Analysis of an M/M/1/N Queue with Balking, Reneging and Server Vacations*

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Abstract *This paper presents an analysis for an M/M/1/N queueing system with balking, reneging and server vacations. Arriving customers balk (do not enter) with a probability and renege (leave the queue after entering) according to a negative exponential distribution. It is assumed that the server has a multiple vacation. By using the Markov process method, we first develop the equations of the steady state probabilities. Then, we derive the matrix form solution of the steady-state probabilities. Next, we give some performance measures of the system. Based on the performance analysis, we formulate a cost model to determine the optimal service rate. Finally, we present some numerical examples to demonstrate how the various parameters of the model influence the behavior of the system.*

Keywords Vacation, Balk, Reneging, Queueing system, Steady-state probability, Cost model

1 Introduction

Many practical queueing systems especially those with balking and reneging have been widely applied to many real-life problems, such as the situations involving impatient telephone switchboard customers, the hospital emergency rooms handling critical patients, and the inventory systems with storage of perishable goods [1]. In this paper, we consider an M/M/1/N queueing system with balking and reneging. We also consider the server to have multiple vacations, i.e., the server leaves for a random length whenever the system becomes empty. At the end of a vacation, the server will take another vacation if the system is still empty.

Queueing systems with balking, reneging, or both have been studied by many researchers. Haight [2] first considered an M/M/1 queue with balking. An M/M/1

*This work was supported in part by the National Natural Science Foundation of China (Grant No. 10271102) and the Natural Science Foundation of Hebei Province (Grant No. A2004000185), China, and was supported in part by GRANT-IN-AID FOR SCIENTIFIC RESEARCH (No. 16560350) and MEXT.ORC (2004-2008), Japan.

queue with customers reneging was also proposed by Haight [3]. The combined effects of balking and reneging in an M/M/1/N queue have been investigated by Ancker and Gafarian [4], [5]. Abou-EI-Ata and Hariri [6] considered the multiple servers queueing system M/M/c/N with balking and reneging. Wang and Chang [7] extended this work to study an M/M/c/N queue with balking, reneging and server breakdowns.

Queueing systems with server vacations have attracted much attention from numerous researchers since the paper was presented by Levy and Yechiali [8]. Server vacations are useful for the system where the server wants to utilize his idle time for different purposes. An excellent survey of queueing systems with server vacations can be found in papers by Doshi [9] and Takagi [10]. However, most of the research works about queueing systems have not considered balking, reneging and server vacations together. There was only one paper [11] that we know to consider an $M^X/G/1$ queue with balking involving multiple vacation. Queueing models with server vacations accommodate the real-world situations more closely. Such model frequently occurs in areas of computer and communications, or manufacturing systems. For example, consider an assembly line where a worker may have some idle time between subsequent jobs. To utilize the time effectively, managers can assign secondary jobs to the worker. However, it is important that the worker must return to do his primary jobs when he completes the secondary jobs. This motivates us to study a queueing system with balking, reneging and server vacations.

The rest of this paper is organized as follows. In the next section, we give a description of the queueing model. In Section 3, we derive the steady-state equations by the Markov process method. By writing the transition rate matrix as block matrix, we get the matrix form solution of the steady-state probabilities and present a procedure for calculating the steady-state probabilities. In Section 4, we give some performance measures of the system. Based on the performance analysis, we formulate a cost model to determine the optimal service rate. Some numerical examples are presented to demonstrate how the various parameters of the model influence the behavior of the system. Conclusions are given in Section 5.

