The Tenth International Symposium on Operations Research and Its Applications (ISORA 2011)
Dunhuang, China, August 28–31, 2011

PSPACE-completeness of the Weighted Poset Game

Hiro Ito Satoshi Takata

School of Informatics, Kyoto University, Kyoto 606-8501, Japan

Abstract Poset game, which includes some famous games. e.g., Nim and Chomp as sub-games, is an important two-player impartial combinatorial game. The rule of the game is as follows: For a given poset (partial ordered set), each player intern chooses an element and the selected element and it's descendants (elements succeeding it) are all removed from the poset. A player who choose the last element is the winner. On the complexity of poset game, although it is clearly in PSPACE, it have not known whether it is in P or NP-hard. Recently a weighted poset game, which is a generalization of poset game, have been presented and it was found that some sub-games of it can be solved in polynomial-time. The complexity of this game is also open. This paper shows that weighted poset game is PSPACE-complete even if the weights are restricted in $\{1,-1\}$, the dag, which represents the poset, is bipartite, and the length of each path in the dag is at most two.

Keywords Poset game, computational complexity, winning strategy, PSPACE-complete, dag

1 Introduction

Poset game [1] is an impartial two-player combinatorial game played on a *poset* (= partial ordered set) $(S,\preceq)$. Each player in turn takes an element of S and the element chosen and it's descendants (elements succeeding it) are all removed from the poset. A player who select the last element is a winner. Poset game is a general game and it includes some well-known games as sub-games, e.g., Nim and Chomp (see [2] for detailes).

If a poset is fixed, clearly one of the players has a winning strategy. For a given poset, which player has a winning strategy can be solved in polynomial space, since the maximum move is at most the number of the elements. Thus poset game is in PSPACE. However whether it is in P or NP-hard is still open. Resolving it seems an important and challenging open problem.

The authors presented a more general game, named *Weighted Poset Game* [2], which is defined as follows. We give an integer weight $w(x)$ for each element $x \in S$ of a poset $(S,\preceq)$, and each player $i \in \{1,2\}$ has his/her own non-negative number L_i of lives. When a player i deletes a set $S' \subseteq S$ of elements, the sum of weights of these elements $\sum_{x\in S'} w(x)$ is decreased from L_i, i.e., L_i is replaced with $L_i - \sum_{x\in S'} w(x)$. A player whose lives become negative first is a loser. An instance of a weighted poset game is represented by $(S,\preceq,w,L_1,L_2)$. An element having a positive weight is called a *poison* element, and an element having a negative weight is called a *medicine* element.

It is easy to see that this game is a generalization of poset game, since if a poset game $(S, \preceq)$ is given, by adding an extra element $s \notin S$ preceding every elements in S, and giving weights and lives as $w(s) = 1$, $w(x) = 0$ for $x \in S$, and $L_1 = L_2 = 0$, the poset game is represented by the weighted form.

WEIGHTED POSET GAME is a problem for determining which player has a winning strategy for a given weighted poset game $(S, \preceq, w, L_1, L_2)$.

Some properties on winning strategies of this game are given in [2], e.g., it gives a polynomial time algorithm for calculating a winning strategy for the case if the partial order is a total order. On the other hand, however, from a viewpoint of lower bounds, nothing without trivial results have been known. Although it is clearly in PSPACE from the same reason of poset game, we don't know whether or not it is PSPACE-complete so far. This paper solves this problem.

Before explaining our theorem, we explain a *dag representation* of a poset. A poset can be represented by a *dag* (= directed acyclic graph). Let $D = (V, A)$ be a given dag, where V is a vertex set and A is an arc set, representing a poset $(S, \preceq)$. An arc (x, y) is in A means that element x precedes element y, i.e., $x \preceq y$. In this paper an arc which can be obtained by using the transitive law (we call it a *redundant arc*) may be omitted for making the representation simple, e.g., if arcs (x, y) and (y, z) are both in A, then from the transitive law arc (x, z) should be in A also, and hence (x, z) is redundant and it can be omitted in the representation. If D includes no redundant arc, then it is called the *canonical dag* of the poset.

This paper proves the following theorem.

Theorem 1.1.
WEIGHTED POSET GAME $(S, \preceq, w, L_1, L_2)$ *is PSPACE-complete even if the following conditions are simultaneously hold, where D is the canonical dag of the poset:*

(1) $w(x) \in \{1, -1\}$ *for all* $x \in S$*, and* $L_1 = L_2 = 0$*,*
(2) D is bipartite, and
(3) the length of each path in D is at most two.

2 Proof

2.1 Formula game

In this section we show a proof of Theorem 1.1. As mentioned before weighted poset game is clearly in PSPACE, and to show PSPACE-hardness is enough to prove it. We use the following *FORMULA GAME*, which is known to be PSPACE-complete [3].

