdots, x_m)$.

$s_i =$ unit cost for shipping products from the manufacturer to retailer i.

$p_i^r =$unit profit (excluding the shipping cost from one retailer to another) when the ith retailer's stock assembled in advance is sold, where $p_i^r > 0$. $p_r = (p_1^r,\, p_2^r,\, \ldots,\, p_m^r)$.

p_j =unit profit (excluding shipping cost from the manufacturer to retailers) when the finished products assembled to order by converting the inventory items at echelon j are sold, $j = 1,2,\cdots,k-1$, and p_k = unit profit (excluding shipping cost) when the finished products assembled in advance at the echelon k are sold. Because in practical situations the net profit (including the downstream shipping cost) of a unit stock sold through a retailer is nondecreasing with its stock point being close to the retailer's customer, it is natural to assume that $0 < p_1 < p_2 < \cdots < p_{k-1} < p_k < p_i^r + s_i$. $p_e = (p_1, p_2, \ldots, p_k)$.

l_j^e =unit liquidation loss for unused inventory at the jth echelon and l_i^r = unit liquidation loss for unused inventory at the ith retailer. It is also natural to assume that $l_1^e \leqslant l_2^e \leqslant \cdots \leqslant l_k^e \leqslant l_i^r$. Define the unit liquidation loss vector at echelons by $l_e = (l_1^e, l_2^e, \ldots, l_k^e) \in R^k$ and the unit liquidation loss vector at retailers by $l_r = (l_1^r, l_2^r, \ldots, l_m^r) \in R^m$.

α_{ji} = fraction of the demand not satisfied by the ith retailer's inventory that will wait for shipments from retailer j, which measure the impacts on the lost sales induced by the various lead times of assembling and shipping processes. From Figure 1, it holds that $0 \leq \alpha_{1i} \leqslant \alpha_{2i} \leqslant \cdots \leqslant \alpha_{ki} \leq 1$.

Here, we neglect the the shipping costs between the echelons because these costs are included in the profit of the sold subassemblies at each echelon. x_j^e and x_i^r are the inventory placement decision variables before the beginning of the selling period and we define the inventory placement policy $x = (x_e,\ x_r) = (x_1^e, \ldots, x_k^e, x_1^r, \ldots, x_m^r)$. When the customer demand D is realized, $min\{x_i^r, D_i\}$ is the supply quantity of the retailer i to meet its customer demand. Let y_{ji} denotes the tth echelon's supply quantity to meet the ith retailer's unsatisfied demand that will wait for the shipment from the echelon j , where all y_{ji} are not nonnegative. Define the shipping decision vector $y = (y_{11}, \cdots, y_{1m}, \cdots, y_{k1}, \cdots, y_{km}) \in \mathbb{R}^{mk}$.

For a given inventory placement policy x , when the actual demand vector D becomes known, the profit of taking the shipping decision y is given by

$$P(x,y,D) = -(\sum_{j=1}^{k} l_j^e x_j^e + \sum_{i=1}^{m} l_i^r x_i^r) + \sum_{i=1}^{m} (p_i^r + l_i^r) \min\{x_i^r, D_i\} + \sum_{j=1}^{k} \sum_{i=1}^{m} (p_j + l_j^e - s_i) y_{ji}$$

Our objective is to determine the most profitable inventory placement policy before the start of the selling period and then make a most reasonable shipping decision when demand is realized so as to maximize the expectation of $P(x,y,D)$. In fact, this illustrates the sequence of events in the single-period inventory problem. In the first stage, the optimal inventory placement policy x is taken in accordance with the maximum expected profit principle before the start of the selling period. Later, the actual demand D becomes known and a second-stage shipping decision y can be taken to realize the actual maximum profit. Therefore, the single-period inventory problem can be formulated mathematically as a two-stage stochastic program problem as follows:

$$\begin{aligned} \max_x \quad & f(x) = E[\max_y P(x,y,D)] \\ \text{s.t.} \quad & \sum_{i=1}^{k} y_{ji} \le x_j^e, \quad j = 1,2,\cdots,m, \\ & \sum_{j=1}^{k} \alpha_{ji}{}^{-1} y_{ji} \le \max\{0, D_i - x_i^r\}, \quad i = 1,2,\cdots,n \\ & x \ge 0, \quad y \ge 0 \end{aligned} \tag{1}$$

