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# Computing an Integer Point in a Class of Polytopes*

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**Abstract** Let P be a polytope satisfying that each row of the defining matrix has at most one positive entry. Determining whether there is an integer point in P is known to be an NP-complete problem. By introducing an integer labeling rule on an augmented set and applying a triangulation of the Euclidean space, we develop in this paper a variable dimension method for computing an integer point in P. The method starts from an arbitrary integer point and follows a finite simplicial path that either leads to an integer point in P or proves no such point exists.

**Keywords** Integer Point; Polytope; Integer Programming; Integer Labeling; Augmented Set; Triangulation; Pivoting Procedure; Simplicial Method

## 1 Introduction

The problem we consider is as follows: Determine whether there is an integer point in a polytope given by $P = \{x \in R^n \mid Ax \leq b\}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

satisfying that each row has at most one positive entry and $b = (b_1, b_2, \ldots, b_m)^\top$. It has been shown in Lagarias (1985) that

**Theorem 1.**
*Determining whether there is an integer point in P is an NP-complete problem.*

To compute an integer point in $P$, a homotopy-like simplicial method was proposed in Dang (2009). The method is simple, but the proof of its finite convergence is rather complicated due to an uneconomical labeling rule and a complex topological structure. To overcome this deficiency, we first introduce an economical labeling rule on an augmented

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set and then develop a variable dimension simplicial method for determining whether there is an integer point in $P$. The method starts from an arbitrary integer point and follows a finite simplicial path that either leads to an integer point in $P$ or proves no such point exists. The topological structure of the method is totally different from that in Dang (2009). The introductions of the labeling rule and the augmented set significantly simplify the analysis of finite convergence. The idea of the method is stimulated from that in Dang (2009) and Dang and Maaren (1998) and has its foundations in simplicial methods for computing fixed points of a continuous mapping that were originated in Scarf (1967) and substantially developed in the literature (e.g., Allgower and Georg, 2000; Dang, 1995; Eaves, 1972; Eaves and Saigal (1972); Kuhn, 1968; van der Laan and Talman, 1979; Merrill, 1972; Scarf, 1973; Todd, 1976).

The rest of this paper is organized as follows. In Section 2, we introduce an integer labeling rule on an augmented set and study its properties. In Section 3, we present a variable dimension method for determining whether there is an integer point in $P$ and prove its finite convergence.

# 2 An Integer Labeling Rule and Its Properties

Let $M = \{1,2,\dots,m\}$, $N = \{1,2,\dots,n\}$, and $N_0 = \{1,2,\dots,n+1\}$. For $i \in M$, let $a_i^\top$ denote the $i$th row of $A$. Then, $A = (a_1,a_2,\dots,a_m)^\top$. Without loss of generality, we assume throughout this paper that $P$ is bounded and full dimensional. As a result of the property of $A$, one can easily obtain that

**Lemma 1.**
*If $x^1 = (x_1^1,x_2^1,\dots,x_n^1)^\top \in P$ and $x^2 = (x_1^2,x_2^2,\dots,x_n^2)^\top \in P$, then $\bar{x} = \max(x^1,x^2) = (\max\{x_1^1,x_1^2\},\max\{x_2^1,x_2^2\},\dots,\max\{x_n^1,x_n^2\})^\top \in P$.*