This problem based on the following game:

1. Play on a given boolean formula ϕ in CNF. Let $x_1, x_2, \ldots, x_k$ be the variables used in ϕ, where$k$ is a positive even number.
2. Two players, called P_T and P_F, take turns selecting the value of the variables $x_1, x_2, \ldots, x_k$. P_T (resp., P_F) selects values with odd (resp., even) quantifiers. The order of play is the same as that of the quantifiers at the beginning of the formula, i.e., first P_T selects a value of x_1, next P_F selects a value of x_2, next P_T selects a value of x_3, ... and so on.

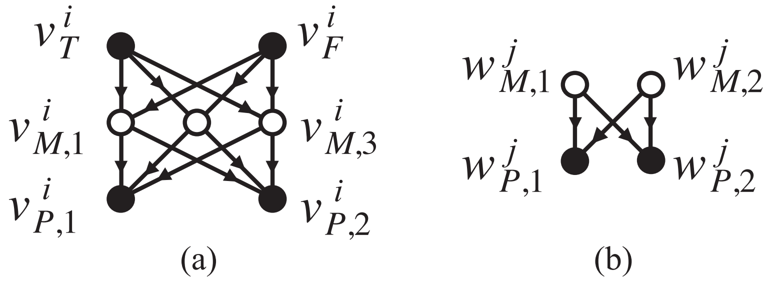

Figure 1: (a) V_i and (b) W_j.

3. P_T wins if ϕ becomes true by using the selected values at the end of play (after fixing all variables), and P_F wins otherwise.

   Example:
   $$\phi = (\overline{x_1} + x_2)(x_1 + x_3)(x_1 + \overline{x_3} + x_4)$$
   If the game goes
   1. P_T sets $x_1 = 0$,
   2. P_F sets $x_2 = 0$,
   3. P_T sets $x_3 = 1$,
   4. P_F sets $x_4 = 0$,

   then $\phi = 0$, which means P_F wins the game.

FORMULA GAME is a problem to determine which player has a winning strategy on this game for a given ϕ.

2.2 Reduction

We give a polynomial-time reduction from FORMULA GAME to WEIGHTED POSET GAME.

For an instance ϕ of FORMULA GAME, let $x_1, \ldots, x_n$ be the variables and $c_1, \ldots, c_m$ be the clauses of ϕ. By reducing ϕ we construct an instance $(D = (V, A), w, 0, 0)$ in the canonical dag representation of WEIGHTED POSET GAME.

$$V = \bigcup_{i=1}^{n} V_i \cup \bigcup_{j=1}^{m} W_j \cup U,$$

where V_i $(i = 1, \ldots, n)$ corresponds to x_i, W_j $(j = 1, \ldots, m)$ corresponds to c_j, and U is a set of extra vertices. For each $i \in \{1, \ldots, n\}$, V_i consists of the following 7 vertices (see Fig. 1 (a)):

$$V_i = \{v^i_t, v^i_f, v^i_{M,1}, v^i_{M,2}, v^i_{M,3}, v^i_{P,1}, v^i_{P,2}\},$$

where $w(v^i_t) = w(v^i_f) = w(v^i_{P,1}) = w(v^i_{P,2}) = 1$, $w(v^i_{M,1}) = w(v^i_{M,2}) = w(v^i_{M,3}) = -1$, and $(v^i_t, v^i_{M,k})$, $(v^i_f, v^i_{M,k})$, $(v^i_{M,k}, v^i_{P,h}) \in A$, $\forall k \in \{1, 2, 3\}$, $\forall h \in \{1, 2\}$.

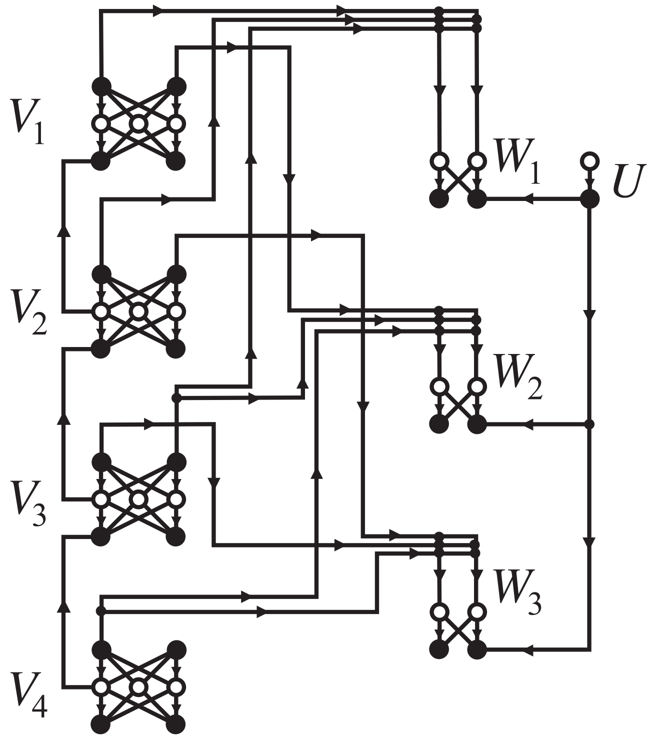

Figure 2: An instance of WEIGHTED POSET GAME reduced from $\phi = (x_1 + x_2 + \overline{x_3})(\overline{x_1} + \overline{x_3} + x_4)(\overline{x_2} + x_3 + x_4)$, where a vertex of filled (resp., blank) circle is a poison (resp., medicine) vertex and it's weight is 1 (resp., -1).