where E represents the mathematical expectation with respect to D. In the above constraints, the first constraints ensure that the total supply quantity of each echelon do not exceed its stock quantity. Since y_{ji} is the supplying quantity from the echelon j to meet the ith retailer's unsatisfied demand, $\sum_{j=1}^{k}(\alpha_{ji})^{-1}y_{ji}$ is the sum of the supplying quantity from echelons to the ith retailer and the corresponding demand loss. Hence, the second constraint ensures that the sum of the supplying quantity meeting the ith retailer's residual demand and the corresponding demand loss do not exceed the ith retailer's residual demand $\max\{0, D_i - x_i^r\}$.

From the above analysis, the optimal solution x^* of the problem (1) is the most profitable inventory placement policy of the single-period inventory system that maximizes the expected profit of the entire system, and the optimal solution y^* to the problem $\max_y P(x^*, y, D)$ is the shipping decision achieving the actual maximal profit for a specific demand realization D. Therefore, the optimal solution x^* of the problem (1) is the optimal inventory placement policy of the single-period stochastic inventory system.

Because of including the nonlinear terms $\min\{x_i^r, D_i\}$ and $\max\{0, D_i - x_i^r\}$, the two-stage stochastic program (1) is a nonlinear stochastic program problem, which makes the problem hard to analyze. Next, by introducing new slack variables, we will transform the nonlinear stochastic program problem into an equivalent two-stage linear stochastic program problem with fixed recourse and thus analyze this problem well.

3 Equivalent Linear Stochastic Program Problem

For the analytical convenience, we denote the vector $l = (l_e, l_r) \in \mathbb{R}^{m+k}$, $\bar{p} = (p_1 + l_1^e - s_1, \cdots, p_1 + l_1^e - s_m, \cdots, p_k + l_k^e - s_1, \cdots, p_k + l_k^e - s_m) \in \mathbb{R}^{mk}$, $p = (\bar{p}, p_r + l_r) \in \mathbb{R}^{m(k+1)}$ and $\min(x_r, D) = (\min\{x_1^r, D_1\}, \min\{x_2^r, D_2\}, \cdots, \min\{x_n^r, D_m\}) \in \mathbb{R}^m$.

In order to remove the nonlinear terms from the problem (1), we introduce a new slack variable y_i^r to denote the supply quantity of the retailer i to meet the customer's demand D_i, where we allow $0 \le y_i^r \le D_i$. And define a new shipping decision vector $z = (y, y_1^r, y_2^r, \cdots, y_m^r) \in \mathbb{R}^{m(k+1)}$ in which y is the shipping decision vector defined by y_{ij}. Thus, we can define a new two-stage linear stochastic program with fixed recourse as follows:

$$\begin{aligned} \max_x \quad & f(x) = -lx' + E[\max_z \ pz'] \\ \text{s.t.} \quad & Mz' \le x', \quad Nz' \le D' \\ & x \ge 0,\ z \ge 0 \end{aligned} \tag{2}$$

in which the matrixes $M = (\ a_{\lambda\iota}\ )_{(m+k)\times m(k+1)}$ and $N = (\ b_{\lambda\iota}\ )_{m\times m(k+1)}$ are defined as follows:

$$a_{\lambda\iota} = \begin{cases} 1 & \lambda = 1,2,\cdots,k,\ (\lambda-1)m < \iota \le \lambda m \\ 1 & \lambda = k+1, k+2, \cdots, k+m,\ \iota = (\lambda - k) + km \\ 0 & \text{otherwise} \end{cases}$$

$$b_{\lambda\iota} = \begin{cases} {\alpha_{t\iota}}^{-1} & \lambda = 1,2,\cdots,m,\ \iota = tm+\lambda, t = 0,1,\cdots,k-1 \\ 1 & \lambda = 1,2,\cdots,m,\ \iota = km+\lambda \\ 0 & \text{otherwise.} \end{cases}$$