Let $e = (1,1,\dots,1)^\top \in R^n$. Lemma 1 implies that $\max_{x\in P} e^\top x$ has a unique solution, which we denote by $x^{\max} = (x_1^{\max},x_2^{\max},\dots,x_n^{\max})^\top$. Let $x^{\min} = (x_1^{\min},x_2^{\min},\dots,x_n^{\min})^\top$ with $x_j^{\min} = \min_{x\in P} x_j$, $j = 1,2,\dots,n$. Clearly, $x^{\min} \le x \le x^{\max}$ for all $x \in P$. For any real number $\alpha$, let $\lfloor \alpha \rfloor$ denote the greatest integer less than or equal to $\alpha$ and $\lceil \alpha \rceil$ the smallest integer greater than or equal to $\alpha$. Let $D(P) = \{x \in R^n \mid x^l \le x \le x^u\}$, where $x^u = \lfloor x^{\max} \rfloor = (\lfloor x_1^{\max} \rfloor, \lfloor x_2^{\max} \rfloor,\dots,\lfloor x_n^{\max} \rfloor)^\top$ and $x^l = \lceil x^{\min} \rceil = (\lceil x_1^{\min} \rceil, \lceil x_2^{\min} \rceil,\dots,\lceil x_n^{\min} \rceil)^\top$. Thus, $D(P)$ contains all integer points in $P$. Without loss of generality, we assume that $x^l \le x^u$.

For $x \in R^n$, let

$$f(x) = \begin{cases} 0 \in R^n & \text{if } x \in P, \\ \sum_{i\in I(x)} \frac{a_i^\top x - b_i}{a_i^\top a_i} a_i & \text{otherwise,} \end{cases}$$

where $I(x) = \{i \in M \mid a_i^\top x - b_i > 0\}$. This mapping can also be found in Dang (2009).

**Lemma 2.**
*For any $x \in R^n$, $f(x) = 0$ if and only if $x \in P$.*

**Lemma 3.**
*If $f(x) \le 0$ and $f(x) \ne 0$, then, for any $y \in P$, there is some $k \in N$ satisfying that $x_k - y_k < 0$.*

Let $x^0 = (x_1^0,x_2^0,\dots,x_n^0)^\top$ be an arbitrary integer point of $D(P)$. For $y \in R^n$ and $C \subseteq R^n$, let $\Gamma(y,C)$ be an augmented set in $R^{n+1}$ given by

$$\Gamma(y,C) = \{t(x,0) + (1-t)(y,1) \mid x \in C \text{ and } 0 \le t \le 1\}.$$

The following integer labeling rule plays an important role in this paper.

**Definition 1 (An Integer Labeling Rule).**
*For each integer point $(x,\gamma) \in \Gamma(x^0,R^n)$, we assign to $(x,\gamma)$ an integer label $l(x,\gamma) \in N_0 \cup \{0\}$ as follows:*

1. $l(x^0,1) = n+1$.
2. *For $(x,0)$ with $x \in D(P)$,*

$$l(x,0) = \begin{cases} 0 & \text{if } x \in P, \\ \max\{k \mid f_k(x) = \max_{j\in N} f_j(x)\} & \text{if } f_j(x) > 0 \text{ for some } j \in N, \\ n+1 & \text{if } f(x) \le 0 \text{ and } f(x) \ne 0. \end{cases}$$

3. *For $(x,0)$ with $x_j > x^u_j$ for some $j \in N$,*

$$l(x,0) = \max\{k \mid x_k - x^u_k = \max_{j\in N} x_j - x^u_j\}.$$

4. *For $(x,0)$ with $x \le x^u$ and $x_j < x^l_j$ for some $j \in N$,*

$$l(x,0) = \begin{cases} n+1 & \text{if } x < x^l, \\ \max\{k \mid x_k - x^l_k = \max_{j\in N} x_j - x^l_j\} & \text{otherwise.} \end{cases}$$

Let $h(n+1) = (1,1,\ldots,1,0)^\top \in R^{n+1}$ and $h(j) = -u^j$, $j = 1,2,\ldots,n$. Let $G(x^0,\emptyset) = \{(x^0,0)\}$ and $G(x^0,N_0) = \Gamma(x^0,R^n)$. For any $K \subset N_0$ with $K \ne \emptyset$, let $G(x^0,K) = \{(x^0,0) + \sum_{j\in K} \lambda_j h(j) \mid 0 \le \lambda_j,\ j \in K\}$. Clearly, $\cup_{j\in N_0} G(x^0,N_0\backslash\{j\}) = R^n \times \{0\}$ and, for any two subsets $K^1 \subset N_0$ and $K^2 \subset N_0$, the intersection of $G(x^0,K^1)$ and $G(x^0,K^2)$, $G(x^0,K^1) \cap G(x^0,K^2)$, is a common face of both of them. Thus, $\{G(x^0,K) \mid K \subset N_0\}$ forms a subdivision of $R^n \times \{0\}$.