2 System Model

In this paper, we consider an M/M/1/N queueing system with balking, reneging and server vacations. The assumptions of the system model are as follows:
(a) Customers arrive at the system one by one according to a Poisson process with rate λ. On arrival a customer either decides to join the queue with probability b_n or balk with probability $1-b_n$ when n customers are ahead of him ($n = 0, 1, ..., N-1$), where N is the maximum number of customers in the system, and

$$0 \le b_{n+1} \le b_n < 1, \qquad 1 \le n \le N-1,$$

$$b_0 = 1, \qquad \text{and} \qquad b_n = 0, \qquad n \ge N.$$

(b) After joining the queue each customer will wait a certain length of time T for service to begin. If it has not begun by then, he will get impatient and leave the queue without getting service. This time T is a random variable whose density function is given by

$$d(t) = \alpha e^{-\alpha t}, \quad t \geq 0, \ \alpha > 0$$

where α is the rate of time T. Since the arrival and the departure of the impatient customers without service are independent, the average reneging rate of the customer can be given by $(n-i)\alpha$. Hence, the function of customer's average reneging rate is given by

$$r(n) = (n-i)\alpha, \quad i \leq n \leq N, \ i = 0, 1,$$

$$r(n) = 0, \quad n > N.$$

(c) The customers are served on a first-come, first served (FCFS) discipline. Once service commences it always proceeds to completion. The service times are assumed to be distributed according to an exponential distribution with density function as follows:

$$s(t) = \mu e^{-\mu t}, \quad t \geq 0, \ \mu > 0$$

where μ is the service rate.
(d) Whenever the system is empty, the server goes on a sequence of vacations for a period of random time V. If the server returns from a vacation to find no customer waiting, he will begin another vacation immediately. It is assumed that V has an exponential distribution with the density function as follows:

$$v(t) = \eta e^{-\eta t}, \quad t \geq 0, \ \eta > 0$$

where η is the vacation rate of a server.

3 Steady-state Probability

In this section, we derive the steady-state probabilities by the Markov process method. Let $p_0(n)$ be the probability that there are n customers in the system when the server is on vacation, $p_1(n)$ be the probability that there are n customers in the system when the server is on available.

Applying the Markov process theory, we obtain the following set of steady-state equations.

$$\eta p_0(1) + (\mu + \alpha)p_1(2) = (\lambda b_1 + \mu)p_1(1), \tag{3.1}$$

$$\lambda b_{n-1} p_1(n-1) + \eta p_0(n) + (\mu + n\alpha)p_1(n+1)$$
$$= [\lambda b_n + \mu + (n-1)\alpha]p_1(n), \quad n = 2, 3, ..., N-1, \tag{3.2}$$

$$\lambda b_{N-1} p_0(N-1) + \eta p_0(N) = [\mu + (N-1)\alpha]p_0(N), \tag{3.3}$$

$$\mu p_1(1) + \alpha p_0(1) = \lambda p_0(0), \tag{3.4}$$

$$\lambda b_{n-1} p_0(n-1) + (n+1)\alpha p_0(n+1) = (n\alpha + \lambda b_n + \eta)p_0(n), \quad n = 1, 2, ..., N-1, \tag{3.5}$$

$$\lambda b_{N-1} p_0(N-1) = (N\alpha + \eta) p_0(N). \tag{3.6}$$

The transition rate matrix $\boldsymbol{Q}$ of the Markov process has the following block form.

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{A} \\ \boldsymbol{C} & \boldsymbol{B}_1 \end{pmatrix} \tag{3.7}$$

where

$$\boldsymbol{C} = \begin{pmatrix} \mu & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \eta & 0 & \cdots & 0 \\ 0 & \eta & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \eta \end{pmatrix},$$

$$\boldsymbol{B}_0 = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha & -c_1 & \lambda b_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\alpha & -c_2 & \lambda b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (N-1)\alpha & -c_{N-1} & \lambda b_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & N\alpha & -(N\alpha+\eta) \end{pmatrix},$$

$$\boldsymbol{B}_1 = \begin{pmatrix} -d_1 & \lambda b_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu+\alpha & -d_2 & \lambda b_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu+2\alpha & -d_3 & \lambda b_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mu+(N-2)\alpha & -d_{N-2} & \lambda b_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu+(N-1)\alpha & -[\mu+(N-1)\alpha] \end{pmatrix}$$