Here, $Mz' \le x'$ expresses the constraints: $y_i^r \le x_i^r$, $\sum_{i=1}^m y_{ji} \le x_j^e$; $Nz' \le D'$ expresses the constraints: $y_i^r + \sum_{j=1}^k {\alpha_{ji}}^{-1} y_{ji} \le D_i$. With the analogous analysis of the stochastic program problem (1), it follows that the new stochastic program problem (2) describes a similar single-period inventory problem to the one discussed above. The only difference is that here the customer's demand arriving at a retailer can be allowed to first satisfied with the stock of other stock points. In the stochastic program (2), the objective function can be also written as $f(x) = E[\max_z(-lx' + pz')]$, in which $-lx' + pz' = \sum_{i=1}^m p_i^r y_i^r - \sum_{i=1}^m l_i^r (x_i^r - y_i^r) + \sum_{j=1}^k p_j \sum_{i=1}^m y_{ji} - \sum_{j=1}^k l_j^e (x_j^e - \sum_{i=1}^m y_{ji}) - \sum_{j=1}^k \sum_{i=1}^m s_i y_{ji}$ is the profit value of taking the shipping decision z for the given inventory placement policy x.

For a fixed x and D, we denote the optimal value of the second-stage program in (2) by

$$SP(x,D) = \max_z \{pz' \mid z \ge 0,\ Mz' \le x',\ Nz' \le D'\}.$$

And let $\mathscr{S}(x) = E[SP(x,D)]$. Then, the above two-stage stochastic program problem (2) is equivalent to the following deterministic program:

$$\begin{aligned} \max_x \quad & f(x) = -lx' + \mathscr{S}(x) \\ \text{s.t.} \quad & x \ge 0 \end{aligned} \tag{3}$$

Theorem 1. *An inventory placement policy x^* is an optimal solution of the problem (1) if and only if it is an optimal solution of the problem (2) (or (3)).*

Proof. Denote $\overline{SP}(x,D) = \max_y \{\bar{p}y' \mid y \ge 0 \text{ and } y \text{ satisfies the constraints of (1) }\}$ and $\bar{\mathscr{S}}(x) = E[\overline{SP}(x,D)]$. Then, the two-stage stochastic program problem (1) is equivalent to the program:

$$\begin{aligned} \max_x \quad & f(x) = -lx' + (p_r + l_r)E[\min(x_r, D)'] + \bar{\mathscr{S}}(x) \\ \text{s.t.} \quad & x \ge 0 \end{aligned} \tag{4}$$

By using the given conditions $0 < p_1 < p_2 < \cdots < p_{k-1} < p_k < p_i^r + s_i$ and $l_1^e \leqslant l_2^e \leqslant \cdots \leqslant l_k^e \leqslant l_i^r$, it can be proved that for any fixed inventory placement policy x and demand D,

$$SP(x,D) = \overline{SP}(x,D) + (p_r + l_r)E[\min(x_r, D)']. \tag{5}$$

Then, we have that

$$\mathscr{S}(x) = \bar{\mathscr{S}}(x) + (p_r + l_r)E[\min(x_r, D)']. \tag{6}$$

Therefore, from (3),(4) and (6), it follows that the theorem holds. □

Furthermore, from the process of the above proof , the following theorem follows directly.

Theorem 2. *For any inventory placement policy x and demand D, $z^* = (y^*, {y^*}_1^r, {y^*}_2^r, \cdots, {y^*}_m^r) \in \mathbb{R}^{m(k+1)}$ is an optimal solution of the problem $SP(x,D)$ if and only if ${y^*}_i^r = \min\{x_i^r, D_i\}$, $i = 1,2,\cdots,m$, and y^* is an optimal solution of the problem $\overline{SP}(x,D)$.*

Theorem 1 and Corollary 2 show that the stochastic program problem (1) is equivalent to the stochastic program problem (2). Due to the fixed coefficient matrices M and N, the stochastic program (2) is a two-stage stochastic linear program problem with fixed recourse. This implies that the single-period stochastic inventory problem is transformed equivalently into a two-stage linear stochastic program problem (2) with fixed recourse, which makes it easy for us to analyze this inventory problem.

4 Optimal Placement Policies and Their Algorithm

We assume further that **the random demand D has finite second moments**. This assumption is very general because it holds for all popular demand distributions.

