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Polyhedral Split Decomposition of Tropical Polytopes for Directed Distances

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Abstract In the last decade, tropical geometry has been attracted a lot of attention in various fields such as the algebraic geometry, computational biology, and physics. The tropical polytope in the tropical geometry was introduced by Develin and Sturmfels as a counterpart of the polytope in the (ordinal) geometry. Recently, in the theory of directed multiflows, it has been shown by Hirai and the author that the dual problem of the μ-weighted maximum multiflow problem on Eulerian networks reduces to a facility location problem on the tropical polytope for μ, where the weight μ is regarded as a directed distance. Moreover, if the dimension of the tropical polytope for μ is at most one, the μ-weighted maximum multiflow problem has an integral optimal multiflow for any Eulerian networks. In this paper, we apply the polyhedral split decomposition to the tropical polytope for a directed distance d. As a result, a tropical polytope of dimension one turns out to be a Minkowski sum of a zonotope, a linear space, and a nonnegative orthant.

Keywords polyhedral split decomposition, tropical polytope, directed distance, subpath distance

1 Introduction

In the last decade, tropical geometry has been attracted a lot of attention in various fields such as the algebraic geometry [7], computational biology [8, 9], and physics [6]. The tropical polytope in the tropical geometry was introduced by Develin and Sturmfels [2] as a counterpart of the polytope in the (ordinal) geometry.

Recently, in the theory of directed multiflows, an unexpected connection between multiflows and tropical polytopes was revealed by Hirai and the author [5]. The dual problem of the μ-weighted maximum multiflow problem on Eulerian networks is reducible to a facility location problem on the tropical polytope $\bar{Q}_\mu$ for μ, where the weight μ is regarded as a directed distance. Moreover, if the dimension of $\bar{Q}_\mu$ is at most one, the μ-weighted maximum multiflow problem has an integral optimal multiflow for any Eulerian networks.

In this paper, we apply the polyhedral split decomposition, which was introduced in [3], to the tropical polytope $\bar{Q}_d$ for a directed distance d. As a result, a tropical polytope of dimension one turns out to be (a projection of the set of minimal points of) a Minkowski sum of a zonotope, a linear space, and a nonnegative orthant. Hence, one can see that such a tropical polytope has a quite simple structure.

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2 Directed distances and tropical polytopes

We denote by $\mathbf{R}_+$ the set of nonnegative real numbers. A function $d : V \times V \to \mathbf{R}_+$ is called a *directed distance* on V if has zero diagonals, i.e., $d(v,v) = 0$ for all $v \in V$. A directed distance d on V is called a *directed metric* if, in addition, d satisfies the triangle inequality, i.e., $d(u,v) \leq d(u,w) + d(w,v)$ for all $u,v,w \in V$. In this paper, since no confusion can arise, directed distances and directed metrics are simply called *distances* and *metrics*, respectively.

Let d be a distance on a finite set V. Let V^c and V^r be copies of V. For an element $v \in V$, the corresponding elements in V^c and V^r are denoted by v^c and v^r, respectively. We denote $V^c \cup V^r$ by V^{cr}. We define the following polyhedral sets:

$$\begin{aligned} P_d &= \{p \in \mathbf{R}^{V^{cr}} \mid p_{u^c} + p_{v^r} \geq d(u,v) \ (u,v \in V)\}, \qquad (1) \\ Q_d &= \text{the set of minimal points of } P_d, \end{aligned}$$

where a point $p \in P_d$ is minimal if there is no $q \in P_d$ with $q \neq p$ and $q_v \leq p_v$ for all $v \in V^{cr}$. Note that P_d has the linearity space $(\mathbf{1},-\mathbf{1})\mathbf{R} := \{\alpha(\mathbf{1},-\mathbf{1}) \mid \alpha \in \mathbf{R}\}$, where $\mathbf{1}$ denotes the all-one vector in $\mathbf{R}^V$. The natural projection of vector $p \in \mathbf{R}^{V^{cr}}$ to $\mathbf{R}^{V^{cr}}/(\mathbf{1},-\mathbf{1})\mathbf{R}$ is denoted by $\bar{p}$. The projection $\bar{Q}_d$ of Q_d coincides with the *tropical polytope* generated by $V \times V$ matrix $(-d(u,v) \mid u,v \in V)$; see [2]. Figure 1 illustrates a tropical polytope.

