Computing an Integer Point of a Class of Polytopes with an Arbitrary Starting Variable Dimension Algorithm

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Abstract *An arbitrary starting variable dimension algorithm is developed for computing an integer point of a polytope, $P = \{x \mid Ax \leq b\}$, which satisfies that each row of A has at most one positive entry. The algorithm is derived from an integer labelling rule and a triangulation of the space. It consists of two phases, one of which forms a variable dimension algorithm and the other a full-dimensional pivoting procedure. Starting at an arbitrary integer point, the algorithm interchanges from one phase to the other, if necessary, and follows a finite simplicial path that either leads to an integer point of the polytope or proves that no such point exists.*

Keywords Integer Point, Polytope, Integer Programming, Integer Labelling, Triangulation, Variable Dimension Algorithm, Full-Dimensional Pivoting Procedure

1 Introduction

The problem we consider in this paper is find an integer point of a polytope given by

$$P = \{x \in R^n \mid Ax \leq b\}, \tag{1}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

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satisfies that each row has at most one positive entry, and $b = (b_1, b_2, \cdots, b_m)^\top$. The problem is very general though it looks special. In fact, if A is an arbitrary integer $(n+1) \times n$ matrix satisfying that there is a positive vector ρ such that $\rho^\top A = 0$ and that any $n \times n$ submatrix of A is nonsingular, a procedure given in [19] shows that applying the following three elementary column operations to A,

1. interchange two columns,

2. multiply a column by -1,

3. add any integer times a column to another column,

one can transform A into a matrix such that each row has at most one positive entry. There seems no way to transform with a unimodular matrix an arbitrary $m \times n$ integer matrix into a matrix having at most one positive entry in each row, however, finding an integer point of a polytope can be reduced to finding an integer point of a simplex through applying aggregation techniques ([28]). The problem in general is an NP-hard problem though, for a special case, a polynomial-time algorithm has been developed in [24].

Simplicial methods were originated by Scarf in [20] for computing fixed points of a continuous mapping. After Scarf's work some substantial developments based on simplicial subdivisions were made (e.g., [1], [2], [3], [9], [10], [11], [12], [14], [15], [16], [17], [18], [20], [21], [25], [26], etc. For a general polytope, by applying primitive sets, Scarf defines in [22] and [23] a path that either leads to an integer point of the polytope or proves that no such point exists. After Scarf's development, the question whether it is possible to find an integer point of a simplex with a simplicial approach was raised. A few attempts were made in [4], [5] and [27]. A positive answer was completely realized by us in [6] only after Scarf brought our attention to Pnueli's work [19] during our visit to Yale in 1994 and we observed a beautiful property given in Lemma 1 of [6]. After this work, some significant improvements were made in [7] and [8].

In this paper we develop an arbitrary starting variable dimension algorithm for computing an integer point of (1). It is derived from an elegant integer labelling rule and a triangulation of the space. The algorithm is composed of two phases, one of which forms a variable dimension algorithm and the other a full-dimensional pivoting procedure. The algorithm starts at an arbitrary integer point, interchanges from one phase to the other, and follows a finite simplicial path that either leads to an integer point of the polytope or proves that no such point exists within a finite number of iterations. Numerical results show that the algorithm is very efficient.

The rest of this paper is organized as follows. In Section 2, we introduce the integer labelling rule. In Section 3, we describe the algorithm and prove its convergence.

2 Integer Labelling

Let $a_i^\top$ denote the ith row of A for $i = 1, 2, \ldots, m$. Let $M = \{1, 2, \cdots, m\}$,$N = \{1, 2, \cdots, n\}$, and $N_0 = \{1, 2, \ldots, n+1\}$. We assume that P is bounded and has an interior point. As a direct consequence of the assumption, one can easily obtain the following lemma.

Lemma 1.

- *For any nonzero vector $\xi \in R^n$, there are i and j satisfying that $a_i^\top \xi < 0$ and $0 < a_j^\top \xi$.*
- *There is a vector $\rho = (\rho_1, \rho_2, \dots, \rho_m)^\top > 0$ satisfying $\rho^\top A = 0$.*

Let $\eta = (\eta_1, \eta_2, \dots, \eta_n)^\top$ be an arbitrary integer point of R^n, which will be the starting point of the algorithm. For $j = 1, 2, \dots, n$, let u^j denote the jth unit vector of R^n. Let $e = (1, 1, \dots, 1)^\top$ and $h(n+1) = e$. For $j = 1, 2, \dots, n$, let $h(j) = -u^j$. For any proper subset $K \subset N_0$, let

$$G(\eta, K) = \{\eta + \sum_{j \in K} \lambda_j h(j) \mid 0 \leq \lambda_j,\ j \in K\}.$$

Clearly, $\cup_{j \in N_0} G(\eta, N_0 \backslash \{j\}) = R^n$, and for any two subsets $K_1 \subset N_0$ and $K_2 \subset N_0$, the intersection of $G(\eta, K_1)$ and $G(\eta, K_2)$, $G(\eta, K_1) \cap G(\eta, K_2)$, is a common face of both of them. Thus, $\{G(\eta, K) \mid K \subset N_0\}$ forms a partition of R^n.

To implement the algorithm, we need a triangulation of R^n that subdivides every integer unit cube of R^n into integer simplices, and $G(\eta, K)$ into integer simplices for any subset $K \subset N_0$. Here, an integer unit cube is a unit cube having only integer vertices and an integer simplex is a simplex having only integer vertices. There are several triangulations of R^n suitable for this purpose, which include the K_1-triangulation in [13], the J_1-triangulation in [25], a modification of the D_1-triangulation in [3], etc. A specific choice of the triangulation plays however no dominant role at all in this paper though efficiency of the algorithm may depends on the underlying triangulation. For simplicity, we choose the K_1-triangulation as an underlying triangulation of the algorithm. For completeness of the algorithm, we introduce the K_1-triangulation here.