For further developments, we need a cubic triangulation, whose restriction on $G(x^0,K)$ is a triangulation of $G(x^0,K)$ for each $K \subseteq N_0$. For simplicity, we choose the $K_1$-triangulation in Freudenthal (1942), which is as follows.

For $j \in N_0$, let $u^j$ denote the $j$th unit vector of $R^{n+1}$. A simplex of the $K_1$-triangulation of $\Gamma(x^0,R^n)$ is the convex hull of $n+2$ vectors, $y^0$, $y^1$, $\ldots$, $y^{n+1}$, given by $y^0 = y$, $y^k = y^{k-1} + u^{\pi(k)}$, $k = 1,2,\ldots,n$, and $y^{n+1} = (x^0,1)$, where $y = (y_1,y_2,\ldots,y_{n+1})^\top$ is an integer point in $R^n \times \{0\}$ and $\pi = (\pi(1),\pi(2),\ldots,\pi(n),\pi(n+1))$ is a permutation of elements of $N_0$ with $\pi(n+1) = n+1$. Let $K_1$ be the set of all such simplices. Then, $K_1$ forms a triangulation of $\Gamma(x^0,R^n)$. Since a simplex of the $K_1$-triangulation is uniquely determined by $y$ and $\pi$, we use $K_1(y,\pi)$ to denote it.

Two simplices of $K_1$ are adjacent if they share a common facet. For any given simplex $\sigma = K_1(y,\pi)$ with vertices $y^0$, $y^1$, $\ldots$, $y^{n+1}$, its adjacent simplex opposite to a vertex, say $y^i$, is given by $K_1(\bar{y},\bar{\pi})$, where $\bar{y}$ and $\bar{\pi}$ are generated according to the pivot rules in the following table.

Pivot Rules of the $K_1$-Triangulation of $\Gamma(x^0,R^n)$

| $i$ | $\bar{y}$ | $\bar{\pi}$ |
|---|---|---|
| $0$ | $y + u^{\pi(1)}$ | $(\pi(2),\ldots,\pi(n),\pi(1),\pi(n+1))$ |
| $1 \le i < n$ | $y$ | $(\pi(1),\ldots,\pi(i+1),\pi(i),\ldots,\pi(n+1))$ |
| $n$ | $y - u^{\pi(n)}$ | $(\pi(n),\pi(1),\ldots,\pi(n-1),\pi(n+1))$ |

Let $\mathscr{K}_1$ be the set of faces of simplices of $K_1$. A $q$-dimensional simplex of $\mathscr{K}_1$ with vertices $y^0$, $y^1$, ..., $y^q$ is denoted by $\langle y^0, y^1, \ldots, y^q \rangle$. For $\sigma \in \mathscr{K}_1$ with $\sigma \subset R^n \times \{0\}$, let $\mathrm{grid}(\sigma) = \max\{\|x-y\| \mid (x,0) \in \sigma \text{ and } (y,0) \in \sigma\}$, where $\|\cdot\|$ denotes the infinity norm. We define $\mathrm{mesh}(K_1) = \max\{\mathrm{grid}(\sigma) \mid \sigma \in \mathscr{K}_1 \text{ and } \sigma \subset R^n \times \{0\}\}$. Then, $\mathrm{mesh}(K_1) = 1$.

For $K \subset N_0$, the restriction of $\mathscr{K}_1$ on $G(x^0, K)$ is given by $\mathscr{K}_1|G(x^0,K) = \{\sigma \in \mathscr{K}_1 \mid \sigma \subset G(x^0,K) \text{ and } \dim(\sigma) = |K|\}$, where $|\cdot|$ denotes the cardinality of a set and $\dim(\cdot)$ the dimension of a set. Obviously, $\mathscr{K}_1|G(x^0,K)$ is a triangulation of $G(x^0,K)$.