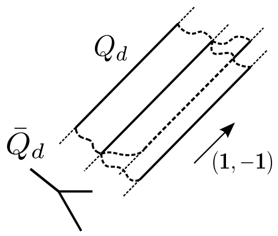

Figure 1: Q_d and $\bar{Q}_d$

As is mentioned in Introduction, the μ-weighted maximum multiflow problem has an integral optimal multiflow for every Eulerian network if the dimension of the tropical polytope $\bar{Q}_\mu$ is at most one, where the weight μ is regarded as a distance. Therefore, distances whose tropical polytopes are of dimension one constitute an important class in the theory of multiflows. Furthermore, those distances are characterized as subpath distances, which are defined as follows.

For a directed graph G, we denote by VG and EG the vertex and edge sets of G, respectively. A directed graph is said to be an *oriented tree* if its underlying undirected graph is a tree. Let Γ be an oriented tree. For two vertices $x,y \in V\Gamma$, let $P[x,y]$ denote the set of edges forming a unique path connecting x and y in the underlying undirected tree of Γ, and let $\vec{P}[x,y]$ be the set of edges in $P[x,y]$ whose directions are the same as the direction from x to y; in particular $P[x,y] = \vec{P}[x,y] \cup \vec{P}[y,x]$ (disjoint union). Given a nonnegative edge-length $\alpha : E\Gamma \to \mathbf{R}_+$, we define $D_{\Gamma,\alpha} : V\Gamma \times V\Gamma \to \mathbf{R}_+$ by $D_{\Gamma,\alpha}(x,y) = \sum\{\alpha(e) \mid e \in \vec{P}[x,y]\}$ for $x,y \in V\Gamma$. It is easy to see that $D_{\Gamma,\alpha}$ is a metric on $V\Gamma$.

A distance d on V is called a *subtree distance* if there is a tuple $(\Gamma,\alpha;\{F_v\}_{v\in V})$, called an *oriented-tree realization* of d, that consists of an oriented tree Γ, a nonnegative edge-length $\alpha : E\Gamma \to \mathbf{R}_+$, and a family $\{F_v\}_{v\in V}$ of subtrees in Γ such that

$$d(u,v) = D_{\Gamma,\alpha}(F_u,F_v) \quad (u,v\in V),$$

where a *subtree* is a subgraph of Γ whose underlying undirected graph is connected, and $D_{\Gamma,\alpha}(F_u,F_v)$ denotes the shortest distance from F_u to F_v, i.e., $\min\{D_{\Gamma,\alpha}(x,y) \mid x\in VF_u, y\in VF_v\}$. A subtree distance having an oriented-tree realization $(\Gamma,\alpha;\{F_v\}_{v\in V})$ is said to be *subpath distance* if every F_v is an oriented path, where an *oriented path* is an oriented tree each of whose vertices has at most one leaving edge and at most one entering edge. In fact, a subpath distance is a distance whose tropical polytope is of dimension one.

Theorem 1 ([4, Theorem 3.2])**.** *For a distance d on V, the dimension of $\bar{Q}_d$ is at most one if and only if d is a subpath distance.*

It is known that a tree metric (in the usual sense) is representable as a sum of cut metrics. A subtree distance has a similar representation with the aid of partial cut distances, which are defined as follows. For two disjoint nonempty subsets A,B of V, the ordered pair (A,B) is called a *partial cut* of V. In particular, a partial cut (A,B) is said to be a *cut* if $A\cup B = V$, and *proper* if $A\cup B \neq V$. For a partial cut (A,B) of V, the *partial cut distance* $\delta_{A,B}$ is defined by

$$\delta_{A,B}(u,v) = \begin{cases} 1 & u\in A \text{ and } v\in B, \\ 0 & \text{otherwise}, \end{cases} \qquad (u,v\in V).$$