Lemma 3. *1). $\mathscr{S}(x)$ is a concave function and is also finite on $\mathbb{R}_+^{n+m+k} = \{x \geq 0 \mid x \in \mathbb{R}^{m+k}\}$.*

2). When the distribution of D is finite, $\mathscr{S}(x)$ is piecewise linear on $\mathbb{R}_+^{m+k}$.

3). If the D's joint distribution function $F(\cdot)$ is absolutely continuous, $\mathscr{S}(x)$ is differentiable on $\mathbb{R}_+^{m+k}$

Proof. It is easy to prove that there exists a positive number B such that $|SP(x,D)| < B\|x\|$. Hence, $|\mathscr{S}(x)| = |E[SP(x,D)]| < B\|x\|$ $(here, \|x\| = \max_\tau |x_\iota|)$. This implies that $\{x \mid \mathscr{S}(x) < \infty\} = \mathbb{R}_+^{m+k}$. It is also known that the D has finite second moments and the stochastic program (2) has the fixed recourse. Therefore, 1), 2) and 3) follow directly from the theory of stochastic programming (see [2] for details). □

The following theorem guarantee that an optimal placement policy of the single-period inventory problem exists.

Theorem 4. *The single-period inventory problem has a finite optimal placement policy* $x^* \in \mathbb{R}_+^{m+k}$.

Proof. $f(x) = -cx' + \mathscr{Z}(x)$ is concave and $\lim_{\|x\| \to \infty} f(x) = \infty$. Hence, the stochastic program problem (2) has a finite optimal value and it is attained for some $x \in \mathbb{R}_+^{m+k}$. By Theorem 1, the theorem holds. □

Next, we characterize the optimal inventory placement policies. The deterministic equivalent program (3) gives us the following results in terms of Karush-Kuhn-Tucker conditions.

Theorem 5. *a placement policy* $x^* \in \mathbb{R}_+^{m+k}$ *is optimal if and only if there exists a subgradient* $\partial\mathscr{S}(x^*)$ *of* $\mathscr{S}(x)$ *at* x^* *such that*

$$\partial\mathscr{S}(x^*)_\kappa = l_\kappa \text{ for } x^*_\kappa > 0 \quad \text{and} \quad \partial\mathscr{S}(x^*)_\kappa \le l_\kappa \text{ for } x^*_\kappa = 0$$

where $\partial\mathscr{S}(x^*)_\kappa,\ l_\kappa$ *and* x^*_κ *are the* κ*th components of the vectors* $\partial\mathscr{S}(x^*),\ l$ *and* x^*, *respectively,* $\kappa = 1, 2, \cdots, m+k$.

Proof. From the optimization of a convex function over a convex region and Karush-Kuhn-Tucker conditions of the problem (3), the theorem follows directly. □

It is known that $\partial\mathscr{S}(x) = \nabla\mathscr{S}(x)$ when $\mathscr{S}(x)$ is differentiable. Thus, when the D's joint distribution function is absolutely continuous, it follows from Lemma 3 that

$$\frac{\partial\mathscr{S}}{\partial x_\kappa}(x^*) = \partial\mathscr{S}(x^*)_\kappa,\ \kappa = 1, 2, \cdots, k+m.$$

Since $\nabla f(x) = \nabla(-lx' + \mathscr{S}(x)) = \nabla\mathscr{S}(x) - l$, $\nabla\mathscr{S}(x) - l$ (or $\partial\mathscr{S}(x) - l$) can be viewed as the marginal expected profit vector of the inventory system at x. It is well-known that many continuous probability distributions are absolutely continuous, including the normal, exponential, uniform, etc. Therefore, Theorem 5 implies that a placement policy $x' \in \mathbb{R}_+^{m+k}$ is optimal if and only if it take zero marginal expected profits on its nonempty stock points and nonpositive marginal expected profits on its empty stock points.