A simplex of the K_1-triangulation of R^n is the convex hull of $n+1$ vectors, y^0, y^1, …, y^n, given by $y^0 = y$ and $y^k = y^{k-1} + u^{\pi(k)}$, $k = 1, 2, \dots, n$, where y is an integer point of R^n and $\pi = (\pi(1), \pi(2), \dots, \pi(n))$ a permutation of elements of $N = \{1, 2, \dots, n\}$. Let K_1 be the set of all such simplices. Since a simplex of the K_1-triangulation is uniquely determined by y and π, we use $K_1(y, \pi)$ to denote it.

We say that two simplices of K_1 are adjacent if they have a common facet. We show how to generate all the adjacent simplices of a simplex of the K_1-triangulation of R^n in the following. For a given simplex $\sigma = K_1(y, \pi)$ with vertices y^0, y^1, …, y^n, its adjacent simplex opposite to a vertex, say y^i, is given by $K_1(\bar{y}, \bar{\pi})$, where $\bar{y}$ and $\bar{\pi}$ are generated in the following table.

Pivot Rules of the K_1-Triangulation

i	$\bar{y}$	$\bar{\pi}$
0	$y + u^{\pi(1)}$	$(\pi(2), \dots, \pi(n), \pi(1))$
$1 < i < n$	y	$(\pi(1), \dots, \pi(i+1), \pi(i), \dots, \pi(n))$
n	$y - u^{\pi(n)}$	$(\pi(n), \pi(1), \dots, \pi(n-1))$

Let $\mathcal{K}_1$ be the set of faces of simplices of K_1. A q-dimensional simplex of $\mathcal{K}_1$ with vertices y^0, y^1, ..., y^q is denoted by $< y^0, y^1, \ldots, y^q >$. The restriction of $\mathcal{K}_1$ on $G(\eta, K)$ for any subset $K \subset N_0$ is given by

$$\mathcal{K}_1|G(\eta,K) = \{\sigma \in \mathcal{K}_1 \mid \sigma \subset G(\eta,K) \text{ and } \dim(\sigma) = |K|\},$$

where $|\cdot|$ denotes the cardinality of a set and $\dim(\cdot)$ the dimension of a set. Obviously, $\mathcal{K}_1|G(\eta,K)$ is a triangulation of $G(\eta,K)$.

For $\sigma \in \mathcal{K}_1$, let $\text{grid}(\sigma) = \max\{\|x - y\| \mid x \in \sigma \text{ and } y \in \sigma\}$, where $\|\cdot\|$ denotes the infinity norm. We define $\text{mesh}(K_1) = \max_{\sigma \in \mathcal{K}_1} \text{grid}(\sigma)$. Then it is clear that $\text{mesh}(K_1) = 1$.

For $x \in R^n$, let

$$f(x) = \begin{cases} 0 \in R^n & \text{if } x \in P, \\ \sum_{j \in J(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j & \text{if } x \notin P, \end{cases}$$

where $J(x) = \{j \in M \mid a_j^\top x - b_j > 0\}$. From the definition of $f(x)$, one can see that

$$f(x) = (\sum_{j \in J(x)} \frac{a_j a_j^\top}{a_j^\top a_j})x - \sum_{j \in J(x)} \frac{b_j}{a_j^\top a_j} a_j.$$

Clearly, $f : R^n \to R^n$ is a continuous piece-wise linear mapping, which is composed of a finite number of affinely linear pieces since there can be only a finite number of different $J(x)$'s. Therefore, there exists some $L > 0$ such that

$$\|f(x) - f(y)\| \le L\|x - y\|$$

for any x and y of R^n.

Lemma 2. *For any given $x^* \in R^n$, $\frac{(x-x^*)^\top f(x)}{\|x\|} \to \infty$ as $\|x\| \to \infty$.*

Proof. Let x^0 be any given point of P. Since P is bounded, there is a ball $B(x^0, r)$ containing P strictly. Let $S(x^0, r)$ be the sphere of the ball. Then, for any $x \notin B(x^0, r)$, there exists some point $y \in S(x^0, r)$ and some number $\rho > 1$ satisfying that $x = x^0 + \rho(y - x^0)$. Thus, for any k,

$$\begin{aligned} & a_k^\top x - b_k \\ = \ & a_k^\top (x^0 + \rho(y - x^0)) - b_k \\ = \ & \rho(a_k^\top y - b_k) + (\rho - 1)(b_k - a_k^\top x^0) \\ \ge \ & \rho(a_k^\top y - b_k) \end{aligned}$$

since $b_k \ge a_k^\top x^0$ and $\rho > 1$. Therefore, if $k \in J(y)$, then $a_k^\top x - b_k$ approaches infinity as $\rho \to \infty$ since $a_k^\top y - b_k > 0$. Observe that $J(y)$ is not empty for any

$y \in S(x^0, r)$ and that, for any $x \notin P$ with $x \neq 0$,

$$\begin{aligned}\frac{(x-x^*)^\top f(x)}{\|x\|} &= \sum_{j \in J(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j \|x\|}(x - x^*)^\top a_j \\ &= \sum_{j \in J(x)} \frac{(a_j^\top x - b_j)^2}{a_j^\top a_j \|x\|} + \sum_{j \in J(x)} \frac{(a_j^\top x - b_j)(b_j - a_j^\top x^*)}{a_j^\top a_j \|x\|}.\end{aligned}$$

Thus, $\frac{(x-x^*)^\top f(x)}{\|x\|} \to \infty$ as $\|x\| \to \infty$. The lemma follows. □

Applying $f(x)$, we obtain the following integer labelling rule for assigning an integer to each point of R^n in the algorithm.

Definition 1. *For $x \in R^n$, we assign to x an integer $l(x)$ given by $l(x) = 0$ if $f(x) = 0$, and*

$$l(x) = \begin{cases} \max\{k \mid f_k(x) = \max_{1 \leq j \leq n} f_j(x)\} & \text{if } f_j(x) > 0 \text{ for some } j \in N, \\ n+1 & \text{if } f(x) \leq 0 \text{ and } f(x) \neq 0. \end{cases}$$

Based on this definition, the next definition gives us a few notations that will be used in our further developments.

Definition 2.