where $c_i = i\alpha + \lambda b_i + \eta$, $d_i = \lambda b_i + \mu + (i-1)\alpha$, $i = 1, 2, ..., N-1$, $\boldsymbol{C}$ is a matrix of order $N \times (N+1)$, $\boldsymbol{A}$ is a matrix of order $(N+1) \times N$, $\boldsymbol{B}_0$ is a square matrix of order $N+1$, $\boldsymbol{B}_1$ is a square matrix of order N. Let $\boldsymbol{P} = \{\boldsymbol{P}_0, \boldsymbol{P}_1\}$ be the corresponding steady-state probability vector of $\boldsymbol{Q}$, where $\boldsymbol{P}_0 = \{p_0(0), p_0(1), p_0(2), ..., p_0(N)\}$ and $\boldsymbol{P}_1 = \{p_1(1), p_1(2), ..., p_1(N)\}$. The steady-state probability vector $\boldsymbol{P}$ must satisfy the following equations:

$$\begin{cases} \boldsymbol{PQ} = \boldsymbol{0} \\ \boldsymbol{Pe} = 1 \end{cases} \tag{3.8}$$

where $\boldsymbol{e}$ is a column vector with each component equal to one. Then, we obtain by some routine substitutions that

$$\boldsymbol{P}_0 \boldsymbol{B}_0 + \boldsymbol{P}_1 \boldsymbol{C} = \boldsymbol{0}, \tag{3.9}$$

$$\boldsymbol{P}_0 \boldsymbol{A} + \boldsymbol{P}_1 \boldsymbol{B}_1 = \boldsymbol{0}, \tag{3.10}$$

$$\boldsymbol{P}_0 \boldsymbol{e}_0 + \boldsymbol{P}_1 \boldsymbol{e}_1 = 1 \tag{3.11}$$

where $\boldsymbol{e}_0$ is a column vector of order $N+1$ with each component equal to one, $\boldsymbol{e}_1$ is a column vector of order N with each component equal to one, $\boldsymbol{B}_0^{-1}$ and $\boldsymbol{B}_1^{-1}$ are the inverse matrix of $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$. Solving Eq. (3.9), we can get

$$\boldsymbol{P}_0 = -\boldsymbol{P}_1\boldsymbol{C}\boldsymbol{B}_0^{-1}. \tag{3.12}$$

Substituting Eq. (3.12) into Eqs. (3.10) and (3.11), we can obtain

$$\boldsymbol{P}_1(\boldsymbol{I} - \boldsymbol{C}\boldsymbol{B}_0^{-1}\boldsymbol{A}\boldsymbol{B}_1^{-1}) = \boldsymbol{0}, \tag{3.13}$$

$$\boldsymbol{P}_1(\boldsymbol{e}_1 - \boldsymbol{C}\boldsymbol{B}_0^{-1}\boldsymbol{e}_0) = 1. \tag{3.14}$$

Solving Eqs. (3.12)−(3.14), we can get the steady-state probabilities of the system.
Theorem 3.1. The steady-state probabilities are given by

$$p_1(1) = (1 - \mu\eta\widetilde{\boldsymbol{\alpha}}\boldsymbol{B}_1^{-1}\boldsymbol{\varepsilon}_1)^{-1}, \tag{3.15}$$

$$p_1(i) = p_1(1)\mu\eta\widetilde{\boldsymbol{\alpha}}\boldsymbol{B}_1^{-1}\boldsymbol{\varepsilon}_i, \qquad i = 2, 3, ..., N, \tag{3.16}$$

$$p_0(i-1) = -p_1(1)\mu a_{1i}^{-1}, \qquad i = 1, 2, ..., N+1 \tag{3.17}$$

where a_{1i}^{-1}, $i = 1, 2, ..., N+1$ is the element of the first row of the matrix $\boldsymbol{B}_0^{-1}$, $\boldsymbol{\alpha} = \left(a_{11}^{-1}\ a_{12}^{-1}\ \cdots\ a_{1N+1}^{-1}\right)$ is the first row vector of the matrix $\boldsymbol{B}_0^{-1}$ and $\widetilde{\boldsymbol{\alpha}} = \left(a_{12}^{-1} \cdots a_{1N+1}^{-1}\right)$ is a row vector of order N, $\boldsymbol{\varepsilon}_i$ $(i = 1, 2, ..., N)$ is a unit vector of order N.
Proof. $\boldsymbol{C}$ and $\boldsymbol{A}$ can be rewritten as the following block matrix:

$$\boldsymbol{C} = \begin{pmatrix} \mu & \boldsymbol{O}_1 \\ \boldsymbol{O}_2 & \boldsymbol{O}_3 \end{pmatrix}_{N\times(N+1)}, \quad \boldsymbol{A} = \begin{pmatrix} \boldsymbol{O}_1 \\ \eta\boldsymbol{I} \end{pmatrix}_{(N+1)\times N}$$

where $\boldsymbol{O}_1$ is a matrix of order $1 \times N$, $\boldsymbol{O}_2$ is a matrix of order $(N-1) \times 1$, $\boldsymbol{O}_3$ is a matrix of order $(N-1) \times N$, $\boldsymbol{I}$ is an identity matrix of order N. Let

$$\boldsymbol{B}_0^{-1} = (a_{ij}^{-1})_{(N+1)\times(N+1)}.$$

Then

$$\boldsymbol{C}\boldsymbol{B}_0^{-1} = \begin{pmatrix} \mu\boldsymbol{\alpha} \\ \boldsymbol{O}_4 \end{pmatrix}_{N\times(N+1)} \tag{3.18}$$

and

$$\boldsymbol{A}\boldsymbol{B}_1^{-1} = \begin{pmatrix} \boldsymbol{O}_1 \\ \eta{\boldsymbol{B}_1}^{-1} \end{pmatrix}_{(N+1)\times N} \tag{3.19}$$

where $\boldsymbol{O}_4$ is a matrix of order $(N-1) \times (N+1)$, and

$$\boldsymbol{\alpha} = \left(a_{11}^{-1}a_{12}^{-1} \cdots a_{1N+1}^{-1}\right)$$

is the first row of $\boldsymbol{B}_0^{-1}$. If we let

$$\widetilde{\boldsymbol{\alpha}} = \left(a_{12}^{-1} \cdots a_{1N+1}^{-1}\right)$$

then $\boldsymbol{C}\boldsymbol{B}_0^{-1}$ can be written as

$$\boldsymbol{C}\boldsymbol{B}_0^{-1} = \begin{pmatrix} \mu a_{11}^{-1} & \mu\widetilde{\boldsymbol{\alpha}} \\ \boldsymbol{O}_2 & \boldsymbol{O}_5 \end{pmatrix}$$

where $\boldsymbol{O}_5$ is a matrix of order $(N-1)\times N$. Thus

$$\boldsymbol{C}\boldsymbol{B}_0^{-1}\boldsymbol{A}\boldsymbol{B}_1^{-1} = \begin{pmatrix} \mu\eta\widetilde{\boldsymbol{\alpha}}\boldsymbol{B}_1^{-1} \\ \boldsymbol{O}_5 \end{pmatrix}. \tag{3.20}$$

Let

$$\widetilde{\boldsymbol{P}}_1 = (p_1(2)p_1(3)\cdots p_1(N)),$$

then, from Eqs. (3.13) and (3.20), we have that

$$\begin{aligned} (p_1(1) \quad \widetilde{\boldsymbol{P}}_1) =& (p_1(1) \quad \widetilde{\boldsymbol{P}}_1)\begin{pmatrix} \mu\eta\widetilde{\boldsymbol{\alpha}}\boldsymbol{B}_1^{-1} \\ \boldsymbol{O}_5 \end{pmatrix} \\ =& p_1(1)\mu\eta\widetilde{\boldsymbol{\alpha}}\boldsymbol{B}_1^{-1}. \end{aligned} \tag{3.21}$$

Hence

$$p_1(1) = p_1(1)\mu\eta\widetilde{\boldsymbol{\alpha}}\boldsymbol{B}_1^{-1}\boldsymbol{\varepsilon}_1.$$