Let d be a subtree distance, and let $(\Gamma,\alpha;\{F_v\}_{v\in V})$ be an oriented-tree realization of d. For a directed edge $e = xy\in E\Gamma$, (A_e,B_e) denotes the partial cut defined in the following way. By deleting the edge e from Γ, we obtain two subtrees Γ_1 and Γ_2 with $x\in V\Gamma_1$ and $y\in V\Gamma_2$. Define the subset A_e (resp. B_e) of V so that $v\in A_e$ (resp. $v\in B_e$) if and only if F_v is a subtree of Γ_1 (resp. Γ_2). By using partial cut distances, the following representation is available.

$$d = \sum_{e\in E\Gamma} \alpha(e)\delta_{A_e,B_e}. \tag{2}$$

For technical reasons, it is convenient to assume that for an edge $e\in E\Gamma$ that induces a *cut* (A_e,B_e), Γ has an edge e' with $(A_{e'},B_{e'}) = (B_e,A_e)$ by letting $\alpha(e') = 0$ if needed.

3 Polyhedral split decomposition

This section briefly describes the polyhedral split decomposition of polyhedral convex functions on the basis of [3]. The polyhedral split decomposition is also applicable to a certain type of discrete functions via their convex extensions.

3.1 Polyhedral split decomposition

For a function $f : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$, the *effective domain of* f, denoted by $\operatorname{dom} f$, is defined by $\operatorname{dom} f = \{x\in \mathbf{R}^n \mid f(x) < +\infty\}$, and the *epigraph of* f, denoted by $\operatorname{epi} f$, is given by $\operatorname{epi} f = \{(x,\alpha)\in \mathbf{R}^n\times\mathbf{R} \mid \alpha \geq f(x)\}$. A convex function $f : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ is said to be *polyhedral* if its epigraph epi f is a polyhedron. Obviously, a polyhedral convex function is piecewise linear.

For $x, y \in \mathbf{R}^n$, let $[x, y]$ denote the closed line segment between x and y. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product of $\mathbf{R}^n$. For $(a, r) \in \mathbf{R}^n \times \mathbf{R}$, we define a hyperplane $H_{a,r} = \{x \in \mathbf{R}^n \mid \langle a, x \rangle = r\}$, and open half spaces $H_{a,r}^{--} = \{x \in \mathbf{R}^n \mid \langle a, x \rangle < r\}$ and $H_{a,r}^{++} = \{x \in \mathbf{R}^n \mid \langle a, x \rangle > r\}$.

For a hyperplane $H_{a,r}$, the *split function* $l_{H_{a,r}} : \mathbf{R}^n \to \mathbf{R}$ associated with $H_{a,r}$ is defined to be the function such that the value $l_{H_{a,r}}(x)$ of each point x in $\mathbf{R}^n$ is $\|a\|/2$ times the distance between the point x and the hyperplane $H_{a,r}$, i.e., $l_{H_{a,r}}$ is given by

$$l_{H_{a,r}}(x) = |\langle a, x \rangle - r|/2 \quad (x \in \mathbf{R}^n).$$

For a polyhedral convex function $f : \mathbf{R}^n \to \mathbf{R}$ and a hyperplane $H_{a,r}$, we define the *quotient* $c_{H_{a,r}}(f)$ of f by $l_{H_{a,r}}$ as

$$c_{H_{a,r}}(f) = \sup\{t \geq 0 \mid f - t l_{H_{a,r}} \text{ is convex}\}.$$

It is easily observed that $c_H(f) l_H$ is independent of the equation representing a hyperplane H. From now on, unless otherwise stated, a hyperplane H is assumed to be represented as $H = H_{a,r}$ for a normal vector a with $\|a\| = 1$.

Suppose that f is a polyhedral convex function. We define the set of hyperplanes $\mathscr{H}(f)$ as

$$\mathscr{H}(f) = \{H : \text{hyperplane} \mid 0 < c_H(f) < +\infty\}.$$

In [3], it is shown that H belongs to $\mathscr{H}(f)$ if and only if $H \cap \mathrm{dom}\, f$ is contained in the set of all points $x \in \mathrm{dom}\, f$ with the property that f is not linear at x. The basic idea for the polyhedral split decomposition of f is to subtract split functions associated with hyperplanes in $\mathscr{H}(f)$ from f successively. In fact, if dom f is full-dimensional, this idea is directly applicable to f since each hyperplane H is unique in its intersection with dom f.