**Definition 2.**

- *A $q$-dimensional simplex $\sigma = \langle y^0, y^1, \ldots, y^q \rangle$ of $\mathscr{K}_1$ is complete if $l(y^i) \neq l(y^j)$ for $0 \leq i < j \leq q$, and $l(y^k) \neq 0$, $k = 0, 1, \ldots, q$.*
- *A $q$-dimensional simplex $\sigma = \langle y^0, y^1, \ldots, y^q \rangle$ of $\mathscr{K}_1$ is 0-complete if $l(y^i) \neq l(y^j)$ for $0 \leq i < j \leq q$, and there is some $k$ satisfying that $l(y^k) = 0$.*
- *A $q$-dimensional simplex $\sigma = \langle y^0, y^1, \ldots, y^q \rangle$ of $\mathscr{K}_1$ is almost complete if labels of $q+1$ vertices of $\sigma$ consist of $q$ different nonzero integers.*

From Definition 2, it is easy to see that an almost complete simplex has exactly two complete facets both carrying the same set of integer labels.

For $y \in R^n$ and $K \subseteq N$, let $H(y,K)$ be the "higher" level cone originated at $y$ along certain directions given in $K$, that is,

$$H(y,K) = \{y + h \in R^n \mid 0 \leq h_j,\ j \in K, \text{ and } h_j = 0,\ j \notin K\}.$$

**Lemma 4.**
*If $P$ has an integer point, then, for any integer point $z^0 \in P$ and any nonempty $K \subseteq N$, each integer point of $H(z^0,K) \times \{0\}$ carries either integer label 0 or an integer label in $K$.*

This lemma plays an essential role in this paper. As a corollary of Lemma 4, we obtain that

**Corollary 1.**
*If $z^0$ is an integer point of $P$,*

1. *there is no complete $n$-dimensional simplex in $H(z^0,N) \times \{0\}$ carrying all integer labels in $N_0$ and,*
2. *for any $j \in N$ and $k \in N_0$, there is no complete $(n-1)$-dimensional simplex in $H(z^0, N \backslash \{j\}) \times \{0\}$ carrying all integer labels in $N_0 \backslash \{k\}$.*

Let $\Omega = \{x \in R^n \mid x^l - e \leq x \leq x^u + e\}$ and $\partial\Omega$ denote the boundary of $\Omega$. Clearly, $D(P) \subset \Omega$. Let $C(x^u)$ be a unit cube given by $C(x^u) = \{x \mid x^u \leq x \leq x^u + e\}$. Then, $C(x^u) = H(x^u, N) \cap \Omega$.

**Lemma 5.**
*$C(x^u) \times \{0\}$ contains all the complete $(n-1)$-dimensional simplices in $\partial\Omega \times \{0\}$ carrying all integer labels in $N$.*

For $y \in R^n$ and $C \subseteq R^n$, let $L(y,C) = \{x \in C \mid x_i \leq y_i \text{ for some } i \in N\}$ and $\partial L(y,C)$ denote the boundary of $L(y,C)$. As a result of Corollary 1 and Lemma 5, we obtain that

**Corollary 2.**
*If $z^0$ is an integer point of $P$, there is no complete $n$-dimensional simplex in $\Gamma(x^0, \partial L(z^0, \Omega))$ carrying all integer labels in $N_0$.*

# 3 A Simplicial Method

Applying the labeling rule and its properties, we develop in this section a variable dimension simplicial method for computing an integer point in $P$, which is as follows.

**Initialization:** Let $K = \emptyset$, $y^0 = (x^0, 0)$, $\sigma_0 = \langle y^0 \rangle$, $y^+ = y^0$, and $k = 0$. Go to **Step 1**.