Consequently

$$p_1(1) = (1-\mu\eta\widetilde{\boldsymbol{\alpha}}\boldsymbol{B}_1^{-1}\boldsymbol{\varepsilon}_1)^{-1},$$

$$p_1(i) = p_1(1)\mu\eta\widetilde{\boldsymbol{\alpha}}\boldsymbol{B}_1^{-1}\boldsymbol{\varepsilon}_i, \qquad i = 2,3,...,N.$$

¿From Eqs. (3.12) and (3.18), we can obtain

$$\begin{aligned} \boldsymbol{P}_0 \quad &= -\begin{pmatrix} p_1(1) & \widetilde{\boldsymbol{P}}_1 \end{pmatrix}\begin{pmatrix} \mu\boldsymbol{\alpha} \\ \boldsymbol{O}_4 \end{pmatrix} \\ &= -p_1(1)\mu\boldsymbol{\alpha}. \end{aligned} \tag{3.22}$$

Thus

$$p_0(i-1) = -p_1(1)\mu a_{1i}^{-1}, \qquad i = 1,2,...,N+1.$$

This completes the proof.

The procedure of calculating the steady-state probabilities of the system is summarized as follows:
(1) Calculating the elements of the first row of $\boldsymbol{B}_0^{-1}$, then we obtain

$$\boldsymbol{C}\boldsymbol{B}_0^{-1} = \begin{pmatrix} \mu\boldsymbol{\alpha} \\ \boldsymbol{O}_4 \end{pmatrix}_{N\times(N+1)}.$$

$$\boldsymbol{C}\boldsymbol{B}_0^{-1} = \begin{pmatrix} \mu\boldsymbol{\alpha} \\ \boldsymbol{O}_4 \end{pmatrix}_{N\times(N+1)}.$$

(2) Calculating $\boldsymbol{B}_1^{-1}$, then we can get

$$\boldsymbol{A}\boldsymbol{B}_1^{-1} = \begin{pmatrix} \boldsymbol{O}_1 \\ \eta \boldsymbol{B}_1^{-1} \end{pmatrix}_{(N+1)\times N}$$

and

$$\boldsymbol{C}\boldsymbol{B}_0^{-1}\boldsymbol{A}\boldsymbol{B}_1^{-1} = \begin{pmatrix} \mu\eta\widetilde{\boldsymbol{\alpha}}\boldsymbol{B}_1^{-1} \\ \boldsymbol{O}_5 \end{pmatrix}.$$

(3) Calculating the steady-state probabilities:

$$p_1(1) = (1 - \mu\eta\widetilde{\boldsymbol{\alpha}}\boldsymbol{B}_1^{-1}\boldsymbol{\varepsilon}_1)^{-1},$$

$$p_1(i) = p_1(1)\mu\eta\widetilde{\boldsymbol{\alpha}}\boldsymbol{B}_1^{-1}\boldsymbol{\varepsilon}_i, \qquad i = 2, 3, ..., N,$$

$$p_0(i-1) = -p_1(1)\mu a_{1i}^{-1}, \qquad i = 1, 2, ..., N+1.$$

Remark 3.1. By the procedure, we need mainly to calculate the elements of the first row of a tridiagonal matrix $\boldsymbol{B}_0^{-1}$ and another tridiagonal matrix $\boldsymbol{B}_1^{-1}$. However, $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ have some special structures. For example, (a) $\boldsymbol{B}_0$ is a reversible tridiagonal matrix in which the sum of the elements in the first row is zero, and the sums of the elements in other rows are all η; (b) $\boldsymbol{B}_1$ is also a reversible tridiagonal matrix in which the sum of the elements in the first row is $-\mu$ and the sums of the elements in other rows are all zero. Thus, it is not difficult to calculate the matrix $\boldsymbol{B}_0^{-1}$ and $\boldsymbol{B}_1^{-1}$.

4 Performance Measures and Cost Model

In this section, we give some performance measures of the system. Based on these performance measures, we develop a cost model to determine the optimal service rate.