Theorem 2 ([3, Theorem 2.2])**.** *A polyhedral convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ whose effective domain is full-dimensional is uniquely decomposable as*

$$f = \sum_{H \in \mathscr{H}(f)} c_H(f) l_H + f', \tag{3}$$

where $f' : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is a polyhedral convex function with $c_{H'}(f') \in \{0, +\infty\}$ for any hyperplane H'.

Figure 2 shows the polyhedral split decomposition of a polyhedral convex function. If the effective domain of f is not full-dimensional, there may exist infinitely many hyperplanes having the same intersection with dom f, and hence the decomposition (3) is unique up to the choice of those hyperplanes.

3.2 Discrete functions and their convex extensions

In this paper, a *discrete function* means a function defined on a finite set of vectors in $\mathbf{R}^n$. Let K be a finite set of vectors in $\mathbf{R}^n$. If K contains the origin $\mathbf{0}$, we assume that $f(\mathbf{0}) = 0$.

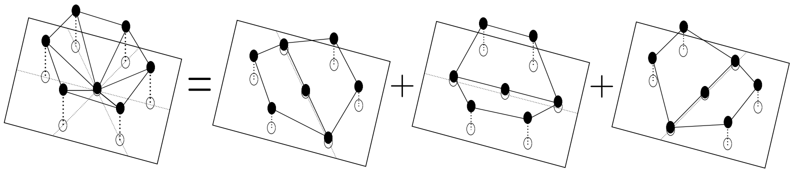

Figure 2: The polyhedral split decomposition of a polyhedral convex function

For a discrete function $f : K \to \mathbf{R}$, the *homogeneous convex closure* of f is defined by

$$\overline{f}(x) = \inf\left\{\sum_{y\in K}\lambda_y f(y) \mid \sum_{y\in K}\lambda_y y = x, \lambda_y \geq 0\ (y \in K)\right\} + \delta_{\mathrm{cone}\,K}(x)\ \ (x \in \mathbf{R}^n).$$

Since K is a finite set, $\overline{f}$ is a polyhedral convex function with $\mathrm{dom}\,f = \mathrm{cone}\,K$. Furthermore, by definition, $\overline{f}$ is positively homogeneous, i.e., $\overline{f}(\alpha x) = \alpha\overline{f}(x)$ holds for $\alpha \geq 0$ and $x \in \mathbf{R}^n$.

For a function $f : \mathbf{R}^n \to \mathbf{R}$, we denote the restriction of f to K by f^K. A discrete function $g : K \to \mathbf{R}$ is said to be *convex-extensible* if it satisfies $\overline{g}^K = g$. In this sense, a convex-extensible function can be identified with its homogeneous convex closure. If f is convex-extensible, we call $\overline{f}$ the *homogeneous convex extension* of f (the *extension* of f for short).

Since the extension of a discrete function is polyhedral, the polyhedral split decomposition is applicable to the extension. What is worth discussing here is a relation between K and $\mathscr{H}(\overline{f})$ for a convex-extensible discrete function f on K. Since $\overline{f}(\mathbf{0}) = 0$, each hyperplane $H \in \mathscr{H}(\overline{f})$ is linear, i.e., $H = H_{a,0}$ for some $a \in \mathbf{R}^n$. Furthermore, by the definition of $\overline{f}$, the set of all points $x \in \mathrm{dom}\,\overline{f}$ such that $\overline{f}$ is not linear at x is a union of cones whose extremal rays are written as αv for some $v \in K$ and $\alpha \in \mathbf{R}_+$. Thus, the hyperplanes in $\mathscr{H}(\overline{f})$ are limited by the vector set K. In fact, a hyperplane in $\mathscr{H}(\overline{f})$ must satisfy the K-admissibility; a hyperplanes H is called *K-admissible* if H satisfies the following:

(A1) H intersects with the relative interior of $\mathrm{cone}\,K$.
(A2) $\mathrm{cone}(H \cap K) = H \cap \mathrm{cone}\,K$.