- *A q-dimensional simplex $\sigma =< y^0, y^1, \ldots, y^q >$ of $\mathcal{K}_1$ is complete if $l(y^i) \neq l(y^j)$ for $0 \leq i < j \leq q$, and $l(y^k) \neq 0$, $k = 0, 1, \ldots, q$.*
- *A q-dimensional simplex $\sigma =< y^0, y^1, \ldots, y^q >$ of $\mathcal{K}_1$ is 0-complete if $l(y^i) \neq l(y^j)$ for $0 \leq i < j \leq q$, and there is some k satisfying that $l(y^k) = 0$.*
- *A q-dimensional simplex $\sigma =< y^0, y^1, \ldots, y^q >$ of $\mathcal{K}_1$ is almost complete if labels of $q+1$ vertices of σ consist of q different nonzero integers.*

From Definition 2, it is easy to see that an almost complete simplex has exactly two complete facets.

Let S be a subset of R^n. For any given $x \in R^n$, the distance between x and S is given by $d(x, S) = \sup_{y \in S} \|x - y\|$. For any two subsets of R^n, S and T, and any nonnegative scalar δ, let

$$\Delta(S, T, \delta) = \{x \in T \mid d(x, S) \leq \delta\}.$$

Theorem 1. *There exists a positive scalar δ satisfying that all the complete n-dimensional simplices of K_1 are contained in $\Delta(x^0, R^n, \delta)$, where x^0 is any given point of P.*

Proof. Let $\sigma =< y^0, y^1, \cdots, y^n >$ be any complete n-dimensional simplex of K_1. Without loss of generality, we assume $l(y^0) = n+1$ and $l(y^i) = i$, $i = 1, 2, \cdots, n$. Let

x be an arbitrary point of σ. Note that $f_i(y^i) > 0$ and $f_i(y^0) \leq 0$, $i = 1, 2, \cdots, n$. Then, for $i = 1, 2, \cdots, n$,

$$f_i(x) = f_i(x) - f_i(y^i) + f_i(y^i) \geq f_i(x) - f_i(y^i) \geq -L\|x - y^i\| \geq -L$$

and

$$f_i(x) = f_i(x) - f_i(y^0) + f_i(y^0) \leq f_i(x) - f_i(y^0) \leq L\|x - y^0\| \leq L$$

since $\text{mesh}(K_1) = 1$. Thus,

$$-L \leq f_i(x) \leq L,\ i = 1, 2, \cdots, n. \tag{2}$$

Therefore,

$$\|f(x)\| \leq L.$$

This implies that any complete n-dimensional simplex is contained in

$$W_c = \{x \mid \|f(x)\| \leq L\}.$$

From (2), we obtain that, for any $x \in W_c$,

$$-L|x_i| \leq x_i f_i(x) \leq L|x_i|$$

for $i = 1, 2, \cdots, n$. Thus, for any $x \in W_c$,

$$-L\sum_{i=1}^{n} |x_i| \leq x^\top f(x) \leq L\sum_{i=1}^{n} |x_i|.$$

Therefore, for any $x \in W_c$,

$$-L \leq \frac{x^\top f(x)}{\|x\|_1} \leq L, \tag{3}$$

where $\|x\|_1 = \sum_{i=1}^{n} |x_i|$. Combining Lemma 2 and (3) together, one can see that W_c is bounded. The theorem follows. □

Lemma 3. *Let*

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

be a matrix such that $q_{ij} \leq 0$ for any $i \neq j$ and $q_{ii} > 0$, $i = 1, 2, \cdots, n$. If there is some $\rho = (\rho_1, \rho_2, \cdots, \rho_n)^\top > 0$ satisfying that $\rho^\top Q > 0$, then Q is nonsingular and $Q^{-1} \geq 0$.

Proof. We prove the lemma by the mathematical induction.

1. When $n = 1$, $Q = (q_{11})$. Since $q_{11} > 0$, hence, Q is nonsingular and $Q^{-1} = (1/q_{11}) \geq 0$. The lemma is true.

2. Assume that the lemma is true for $n = m - 1$. Consider $n = m$. Let

$$U = \begin{pmatrix} 1 & & & \\ -q_{21}/q_{11} & 1 & & \\ \vdots & & \ddots & \\ -q_{n1}/q_{11} & & & 1 \end{pmatrix}$$

and

$$W = \begin{pmatrix} 1 & -q_{12}/q_{11} & \cdots & -q_{1n}/q_{11} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

The inverse matrix of U is given by

$$U^{-1} = \begin{pmatrix} 1 & & & \\ q_{21}/q_{11} & 1 & & \\ \vdots & & \ddots & \\ q_{n1}/q_{11} & & & 1 \end{pmatrix}.$$

Note that $-q_{i1}/q_{11} \geq 0$ and $-q_{1i}/q_{11} \geq 0$, $i = 2, 3, \cdots, n$. Multiplying U to the left side of Q, we obtain

$$UQ = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ 0 & q_{22} - \frac{q_{21}q_{12}}{q_{11}} & \cdots & q_{2n} - \frac{q_{21}q_{1n}}{q_{11}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & q_{n2} - \frac{q_{n1}q_{12}}{q_{11}} & \cdots & q_{nn} - \frac{q_{n1}q_{1n}}{q_{11}} \end{pmatrix}.$$

Clearly, all the entries of UQ except its diagonal entries are non-positive. Multiplying ρ to U^{-1}, we obtain

$$\rho^\top U^{-1} = (\rho_1 + \sum_{i=2}^{n} \frac{q_{i1}\rho_i}{q_{11}}, \rho_2, \cdots, \rho_n)^\top.$$

From $0 < q_{11}$ and $0 < \rho^\top Q = (\rho^\top U^{-1})(UQ)$, we derive that

$$\rho_1 + \sum_{i=2}^{n} \frac{q_{i1}\rho_i}{q_{11}} > 0$$

and all the diagonal entries of UQ are positive. Let $\bar{\rho} = (\rho_2, \cdots, \rho_n)^\top$. By deleting the first row and the first column of UQ, we obtain an $(m-1) \times (m-1)$ matrix,

$$\bar{Q} = \begin{pmatrix} q_{22} - \frac{q_{21}q_{12}}{q_{11}} & \cdots & q_{2n} - \frac{q_{21}q_{1n}}{q_{11}} \\ \vdots & \ddots & \vdots \\ q_{n2} - \frac{q_{n1}q_{12}}{q_{11}} & \cdots & q_{nn} - \frac{q_{n1}q_{1n}}{q_{11}} \end{pmatrix}.$$