$\partial\mathscr{S}(x^*)$ in Theorem 5 can be decomposed into subgradients of $SP(x, D)$ for each realization of D. Because $\{x \mid \mathscr{S}(x) < \infty\} = \mathbb{R}_+^{k+m}$, it follows from Corollary 3.12 in [2] that $\partial\mathscr{S}(x) = E\partial\mathscr{S}(x, D)$, for any $x \in \mathbb{R}_+^{m+k}$. Because $SP(x, D)$ is a piecewise linear concave function in x, $E\partial SP(x, D)$ is easier to compute than $\partial\mathscr{S}(x)$, which will be used in the following algorithm finding optimal placement policies. In addition, it is shown below that $E\partial SP(x, D)$ is just the expectation of optimal dual value of the problem $SP(x, D) = \max_z\{pz' \mid z \ge 0, Mz' \le x',\ Nz' \le D'\}$. The $SP(x, D)$'s dual problem can be written as

$$\begin{aligned} \mathscr{D}(x, D) = \min_{\mathscr{X} = (\mathscr{X}^1,\ \mathscr{X}^2)} \quad & \mathscr{X}^1 x' + \mathscr{X}^2 D' \\ \text{s.t.} \quad & M'\mathscr{X}^{1'} + N'\mathscr{X}^{2'} \ge p' \\ & \mathscr{X}^1 \ge 0,\ \mathscr{X}^2 \ge 0. \end{aligned} \tag{7}$$

Let $\mathscr{X}^{1^*}$ and $\mathscr{X}^{2^*}$ denote the optimal values of the dual problem (7) corresponding to x and D respectively. This implies that $E\partial SP(x,D) = E\mathscr{X}^{1^*}$. The following theorem reveals this relationship and gives new necessary and sufficient conditions to characterize an optimal placement policy.

Theorem 6. *A placement policy $x \in \mathbb{R}_+^{m+k}$ is optimal if and only if there exists an optimal solution $(\mathscr{X}^{1^*}, \mathscr{X}^{2^*})$ of the dual problem $\mathscr{D}(x,D)$ (7) such that*

$$E\mathscr{X}^{1^*}_\kappa = l_\kappa \text{ for } x_\kappa > 0 \quad \text{and} \quad E\mathscr{X}^{1^*}_\kappa \leq l_\kappa \text{ for } x_\kappa = 0 \tag{8}$$

where $E\mathscr{X}^{1^*}_\kappa$ denotes the κth component of $E\mathscr{X}^{1^*}$ and l_κ is the κth components of the unit liquidation cost vector l , $\kappa = 1,2,\cdots,k+m$.

Proof. Since $SP(x,D)$ has an optimal solution, so does its dual problem (7) and at optimality their objective values are equal. Namely, $SP(x,D) = \mathscr{X}^{1^*}x' + \mathscr{X}^{2^*}D'$.

Because of the existence of finitely many different optimal bases for the dual problem (7), $(\mathscr{X}^{1^*}, \mathscr{X}^{2^*})$ can take finitely many different values for all D, $x \in \mathbb{R}_+^{m+k}$ and thus $\mathscr{X}^{1^*}x' + \mathscr{X}^{2^*}D'$ is piecewise linear. It follows that $E\partial SP(x,D) = E\mathscr{X}^{1^*}$. Therefore, this theorem follows from Theorem 5. □

From linear program theory, the optimal dual value $\mathscr{X}^{1^*}$ is the shadow prices of the inventory x under the demand D, and thus $E\mathscr{X}^{1^*}$ is the expected shadow prices of the inventory x. Therefore, Theorem 6 shows that a placement policy $x^* \in \mathbb{R}_+^{m+k}$ is optimal if and only if, at each stock point, its inventory has an expected shadow price which is not greater than the inventory's unit liquidation loss when the inventory of this stock point is placed on zero and equal to the inventory's unit liquidation loss when the inventory of this stock point is nonempty. Theorem 5 and Theorem6 not only characterize the optimal placement policies but also provide a method for finding a optimal inventory placement policy. Next, we consider the algorithms solving the optimal inventory placement policies.