In addition, we define the set of linear hyperplanes $\mathscr{H}_K$ as

$$\mathscr{H}_K = \{H \mid H : \text{a } K\text{-admissible linear hyperplane}\}.$$

Then, for any discrete function $f : K \to \mathbf{R}$, we have $\mathscr{H}(\overline{f}) \subseteq \mathscr{H}_K$.

The next theorem implies that the quotient $c_H(\overline{f})$ can be calculated without the explicit construction of $\overline{f}$.

Theorem 3 ([3, Theorem 3.4])**.** *For a discrete function $f : K \to \mathbf{R}$ and a hyperplane $H \in \mathscr{H}_K$, let $\tilde{c}_H(f)$ be defined by*

$$\tilde{c}_H(f) = \frac{1}{2}\inf\left\{\frac{f(x) - \overline{f^{K\cap H}(w)}}{l_H(x)} + \frac{f(y) - \overline{f^{K\cap H}(w)}}{l_H(y)} \;\middle|\; \begin{array}{c} x \in K \cap H^{++}, \\ y \in K \cap H^{--}, \\ \{w\} = [x,y] \cap H \end{array}\right\}.$$

Then we have $c_H(\overline{f}) = \max\{0, \tilde{c}_H(f)\}$.

4 Polyhedral split decomposition of directed distances

For a set $U \subseteq V^{cr}$, we denote by $\mathbf{1}_U$ the incidence vector of U, i.e., $(\mathbf{1}_U)_i = 1$ if $i \in U$; $(\mathbf{1}_U)_i = 0$ otherwise. For simplicity of notation, we write $\mathbf{1}_{uv}$ instead of $\mathbf{1}_{\{u^c,v^r\}}$. Let X be the set of vectors $\{-\mathbf{1}_{uv} \mid u,v \in V\}$.

By the correspondence $d(-\mathbf{1}_{uv}) \leftarrow d(u,v)$, we regard a distance d on V as a discrete function on X. Since any point in X cannot be represented as a conical combination of other points in X, $-d$ is convex-extensible. Then, the extension of $-d$ is given by, for $x \in \operatorname{cone} X$,

$$\overline{-d}(x) = \inf\left\{ \sum_{u,v\in V} \lambda_{uv}(-d(-\mathbf{1}_{uv})) \;\middle|\; \sum_{u,v\in V} \lambda_{uv}(-\mathbf{1}_{uv}) = x,\, \lambda_{uv} \geq 0\ (u,v\in V) \right\}.$$

By the duality, $\overline{-d}$ is representable as $\overline{-d}(x) = \sup\{\langle p,x\rangle \mid p \in \mathbf{R}^{V^{cr}}, \langle p, \mathbf{1}_{uv}\rangle \geq d(-\mathbf{1}_{uv})\}$. From this representation, we note that $\overline{-d}$ is the support function of the polyhedron P_d in (1). We will apply the polyhedral split decomposition to $\overline{-d}$. For this purpose, we deal with the set $\mathscr{H}_X$ and the quotient $\tilde{c}_H(-d)$ for $H \in \mathscr{H}_X$.

For a partial cut (A,B) of V, we denote by $H_{A,B}$ the hyperplane $H_{a,0}$ with $a = \mathbf{1}_{A^c} - \mathbf{1}_{B^r}$. Note that we have

$$\langle \mathbf{1}_{A^c} - \mathbf{1}_{B^r}, -\mathbf{1}_{uv}\rangle \begin{cases} < 0 & \text{if } u \in A, v \in V\setminus B \\ = 0 & \text{if } u \in A, v \in B \text{ or } u \in V\setminus A, v \in V\setminus B \\ > 0 & \text{if } u \in V\setminus A, v \in B \end{cases}.$$

From this, the following lemma is easily established, and we omit the proof.

Lemma 4. *For a partial cut (A,B) of V, $H_{A,B}$ belongs to $\mathscr{H}_X$.*

Then, we apply Theorem 3 to the hyperplane $H_{A,B}$. We define $b^d_{A,B}$ by

$$b^d_{A,B} = \min\left\{ d(i,l) + d(k,j) - d(i,j) - d(k,l) \;\middle|\; \begin{array}{l} i \in A, j \in V\setminus B \\ k \in V\setminus A, l \in B \end{array} \right\}.$$

This index $b^d_{A,B}$ can be interpreted as a directed version of the Buneman index for (ordinal) metrics [1].