Since $0 < \bar{\rho}$, $0 < \bar{\rho}^\top \bar{Q}$, and $\bar{Q}$ is an $(m-1) \times (m-1)$ matrix, it follows from the hypothesis that $\bar{Q}$ is nonsingular and $\bar{Q}^{-1} \geq 0$. Multiplying W to the right side of UQ, we obtain

$$UQW = \begin{pmatrix} q_{11} & 0 \\ 0 & \bar{Q} \end{pmatrix}.$$

Thus, Q is nonsingular and

$$Q^{-1} = W \begin{pmatrix} 1/q_{11} & 0 \\ 0 & \bar{Q}^{-1} \end{pmatrix} U.$$

Therefore, $Q^{-1} \geq 0$ since $q_{11} > 0$, $\bar{Q}^{-1} \geq 0$, $U \geq 0$, and $W \geq 0$. The lemma follows.

□

As a corollary of Lemma 3, we obtain the following result.

Corollary 1. *For any $x \in R^n$, if $0 < f(x)$, then $0 < x - x^0$ for any $x^0 \in P$.*

Proof. Let $J_i(x) = \{j \in J(x) \mid a_{ji} > 0\}$, $i = 1, 2, \cdots, n$, and $J_{n+1}(x) = \{j \in J(x) \mid a_j \leq 0\}$. Then, $J_1(x)$, $J_2(x)$, $\cdots$, $J_{n+1}(x)$ forms a partition of $J(x)$. Since $f(x) > 0$, hence, $J_i(x) \neq \emptyset$, $i = 1, 2, \cdots, n$. Let

$$r_i(x) = \sum_{j \in J_i(x)} \frac{(a_j^\top x - b_j)^2}{a_j^\top a_j}$$

$i = 1, 2, \cdots, n$, and $r(x) = (r_1(x), r_2(x), \cdots, r_n(x))^\top$. Clearly, $r(x) > 0$. Let

$$\bar{a}_i(x) = \frac{\sum_{j \in J_i(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j}{r_i(x)},$$

$i = 1, 2, \cdots, n$, and $\bar{A}(x) = (\bar{a}_1(x), \bar{a}_2(x), \cdots, \bar{a}_n(x))$. Since $0 < f(x)$ and

$$\sum_{j \in J_{n+1}(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j \leq 0,$$

hence,

$$\begin{aligned} 0 < \quad & f(x) - \textstyle\sum_{j \in J_{n+1}(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j = \sum_{j \in J(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j - \sum_{j \in J_{n+1}(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j \\ = \quad & \textstyle\sum_{i=1}^n \sum_{j \in J_i(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j = \sum_{i=1}^n \frac{\sum_{j \in J_i(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j}{r_i(x)} r_i(x) \\ = \quad & \textstyle\sum_{i=1}^n \bar{a}_i(x) r_i(x) = \bar{A}(x) r(x). \end{aligned}$$

Note that each row and each column of $\bar{A}(x)$ have exactly one positive entry. According to Lemma 3, $\bar{A}(x)$ is nonsingular and $\bar{A}(x)^{-1} \geq 0$. From the definition of $r(x)$, we obtain that

$$r(x) = \begin{pmatrix} \sum_{j\in J_1(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j}(a_j^\top(x-x^0) + a_j^\top x^0 - b_j) \\ \sum_{j\in J_2(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j}(a_j^\top(x-x^0) + a_j^\top x^0 - b_j) \\ \vdots \\ \sum_{j\in J_n(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j}(a_j^\top(x-x^0) + a_j^\top x^0 - b_j) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j\in J_1(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top \\ \sum_{j\in J_2(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top \\ \vdots \\ \sum_{j\in J_n(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top \end{pmatrix}(x-x^0) + \begin{pmatrix} \sum_{j\in J_1(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j}(a_j^\top x^0 - b_j) \\ \sum_{j\in J_2(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j}(a_j^\top x^0 - b_j) \\ \vdots \\ \sum_{j\in J_n(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j}(a_j^\top x^0 - b_j) \end{pmatrix}.$$

Let

$$s(x^0) = \begin{pmatrix} \sum_{j\in J_1(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j}(a_j^\top x^0 - b_j) \\ \sum_{j\in J_2(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j}(a_j^\top x^0 - b_j) \\ \vdots \\ \sum_{j\in J_n(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j}(a_j^\top x^0 - b_j) \end{pmatrix}.$$

Then, $s(x^0) \leq 0$ since $x^0 \in P$. Thus,

$$0 < r(x) - s(x^0) = \begin{pmatrix} \sum_{j\in J_1(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top \\ \sum_{j\in J_2(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top \\ \vdots \\ \sum_{j\in J_n(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top \end{pmatrix}(x-x^0).$$

Let

$$R(x) = \begin{pmatrix} r_1(x) & & & \\ & r_2(x) & & \\ & & \ddots & \\ & & & r_n(x) \end{pmatrix}.$$

Then,

$$R(x)\bar{A}(x)^\top = \begin{pmatrix} \sum_{j\in J_1(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top \\ \sum_{j\in J_2(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top \\ \vdots \\ \sum_{j\in J_n(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top \end{pmatrix}.$$

Therefore,

$$\begin{aligned} x - x^0 &= \begin{pmatrix} \sum_{j\in J_1(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top \\ \sum_{j\in J_2(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top \\ \vdots \\ \sum_{j\in J_n(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top \end{pmatrix}^{-1} (r(x) - s(x^0)) \\ &= (\bar{A}(x)^{-1})^\top R(x)^{-1}(r(x) - s(x^0)) > 0 \end{aligned}$$

since $\bar{A}(x)^{-1} \geq 0$. The corollary follows. □

Lemma 4. *For any $x^0 \in P$, $C(x^0) = R^n \backslash \{x \in R^n \mid x^0 \leq x\}$ contains at most a finite number of almost complete n-dimensional simplices carrying only integer labels in N.*