**Step 1:** Compute $l(y^+)$. If $l(y^+) = 0$, the method terminates, and an integer point of $P$ has been found. If $K = N_0$ and $y^+ \geq (x^u, 0)$ with $y^+_{n+1} = 0$, the method terminates, and $P$ has no integer point. If $l(y^+) \in K$, let $y^-$ be the vertex of $\sigma_k$ other than $y^+$ and carrying integer label $l(y^+)$, and $\tau_{k+1}$ the facet of $\sigma_k$ opposite to $y^-$, and go to **Step 2**. If $l(y^+) \notin K$, go to **Step 3**.

**Step 2:** If $\tau_{k+1} \subset G(x^0, K \backslash \{j\})$ for some $j \in K$, let $K = K \backslash \{j\}$ and go to **Step 4**. Otherwise, proceed as follows: Let $\sigma_{k+1}$ be the unique simplex that is adjacent to $\sigma_k$ and has $\tau_{k+1}$ as a facet. Let $y^+$ be the vertex of $\sigma_{k+1}$ opposite to $\tau_{k+1}$ and $k = k+1$. Go to **Step 1**.

**Step 3:** Let $K = K \cup \{l(y^+)\}$ and $\tau_{k+1} = \sigma_k$. Let $\sigma_{k+1}$ be the unique $|K|$-dimensional simplex in $G(x^0, K)$ having $\tau_{k+1}$ as a facet, and $y^+$ the vertex of $\sigma_{k+1}$ opposite to $\tau_{k+1}$. Let $k = k+1$ and go to **Step 1**.

**Step 4:** Let $\sigma_{k+1} = \tau_{k+1}$, $y^-$ be the vertex of $\sigma_{k+1}$ carrying integer label $j$, and $\tau_{k+2}$ the facet of $\sigma_{k+1}$ opposite to $y^-$. Let $k = k+1$ and go to **Step 2**.

**Theorem 2.**
*Within a finite number of iterations, the method either yields an integer point in P or proves no such point exists.*

**Example 1.**
*Find an integer point in*

$$P = \left\{ x = (x_1, x_2)^\top \mid -x_1 + x_2 \leq \frac{1}{2},\ x_1 - x_2 \leq \frac{1}{2},\ x_1 \leq \frac{4}{5},\ -x_1 \leq \frac{4}{5} \right\}.$$

*Given this polytope, we obtain that* $x^u = (1,0)^\top$ *and* $x^l = (-1,0)^\top$. *Let* $x^0 = (-1,0)^\top$, $K = \emptyset$, $y^0 = (x^0, 0)$, $\sigma_0 = \langle y^0 \rangle$, *and* $y^+ = y^0$.

**Iteration 1:** $l(y^+) = 2 \notin K$. *Let* $K = K \cup \{2\} = \{2\}$, $\tau_1 = \sigma_0$, $y^1 = (-1,-1,0)^\top$, $\sigma_1 = \langle y^0, y^1 \rangle$, *and* $y^+ = y^1$.

**Iteration 2:** $l(y^+) = 1 \notin K$. *Let* $K = K \cup \{1\} = \{1,2\}$, $\tau_2 = \sigma_1$, $y^2 = (-2,-1,0)^\top$, $\sigma_2 = \langle y^0, y^1, y^2 \rangle$, *and* $y^+ = y^2$.

**Iteration 3:** $l(y^+) = 3 \notin K$. *Let* $K = K \cup \{l(y^+)\} = \{1,2,3\} = N_0$, $\tau_3 = \sigma_2$, $y^3 = (-1,0,1)^\top$, $\sigma_3 = \langle y^0, y^1, y^2, y^3 \rangle$, *and* $y^+ = y^3$.

**Iteration 4:** $l(y^+) = 3 = l(y^2) \in K$. *Let* $\tau_4 = \langle y^0, y^1, y^3 \rangle$, $y^2 = (0,0,0)^\top$, $\sigma_4 = \langle y^0, y^1, y^2, y^3 \rangle$, *and* $y^+ = y^2$.

**Iteration 5:** $l(y^+) = 0$. *An integer point of P has been found.*

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