For each $j \in \{1, \ldots, m\}$, W_j consists of the following 4 vertices (see Fig. 1 (b)):

$$W_j = \{w^j_{M,1}, w^j_{M,2}, w^j_{P,1}, w^j_{P,2}\},$$

where $w(w^j_{M,1}) = w(w^j_{M,2}) = -1$, $w(w^i_{P,1}) = w(w^j_{P,2}) = 1$, and $(w^j_{M,k}, w^j_{P,h}) \in A$, $\forall k, h \in \{1, 2\}$.

For representing that literal x_i is included in clause c_j, add the following arcs: $(v^i_t, w^j_{M,1})$, $(v^i_t, w^j_{M,2}) \in A$. For representing that literal $\overline{x_i}$ is included in clause c_j, add the following arcs: $(v^i_f, w^j_{M,1})$, $(v^i_f, w^j_{M,2}) \in A$.

To force the order of selecting the value of variables, cascade $V_1, \ldots, V_n$ by the following arcs: $(v^{i+1}_{M,1}, v^i_{P,1}) \in A$, $\forall i \in \{1, \ldots, n-1\}$.

Finally, let $U = \{u_M, u_P\}$, where $w(u_M) = -1$, $w(u_P) = 1$, and (u_M, u_P), $(u_P, w^j_{P,1}) \in A$, $\forall j = \in \{1, \ldots, m\}$.

See Fig. 2 for an example of the reduction.

2.3 Correctness of the reduction

Now we show the equivalence of the two instances:

Lemma 2.1.
P_T has a winning strategy in ϕ if and anly if player 1 has a winning strategy in $(D = (V,A), w, 0, 0)$.

Proof: In $(D = (V,A), w, L_1, L_2)$, the first choice (of player 1) is restricted in v_t^1 or v_f^1, since if the player chooses another vertex, the sum of the weights of the vertex and it's descendants is greater than zero, and he/she immediately loses (note $L_1 = L_2 = 0$). If v_t^1 or v_f^1 is chosen, the next player (2) can choose v_t^2 or v_f^2, since a poison vertex $v_{P,1}^1$, which is a descendant of them, have been deleted. In the same argument, v_t^3 or v_f^3, v_t^4 or v_f^4, ... are chosen in this order. Vertices in W_j are deleted if and only if a vertex v_t^i with $(v_t^i, w_{M,1}^j)$, $(v_t^i, w_{M,2}^j) \in A$ or a vertex v_f^i with $(v_f^i, w_{M,1}^j)$, $(v_f^i, w_{M,2}^j) \in A$ is deleted. Note that $(v_t^i, w_{M,1}^j)$, $(v_t^i, w_{M,2}^j) \in A$ means $x_i \in c_j$, i.e., c_j is satisfied by letting $x_i = 1$; and $(v_f^i, w_{M,1}^j)$, $(v_f^i, w_{M,2}^j) \in A$ means $\overline{x_i} \in c_j$, i.e., c_j is satisfied by letting $x_i = 0$.

After player choses v_t^n or v_f^n, if all $W_1, \ldots, W_m$ have been deleted, player 1 can choose u_M and wins (since the remaining vertices for player 2 are all poison vertices), otherwise player 1 loses (since every selection for him/her has a positive weight).

Hence if P_T has a winning strategy of ϕ, player 1 can simulate the strategy by choosing v_t^i if $x_i = 1$ or v_f^i if $x_i = 0$ and wins. Contrary, if if P_F has a winning strategy of ϕ, player 2 can simulate the strategy by choosing v_t^i if $x_i = 1$ or v_f^i if $x_i = 0$ and wins. □

Proof of Theorem 1.1: Obvious from the reduction shown in **2.2** and Lemma 2.1. □

3 Conclusion

This paper shows that deciding a winner of weighted poset game is PSPACE-complete even under strong restrictions. The most important open problem is how about (unweighted) poset game? Although it is clearly in PSPACE, we have not known whether it is in P or NP-hard.

Acknowledgements

We would like to give a sincerely gratitude to Professor Gisaku Nakamura for his invaluable comments and helps.

References

[1] Steven Byrnes, Poset game periodicity, INTEGERS: The Electoronic Journal of Combinatorial Number Theory 3, G3, 2003.

[2] Hiro Ito, Gisaku Nakamura, and Satoshi Takata, Winning ways of weighted poset games, Graph and Combinatorics, Springer, 23, 291–306, 2007.

[3] Michael Sipser, Introduction to the Theory of Computation, Thomson Course Technology, 2nd edt., 2006.