Proof. Let $\sigma =< y^0, y^1, \cdots, y^n >$ be any almost complete n-dimensional simplex of K_1 carrying only integer labels in N. Without loss of generality, we assume that $l(y^0) = k$ and $l(y^i) = i$, $i = 1, 2, \cdots, n$, and that $f_k(y^0) \leq f_i(y^i)$, $i = 1, 2, \cdots, n$. Note that $f_k(y^0) > 0$ and $f_i(y^i) > 0$, $i = 1, 2, \cdots, n$. Let x be an arbitrary point of σ. Then, for $i = 1, 2, \cdots, n$,

$$f_i(x) - f_k(y^0) = f_i(x) - f_i(y^i) + f_i(y^i) - f_k(y^0) \geq f_i(x) - f_i(y^i) \geq -L\|x - y^i\| \geq -L$$

and

$$f_i(x) - f_k(y^0) = f_i(x) - f_i(y^0) + f_i(y^0) - f_k(y^0) \leq f_i(x) - f_i(y^0) \leq L\|x - y^0\| \leq L$$

since $f_i(y^i) - f_k(y^0) \geq 0$, $f_i(y^0) - f_k(y^0) \leq 0$, and $\text{mesh}(K_1) = 1$. Thus,

$$-L \leq f_i(x) - f_k(y^0) \leq L, \; i = 1, 2, \cdots, n. \tag{4}$$

Therefore,

$$\|f(x) - f_k(y^0)e\| \leq L.$$

This implies that any almost complete n-dimensional simplex carrying only integer labels in N is contained in

$$W_a = \{x \mid \|f(x) - \mu e\| \leq L \text{ for some } \mu > 0\}.$$

Let x be any point of R^n satisfying $f(x) = \mu e > 0$. From Corollary 1, we know that $x > x^0$. Thus, $a_j^\top x \leq a_j^\top x^0 \leq b_j$ for any $a_j \leq 0$. Let $J_i(x) = \{j \in J(x) \mid a_{ji} > 0\}$, $i = 1, 2, \cdots, n$. Then, $J_1(x)$, $J_2(x)$, $\cdots$, $J_n(x)$ forms a partition of $J(x)$. Since $f(x) > 0$, hence, $J_i(x) \neq \emptyset$, $i = 1, 2, \cdots, n$. For $i = 1, 2, \cdots, n$, let j_i be any given index of $J_i(x)$ satisfying $a_{j_i}^\top x - b_{j_i} = \min_{j \in J_i(x)} a_j^\top x - b_j$. Clearly, for any $j \in J_i(x)$, there exists $r_j \geq 1$ satisfying $a_j^\top x - b_j = r_j(a_{j_i}^\top x - b_{j_i})$. Thus,

$$\sum_{j \in J_i(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j = \sum_{j \in J_i(x)} \frac{r_j}{a_j^\top a_j} a_j (a_{j_i}^\top x - b_{j_i}) = (\sum_{j \in J_i(x)} \frac{r_j}{a_j^\top a_j} a_j)(a_{j_i}^\top x - b_{j_i}).$$

Let $d_{j_i} = \sum_{j \in J_i(x)} \frac{r_j}{a_j^\top a_j} a_j$ and $D = (d_{j_1}, d_{j_2}, \cdots, d_{j_n})^\top$. Note that each row and each column of D have exactly one positive entry. From Lemma 3, we know that D is nonsingular and $D^{-1} \geq 0$. From $f(x) = \mu e > 0$, we get that

$$\begin{aligned} 0 < \mu e = \quad & \textstyle\sum_{j \in J(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j \\ = \quad & \textstyle\sum_{i=1}^n \sum_{j \in J_i(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j \\ = \quad & \textstyle\sum_{i=1}^n (\sum_{j \in J_i(x)} \frac{r_j}{a_j^\top a_j} a_j)(a_{j_i}^\top x - b_{j_i}) \\ = \quad & D(a_{j_1}^\top x - b_{j_1}, a_{j_2}^\top x - b_{j_2}, \cdots, a_{j_n}^\top x - b_{j_n})^\top. \end{aligned}$$

Therefore,

$$(a_{j_1}^\top x - b_{j_1}, a_{j_2}^\top x - b_{j_2}, \cdots, a_{j_n}^\top x - b_{j_n})^\top = \mu D^{-1} e > 0$$

since $D^{-1} \geq 0$. Let $\bar{A}^\top = (a_{j_1}, a_{j_2}, \cdots, a_{j_n})$ and $\bar{b} = (b_{j_1}, b_{j_2}, \cdots, b_{j_n})^\top$. Then,

$$\bar{A}x - \bar{b} = (a_{j_1}^\top x - b_{j_1}, a_{j_2}^\top x - b_{j_2}, \cdots, a_{j_n}^\top x - b_{j_n})^\top.$$

Thus,

$$\bar{A}x = \mu D^{-1} e + \bar{b}.$$

Since each row and each column of $\bar{A}$ have exactly one positive entry, from Lemma 3, we obtain that $\bar{A}$ is nonsingular, $\bar{A}^{-1} \geq 0$, and $\bar{A}^{-1} D^{-1} e > 0$. Therefore,

$$x = \mu \bar{A}^{-1} D^{-1} e + \bar{A}^{-1} \bar{b}. \tag{5}$$

Let $\Gamma = \{x \mid f(x) = \mu e,\ \mu > 0\}$, and $\Gamma(\mu) = \{x \mid f(x) = \mu e\}$ for any given $\mu > 0$. Note that $\Gamma = \cup_{\mu > 0} \Gamma(\mu)$. From (5), it is clear that $\Gamma(\mu)$ contains a finite number of points for any given $\mu > 0$. Let ϵ be a given positive number such that

$$\{z \mid \|x - z\| < \epsilon\} \cap \{z \mid \|y - z\| < \epsilon\} = \emptyset$$

for any x and y of $\Gamma(\mu)$ with $x \neq y$. For any given $\mu > 0$, let

$$W_a(\mu) = \{x \mid \|f(x) - \mu e\| \leq L\}.$$

Then,

$$W_a = \cup_{0<\mu} W_a(\mu).$$

Let

$$W_a(\mu, \epsilon) = W_a(\mu) \backslash (\cup_{x \in \Gamma(\mu)} \{z \mid \|z - x\| < \epsilon\}).$$