Since the inventory placement problem with continuous or infinitely discrete demand requires some form of approximation and these approximations are built on the algorithms of the finite demand's realizations case (for example, Monte Carlo sampling approximation approach [see [2] and [9] for details]) , we here mainly focuses on the algorithms for solving the inventory problem with finite demand's realizations. Let $d^1, d^2, \cdots, d^T$ denote the possible realizations of the demand D and let $q_1, q_2, \cdots, q_T$ be their probabilities. By using Lemma 3 and Theorem 5–6, we can give the following algorithm for finding optimal inventory placement policies, which is really a modified L-shaped method of stochastic programming for the special inventory problem. It proceeds as follows.

Algorithm:

$\mathbb{1}^\circ$. Set $t = \varsigma = 0$.

2°. Set $\varsigma = \varsigma + 1$. Solve the following linear program

$$\begin{aligned} \max \quad & f(x,\varepsilon) = -lx' + \varepsilon \\ \text{s.t.} \quad & -H_\gamma x' + \varepsilon \le h_\gamma, \gamma = 1,2,\cdots,t \\ & x \ge 0. \end{aligned} \tag{9}$$

Let $(x^\varsigma, \varepsilon^\varsigma)$ be the optimal solution. When $t = 0$, the constraint (9) is not considered in the computation of x^ς and ε^ς is set equal to $+\infty$.

3°. For $\kappa = 1,2,\cdots,T$, solve the linear program

$$\begin{aligned} \mathscr{D}(x^\varsigma, d^\kappa) = & \min_{\mathscr{X}=(\mathscr{X}^1,\ \mathscr{X}^2)} \mathscr{X}^1 x^{\varsigma\prime} + \mathscr{X}^2 d^{\kappa\prime} \\ \text{s.t.} \quad & M'\mathscr{X}^{1\prime} + N'\mathscr{X}^{2\prime} \ge p' \\ & \mathscr{X}^1 \ge 0,\ \mathscr{X}^2 \ge 0. \end{aligned} \tag{10}$$

Let $\mathscr{X}^\varsigma = (\mathscr{X}^1_\varsigma,\ \mathscr{X}^2_\varsigma)$ be the optimal solution of Problem κ of type (10). Define

$$H_{t+1} = \sum_{\kappa=1}^{T} q_\kappa \cdot \mathscr{X}^1_\varsigma, \quad \text{and} \quad h_{t+1} = \sum_{\kappa=1}^{T} q_\kappa \cdot \mathscr{X}^2_\varsigma d^{\kappa\prime}.$$

4°. Let $\omega^\varsigma = h_{t+1} + H_{t+1} x^{\varsigma\prime}$. If $\varepsilon^\varsigma < \omega^\varsigma$, stop; x^ς is an optimal solution. Otherwise, set $t = t+1$, add to the constraint set (9), and return to Step 1°.

For the continuous or infinitely discrete demand, based on the above finite realization algorithm, Monte Carlo sampling approximation approach can be used to solve the inventory problem. The basic ideas of the approach are to generates a finite number of samples according to the given demand distribution and then use the above finite realization algorithm to obtain computable approximation. Besides these approaches, other methods may be also used to solve the inventory problem, such as the bounding approximation approach, the subgradient method and other nonlinear convex optimization methods, which may be more effective for some special cases.

5 Conclusions

This paper addresses a two-stage stochastic inventory model for a single-period seasonal or fashion product inventory system consisting of multiple stocking echelons in series and retailers. In the inventory system, the joint distribution of the demands arriving retailers is a general distribution. The customer demand arriving at a retailer is first satisfied with the retailer's stock. In case a shortage occurs at the retailer's inventory, the excess demand can be partially satisfied by shipping and converting the upstream echelon's inventory items based on the customers' different willingness to wait. The objective is to seek initial stock quantities placed at various stock points so as to maximize the expected profit of the entire system. By transforming model, an equivalent two-stage stochastic linear program model is developed to solve this nonlinear stochastic inventory problem, which makes the inventory problem easier to analyze. We characterize an optimal placement policy of this inventory system, prove its existence and give its necessary and sufficient conditions. Finally, we propose an effective algorithm for finding optimal placement policies.

In a few cases, perhaps some retailers are allowed to obtain the finished products from its neighboring retailers. By some minor modifications (only adding some linear parts of new shipping variables representing the shipments between these retailers into the corresponding constraints and objective function) to our inventory model, our model can include this case and the results and algorithm proposed in this paper still hold.

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