4.1 Performance measures

Using the steady-state probability presented in Sec. 3, we can obtain some performance measures of the system, such as the busy probability of the server P_B, the vacation probability of the server P_V, the expected number of the waiting customers $E(N_q)$ and the expected number of the customers in the system $E(N)$ as follows:

$$P_B = \sum_{n=1}^{N} p_1(n), \tag{4.1}$$

$$P_V = \sum_{n=0}^{N} p_0(n) = 1 - P_B, \tag{4.2}$$

$$E(N_q) = \sum_{i=0}^{1}\sum_{n=0}^{N}(n-i)p_i(n), \tag{4.3}$$

$$E(N) = \sum_{n=1}^{N} np_1(n) + \sum_{n=0}^{N} np_0(n). \tag{4.4}$$

Using the concept of Ancker and Gafarian [4], [5], we can obtain the average balking rate $B.R.$, the average reneging rate $R.R.$ and the average rate $L.R.$ of customer loss because of impatient as follows:

$$B.R. = \sum_{i=0}^{1}\sum_{n=0}^{N}[\lambda(1-b_n)]p_i(n), \tag{4.5}$$

$$R.R. = \sum_{i=0}^{1}\sum_{n=0}^{N}(n-i)\alpha p_i(n), \tag{4.6}$$

$$L.R. = B.R. + R.R. \tag{4.7}$$

where $\lambda(1-b_n)$ is the instantaneous balking rate and $(n-i)\alpha$ is the instantaneous reneging rate.

4.2 Cost model

In this subsection, we develop an expected cost model, in which service rate μ is the control variable. Our objective is to control the service rate to minimize the system's total average cost per unit. Let

$C_1 \equiv$ cost per unit time when the server is busy,

$C_2 \equiv$ cost per unit time when the server is on vacation,

$C_3 \equiv$ cost per unit time when a customer joins in the queue and waits for service,

$C_4 \equiv$ cost per unit time when a customer balks or reneges.

Using the definitions of each cost element listed above, the total expected cost function per unit time is given by

$$F(\mu) = C_1P_B + C_2P_V + C_3E(N_q) + C_4L.R.$$

where $P_B, P_V, E(N_q), L.R.$ are given in Eqs. (4.1) – (4.3) and (4.7). The first two items are the cost incurred by the server. The third item $C_3E(N_q)$ is the cost incurred by the customer's waiting. The last item $C_4L.R.$ is the cost incurred by the customer loss.

4.3 Numerical results

In this subsection, we present some numerical examples to demonstrate how the various parameters of the model influence the optimal service rate μ^*, the optimal expected cost of the system $F(\mu^*)$ and other performance measures of the system. We fix the maximum number of customers in the system $N = 3$, the probability $b_n = 1/(n+1)$ and the cost elements $C_1 = 15, C_2 = 12, C_3 = 18, C_4 = 12$.

Table 1. The case for $\alpha = 0.1$ and $\eta = 0.1$.

λ	0.4	0.5	0.6	0.7	0.8	0.9
μ^*	0.2642	0.2590	0.2574	0.2579	0.2599	0.2628
$F(\mu^*)$	33.9798	37.1878	40.0079	42.5363	44.8401	46.9680
P_B	0.3732	0.4298	0.4756	0.5134	0.5448	0.5714
P_V	0.6268	0.5702	0.5244	0.4866	0.4552	0.4286
$E(N_q)$	0.9580	1.0686	1.1583	1.2325	1.2947	1.3475
$E(N)$	1.3312	1.4983	1.6340	1.7459	1.8395	1.9189
$L.R.$	0.3014	0.3887	0.4776	0.5676	0.6584	0.7498

Table 2. The case for $\lambda = 0.5$, $\alpha = 0.1$.