Then, for any $y \in W_a(\mu, \epsilon)$ and $x \in \Gamma(\mu)$, there exists some positive number ν_ϵ satisfying

$$\|f(y) - f(x)\| \geq \nu_\epsilon \|y - x\|,$$

where

$$\nu_\epsilon = \text{argmin}\{\nu \mid \|f(y) - f(x)\| = \nu \|x - y\| \text{ for some } y \in W_a(\mu, \epsilon) \text{ and } x \in \Gamma(\mu)\}$$

is independent of μ. Since $\|f(y) - \mu e\| \leq L$ for any $y \in W_a(\mu, \epsilon)$, hence, for any $y \in W_a(\mu, \epsilon)$ and $x \in \Gamma(\mu)$,

$$L \geq \|f(y) - \mu e\| = \|f(y) - f(x)\| \geq \nu_\epsilon \|y - x\|.$$

Therefore, for any $y \in W_a(\mu, \epsilon)$ and $x \in \Gamma(\mu)$,

$$\|x - y\| \leq \frac{L}{\nu_\epsilon}.$$

Let $\delta = \max\{\epsilon, \frac{L}{\nu_\epsilon}\}$, which is independent of μ. Then,

$$W_a(\mu) \subseteq \Delta(\Gamma(\mu), R^n, \delta).$$

Thus,

$$W_a = \cup_{\mu>0} W_a(\mu) \subseteq \cup_{\mu>0} \Delta(\Gamma(\mu), R^n, \delta) \subseteq \Delta(\Gamma, R^n, \delta).$$

Therefore, all the almost complete n-dimensional simplices carrying only integer labels in N are contained in $\Delta(\Gamma, R^n, \delta)$. Consider $C(x^0) \cap \Delta(\Gamma, R^n, \delta)$. From (5), we know that, as $\mu \to \infty$, every component of x satisfying $f(x) = \mu e$ approaches infinity. It implies that $C(x^0) \cap \Delta(\Gamma, R^n, \delta)$ is bounded. The lemma follows. □

For any positive number δ and any nonempty subset $K \subset N_0$, let

$$\Lambda(\eta, \delta, K) = \{x \in G(\eta, K) \mid \|x - \eta\| \leq \delta\}.$$

Lemma 5. *There is a sufficiently large positive number δ_0 satisfying that, for any nonempty subset $K \subset N_0$, all the almost complete $|K|$-dimensional simplices in $G(\eta, K)$ carrying only the integer labels in K are contained in $\Lambda(\eta, \delta_0, K)$.*

Proof. Consider $n + 1 \notin K$. Let

$$D(K) = \{x \in G(\eta, K) \mid f_k(x) = \mu,\ k \in K,\ \text{for some } \mu > 0\}.$$

From the definition of $G(\eta, K)$, we know that $x_k - \eta_k \leq 0$ for any $k \in K$ and $x_k - \eta_k = 0$ for any $k \notin K$. Then, for any $x \in G(\eta, K)$,

$$(x - \eta)^\top f(x) = \sum_{k \in K} (x_k - \eta_k) f_k(x).$$

Thus, for any $x \in D(K)$,

$$(x - \eta)^\top f(x) \leq 0$$

since $f_k(x) > 0$ and $x_k - \eta_k \leq 0$ for any $k \in K$. According to Lemma 2, $(x - \eta)^\top f(x) > 0$ when $\|x\|$ is sufficiently large. Therefore, $D(K)$ is bounded. Similar to the proof in Lemma 4, one can show that, when δ is sufficiently large, all the almost complete $|K|$-dimensional simplices in $G(\eta, K)$ carrying only the integer labels in K are contained in

$$\Delta(D(K), G(\eta, K), \delta) = \{x \in G(\eta, K) \mid d(x, D(k)) \leq \delta\}.$$

The boundedness of $D(K)$ implies that $\Delta(D(K), G(\eta, K), \delta)$ is bounded.

Consider $n + 1 \in K$. Let

$$C(K) = \{x \in G(\eta, K) \mid f_k(x) = 0,\ k \in K \backslash \{n + 1\},\ \text{and } f_k(x) \leq 0,\ k \notin K\}.$$

From the definition of $G(\eta, K)$, we know that, for any $x \in G(\eta, K)$, there are $0 \leq \lambda(x)$ and $0 \leq \gamma_k(x)$, $k \in K\backslash\{n + 1\}$, such that $x_k - \eta_k = \lambda(x) - \gamma_k(x)$ for any $k \in K\backslash\{n + 1\}$ and $x_k - \eta_k = \lambda(x)$ for any $k \notin K$. Then, for any $x \in G(\eta, K)$,

$$(x - \eta)^\top f(x) = \sum_{k \in K\backslash\{n+1\}} (\lambda(x) - \gamma_k(x)) f_k(x) + \sum_{k \notin K} \lambda(x) f_k(x).$$

Thus, for any $x \in C(K)$,

$$(x - \eta)^\top f(x) = \sum_{k \notin K} \lambda(x) f_k(x) \leq 0$$

since $\lambda(x) \geq 0$ and $f_k(x) \leq 0$, $k \notin K$. According to Lemma 2, $(x - \eta)^\top f(x) > 0$ when $\|x\|$ is sufficiently large. Therefore, $C(K)$ is bounded. Similar to the proof in Lemma 4, one can show that, when δ is sufficiently large, all the almost complete $|K|$-dimensional simplices in $G(\eta, K)$ carrying only the integer labels in K are contained in

$$\Delta(C(K), G(\eta, K), \delta) = \{x \in G(\eta, K) \mid d(x, C(K)) \leq \delta\}.$$

The boundedness of $C(K)$ implies that $\Delta(C(K), G(\eta, K), \delta)$ is bounded. The lemma follows. □

Lemma 6. *If $f(x) \leq 0$ and $f(x) \neq 0$, then, for any $x^0 \in P$, there is some k satisfying $x_k - x_k^0 < 0$.*

Proof. Suppose that $x - x^0 \geq 0$. Then,

$$\begin{aligned} 0 \geq (x - x^0)^\top f(x) &= \textstyle\sum_{j \in J(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} a_j^\top (x - x^0) \\ &= \textstyle\sum_{j \in J(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} (a_j^\top x - b_j + b_j - a_j^\top x^0) \\ &\geq \textstyle\sum_{j \in J(x)} \frac{a_j^\top x - b_j}{a_j^\top a_j} (a_j^\top x - b_j) > 0. \end{aligned}$$

A contradiction occurs. The lemma follows. □

For $\xi \in R^n$ and $K \subseteq N$, let

$$H(\xi, K) = \{\xi + x \in R^n \mid 0 \leq x_i,\ i \in K,\ \text{and } x_i = 0,\ i \notin K\}.$$

Lemma 7. *If z^0 is an integer point of P, then, for any $K \subseteq N$, every point $x \in H(z^0, K)$ carries a label of either 0 or an integer of K.*

Proof. From Lemma 6, we know that no point of $H(z^0, K)$ carries integer label $n + 1$. For $x \in H(z^0, K)$, let $\lambda = x - z^0$. Then, $0 \leq \lambda_j$, $j \in K$, and $\lambda_j = 0$, $j \notin K$. Thus, for i with $a_{ij} \leq 0$ for any $j \in K$,

$$\begin{aligned} a_i^\top x &= a_i^\top z^0 + a_i^\top \lambda \\ &\leq b_i + a_i^\top \lambda \\ &= b_i + \textstyle\sum_{j \in K} a_{ij} \lambda_j \\ &\leq b_i. \end{aligned}$$

Therefore, according to Definition 1, no point in $H(z^0, K)$ carries an integer label in $N_0 \backslash K$. The lemma follows. □

As a direct result of Lemma 7, we obtain the following result.

Corollary 2. *If z^0 is an integer point of P, there is no complete n-dimensional simplex in $H(z^0, N)$, and for any $j \in N$, there is no complete $(n-1)$-dimensional simplex in $H(z^0, N \backslash \{j\})$ carrying all integer labels in N.*

3 The Algorithm

Let $x^{\max}$ denote the unique solution of $\max_{x \in P} e^\top x$.

Lemma 8. *For any $x \in P$, $x \leq x^{\max}$.*

Proof. Let $x^1 = (x_1^1, x_2^1, \cdots, x_n^1)^\top$ and $x^2 = (x_1^2, x_2^2, \cdots, x_n^2)^\top$ be two arbitrary points of P. Let $\bar{x} = (\max\{x_1^1, x_1^2\}, \max\{x_2^1, x_2^2\}, \cdots, \max\{x_n^1, x_n^2\})^\top$. Let j be an arbitrary index of M. From the assumption on P, we know that each row of A has at most one positive entry. If $a_j \leq 0$, then

$$a_j^\top \bar{x} \leq a_j^\top x^1 \leq b_j.$$

Consider a_j with a positive entry, say a_{ji}. Without loss of generality, assume $\bar{x}_i = x_i^1$. Since $a_{jk} \leq 0$ for any $k \neq i$, hence,

$$a_j^\top \bar{x} = a_{ji}\bar{x}_i + \sum_{k\neq i} a_{jk}\bar{x}_k = a_{ji}x_i^1 + \sum_{k\neq i} a_{jk}\bar{x}_k \leq a_{ji}x_i^1 + \sum_{k\neq i} a_{jk}x_k^1 = a_j^\top x^1 \leq b_j.$$

Thus, $\bar{x} \in P$.

Suppose that there is a point $\hat{x} \in P$ satisfying $\hat{x}_q > x_q^{\max}$ for some $q \in N$. Let $x = (\max\{\hat{x}_1, x_1^{\max}\}, \max\{\hat{x}_2, x_2^{\max}\}, \cdots, \max\{\hat{x}_n, x_n^{\max}\})^\top$. Then, $x \in P$. Clearly, $e^\top x > e^\top x^{\max}$, which contradicts that $e^\top x^{\max} = \max_{x\in P} e^\top x$. The lemma follows. □

For any number α, let $\lceil \alpha \rceil$ denote the smallest integer greater than or equal to α. We define $x^u = (x_1^u, x_2^u, \ldots, x_n^u)^\top$ with

$$x_i^u = \begin{cases} \lceil x_i^{\max} \rceil & \text{if } x_i^{\max} < \lceil x_i^{\max} \rceil, \\ 1 + \lceil x_i^{\max} \rceil & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$. Then, $x < x^u$ for any $x \in P$.

Applying the results in the previous section and the above results, we have developed an arbitrary starting variable dimension algorithm for computing an integer point of P, which is as follows.

Initialization: Let $K = \emptyset$, $y^0 = \eta$, $\sigma_0 =< y^0 >$, $y^+ = y^0$, $p = 1$, and $k = 0$. Go to **Step 1**.

Step 1: Compute $l(y^+)$. If $l(y^+) = 0$ then the algorithm terminates, and an integer point of P has been found. If $l(y^+) \in K$ then let y^- be the vertex of σ_k other than y^+ and carrying integer label $l(y^+)$, and τ_{k+1} the facet of σ_k opposite to y^-, and go to **Step 2**. If $l(y^+) \notin K$ then go to **Step 3**.

Step 2: If $\tau_{k+1} \subset G(\eta, K\backslash\{j\})$ for some $j \in K$ then $K = K\backslash\{j\}$ and go to **Step 4**. Otherwise, do as follows: Let σ_{k+1} be the unique simplex that is adjacent to σ_k and has τ_{k+1} as a facet. Let y^+ be the vertex of σ_{k+1} opposite to τ_{k+1} and $k = k + 1$. Go to **Step 1**.