η	0.03	0.05	0.1	0.15	0.2	0.25
μ^*	0.0439	0.1045	0.2590	0.4349	0.6437	0.8974
$F(\mu^*)$	41.8106	40.3539	37.1878	34.6789	32.6311	30.9194
P_B	0.7194	0.5792	0.4298	0.3444	0.2817	0.2317
P_V	0.2806	0.4208	0.5702	0.6556	0.7183	0.7683
$E(N_q)$	1.2240	1.1857	1.0686	0.9691	0.8868	0.8178
$E(N)$	1.9433	1.7649	1.4983	1.3135	1.1684	1.0495
$L.R.$	0.4684	0.4395	0.3887	0.3502	0.3187	0.2920

Table 3. The case for $\lambda = 0.5$, $\eta = 0.1$.

α	0.05	0.1	0.2	0.3	0.4	0.5
μ^*	0.2642	0.2590	0.2476	0.2415	0.2497	0.2810
$F(\mu^*)$	39.8798	37.1878	33.4391	30.9770	29.2295	27.9156
P_B	0.4851	0.4298	0.3562	0.3045	0.2562	0.2047
P_V	0.5149	0.5702	0.6438	0.6955	0.7438	0.7953
$E(N_q)$	1.2201	1.0686	0.8572	0.7192	0.6238	0.5551
$E(N)$	1.7052	1.4983	1.2133	1.0237	0.8800	0.7598
$L.R.$	0.3718	0.3887	0.4118	0.4265	0.4360	0.4425

First, we select the rate of the waiting time $\alpha = 0.1$, the rate of the vacation time $\eta = 0.1$, and change values of arrival rate of customers λ. The numerical results are summarized in Table 1. Table 1 shows that: (i) the optimal service rate μ^* first decreases and then increases slightly with the increasing of λ, and its minimum expected cost $F(\mu^*)$ increases greatly with the increasing of λ; (ii) the busy probability of the server P_B, the expected number of the waiting customers

$E(N_q)$, the expected number of customers in the system $E(N)$ and the average rate of customer loss $L.R.$ all increase with the increasing of λ, while the vacation probability of the server P_V decreases with the increasing of λ. This is because the number of the customers in the system increases with the increasing of λ. Thus, P_B, $E(N_q)$ and $L.R.$ all increase which result in the increasing of the optimal cost.

Next, we select $\alpha = 0.1$, $\lambda = 0.5$, and change values of η. The numerical results are summarized in Table 2. Table 2 shows that: (i) the optimal service rate μ^* increases greatly with the slightly increasing of η, and its minimum expected cost $F(\mu^*)$ decreases with the increasing of η; (ii) P_B, $E(N_q)$, $E(N)$ and $L.R.$ all decrease with the increasing of η, and P_V increases with the increasing of η. This is because the mean vacation time of the server $1/\eta$ decreases with the increasing of η. Thus, P_B, $E(N_q)$ and $L.R.$ all decrease which result in the decreasing of the optimal cost.

Finally, we select $\lambda = 0.5$, $\eta = 0.1$, and change values of α. The numerical results are summarized in Table 3. Table 3 shows that: (i) the optimal service rate μ^* slightly change with the increasing of α, while its minimum expected cost $F(\mu^*)$ decreases with the increasing of α; (ii) $E(N_q)$ and $E(N)$ decrease with the increasing of α; $L.R.$ and P_V increases with the increasing of α, and P_B decrease with the increasing of α. This is because the mean waiting time of impatient customers decreases with the increasing of α. Thus, the average rate of customers loss $L.R.$ increases, while the expected number of waiting customers $E(N_q)$ and the busy probability P_B decreases which result in the decreasing of the optimal cost.

5 Conclusions

In this paper, we considered an M/M/1/N queueing system with balking, reneging and server vacations. We developed the equations of the steady state probabilities and derived the matrix form solution of the steady-state probabilities. We also gave some performance measures of the system, and formulated a cost model to determine the optimal service rate. Although the function of the cost is too complicated to derive the explicit expression of the optimal service rate, the performance measures and the optimal service rate can be numerically evaluated by the formula in Section 4. Some numerical examples were presented to demonstrate how the various parameters of the model influence the behavior of the system.

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