Step 3: If $|K| = n$ then go to **Step 5**. Otherwise, do as follows: Let $K = K \cup \{l(y^+)\}$ and $\tau_{k+1} = \sigma_k$. Let σ_{k+1} be the unique $|K|$-dimensional simplex in $G(\eta, K)$ having τ_{k+1} as a facet, and y^+ the vertex of σ_{k+1} opposite to τ_{k+1}. Let $k = k + 1$ and go to **Step 1**.

Step 4: Let $\sigma_{k+1} = \tau_{k+1}$, y^- be the vertex of σ_{k+1} carrying integer label j, and τ_{k+2} the facet of σ_{k+1} opposite to y^-. Let $k = k+1$ and go to **Step 2**.

Step 5: If p is even then let $p = p+1$, j be the index of N_0 satisfying that $\sigma_k \subset G(\eta, N_0 \backslash \{j\})$, y^- the vertex of σ_k carrying integer label j, τ_{k+1} the facet of σ_k opposite to y^-, and $K = N_0 \backslash \{j\}$, and go to **Step 2**. If p is odd, do as follows: Let $p = p+1$, y^- be the vertex of σ_k carrying integer label $n+1$ and τ_{k+1} the facet of σ_k opposite to y^-. Go to **Step 6**.

Step 6: Let σ_{k+1} be the unique simplex that is adjacent to σ_k and has τ_{k+1} as a facet, and y^+ the vertex of σ_{k+1} opposite to τ_{k+1}. Let $k = k+1$ and go to **Step 7**.

Step 7: Compute $l(y^+)$. If $l(y^+) = 0$ then the algorithm terminates, and an integer point of P has been found. If $x^u \leq y^+$ then the algorithm terminates, and there is no integer point in P. If $l(y^+) = n+1$ then go to **Step 5**. If $l(y^+) \neq n+1$ then let y^- be the vertex of σ_k other than y^+ and carrying integer label $l(y^+)$, and τ_{k+1} the facet of σ_k opposite to y^-. Go to **Step 6**.

Note that the first phase (Steps 1-4) of the algorithm comes from Laan and Talman's $(n+1)$-ray algorithm ([17]). In the following we prove the finite convergence of the algorithm.

Theorem 2. *Within a finite number of iterations, the algorithm either yields an integer point of P or proves that no such point exists.*

Proof. For any positive integer μ, let

$$\Lambda(\eta, \mu) = \{x \in R^n \mid \|x - \eta\| \leq \mu\}.$$

Lemma 5 implies that there exists some $\mu_1 > 0$ such that all the simplices generated by the algorithm in Steps 1, 2, 3 and 4 are contained in $\Lambda(\eta, \mu_1)$. Lemma 4 implies there exists some $\mu_2 > 0$ such that all the simplices generated by the algorithm in Steps 5, 6, and 7 are contained in $\Lambda(\eta, \mu_2)$. To show that the algorithm does not cycle, one can simply introduces two undirected graphes in the same way as that in [7]. Then, applying Theorem 1 and Corollary 2 and following a similar argument to that in [7], one can readily obtain the theorem. □

The following example illustrates how the algorithm works.

Example 1. *Find an integer point of P given by*

$$P = \left\{ x = (x_1, x_2)^\top \;\middle|\; \begin{array}{l} -x_1 + x_2 \leq 1/2, \\ x_1 - x_2 \leq 1/2, \\ x_1 \leq 1/2, \\ -x_1 \leq 1/2, \\ x_2 \leq 1/2, \\ -x_2 \leq 1/2 \end{array} \right\}.$$

Let $\eta = (-2, 1)^\top$, $K = \emptyset$, $y^0 = \eta$, $\sigma_0 =< y^0 >$, $y^+ = y^0$, and $p = 1$.

Iteration 1: *Computing $l(y^+)$, we obtain $l(y^+) = 2$. Then $l(y^+) \notin K$. Let $K = K \cup \{2\} = \{2\}$, $\tau_1 = \sigma_0$, $y^1 = (-2, 0)^\top$, $\sigma_1 =< y^0, y^1 >$, and $y^+ = y^1 = (-2, 0)^\top$.*

Iteration 2: *Computing $l(y^+)$, we obtain $l(y^+) = 2$. Then $l(y^+) = l(y^0) \in K$. Let $\tau_2 =< y^1 >$, $y^0 = y^1$, $y^1 = (-2, -1)^\top$, $\sigma_2 =< y^0, y^1 >$, and $y^+ = y^1 = (-2, -1)^\top$.*

Iteration 3: *Computing $l(y^+)$, we obtain $l(y^+) = 3$. Then $l(y^+) \notin K$. Let $K = K \cup \{l(y^+)\} = \{2, 3\}$, $\tau_3 = \sigma_2$, $y^2 = (-1, 0)^\top$, $\sigma_3 =< y^0, y^1, y^2 >$, and $y^+ = y^2 = (-1, 0)^\top$.*

Iteration 4: *Computing $l(y^+)$, we obtain $l(y^+) = 2$. Then $l(y^+) = l(y^0) \in K$. Let $\tau_4 =< y^1, y^2 >$, $y^0 = y^1$, $y^1 = y^2$, $y^2 = (-1, -1)^\top$, $\sigma_4 =< y^0, y^1, y^2 >$, and $y^+ = y^2 = (-1, -1)^\top$.*

Iteration 5: *Computing $l(y^+)$, we obtain $l(y^+) = 3$. Then $l(y^+) = l(y^0) \in K$. Let $\tau_5 =< y^1, y^2 >$, $y^0 = y^1$, $y^1 = y^2$, $y^2 = (0, 0)^\top$, $\sigma_5 =< y^0, y^1, y^2 >$, and $y^+ = y^2 = (0, 0)^\top$.*

Iteration 6: *Computing $l(y^+)$, we obtain $l(y^+) = 0$. An integer point of P has been found.*

One may see from this example that the algorithm will never perform Steps 4, 5, 6, and 7 when $n = 2$. However, the situation is far more complicated when $n > 2$.

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