Theorem 5. *Let d be a distance on V. For a partial cut (A,B) of V, we have $\tilde{c}_{H_{A,B}}(-d) = b^d_{A,B}$.*

Proof. By Theorem 3, $\tilde{c}_{H_{A,B}}(-d)$ is equal to the minimum of

$$\frac{-d(-\mathbf{1}_{ij}) - \overline{(-d^{X\cap H_{A,B}})}(w)}{2l_{H_{A,B}}(-\mathbf{1}_{ij})} + \frac{-d(-\mathbf{1}_{kl}) - \overline{(-d^{X\cap H_{A,B}})}(w)}{2l_{H_{A,B}}(-\mathbf{1}_{kl})}$$

where $i \in A, j \in V\setminus B, k \in V\setminus A, l \in B$ and $\{w\} = H_{A,B} \cap [-\mathbf{1}_{ij}, -\mathbf{1}_{kl}]$. It is easy to see that $w = (-\mathbf{1}_{ij} - \mathbf{1}_{kl})/2$. Hence, $\tilde{c}_{H_{A,B}}(-d)$ is given by

$$\min\left\{ -d(-\mathbf{1}_{ij}) - d(-\mathbf{1}_{kl}) - 2\overline{(-d^{X\cap H_{A,B}})}\left(\frac{-\mathbf{1}_{ij} - \mathbf{1}_{kl}}{2}\right) \;\middle|\; \begin{array}{l} i \in A, j \in V\setminus B \\ k \in V\setminus A, l \in B \end{array} \right\}.$$

Furthermore, since $w = (-\mathbf{1}_{il} - \mathbf{1}_{kj})/2$ is a unique representation of w as a conical combination of points in $X \cap H_{A,B}$, we obtain

$$\overline{(-d^{X\cap H_{A,B}})}\left(\frac{-\mathbf{1}_{ij} - \mathbf{1}_{kl}}{2}\right) = \frac{1}{2}(-d(-\mathbf{1}_{il}) - d(-\mathbf{1}_{kj})).$$

Thus, we have $\tilde{c}_{H_{A,B}}(-d) = b^d_{A,B}$. □

The following lemma does not hold for subtree distances in general.

Lemma 6. *Let d be a subpath distance with an oriented-tree realization* $(\Gamma, \alpha; \{F_v\}_{v\in V})$, *and let e be an edge of* Γ. *If the partial cut* (A_e, B_e) *is proper, then* $b^d_{A_e,B_e} = \alpha(e)$. *If* (A_e, B_e) *is a cut, then* $b^d_{A_e,B_e} = \alpha(e) + \alpha(e')$, *where* $e' \in E\Gamma$ *is the edge with* $(A_{e'}, B_{e'}) = (B_e, A_e)$.

Sketch of the proof. We show that $b^d_{A_e,B_e} \geq \alpha(e)$ for an edge $e \in E\Gamma$ whose corresponding partial cut (A_e, B_e) is proper. Let $(A, B) = (A_e, B_e)$ and $C = V \setminus (A \cup B)$. Suppose that $i \in A, j \in A \cup C, k \in B \cup C$, and $l \in B$. We denote by $I[u, v]$ the set of edges which provides $D_{\Gamma,\alpha}(F_u, F_v)$. For the choice of j, k, there are four cases: (i) $j \in A, k \in B$, (ii) $j \in A, k \in C$, (iii) $j \in C, k \in B$, and (iv) $j, k \in C$. For checking $d(i,l) + d(k, j) - d(i, j) - d(k,l) \geq \alpha(e)$, it is sufficient to show that (a) $I[i, j] \cup I[k,l] \subseteq I[i,l] \cup I[k, j]$ and (b) $e \in (I[i,l] \cup I[k, j]) \setminus (I[i, j] \cup I[k,l])$. Note that $I[i,l]$ contains e in all cases. We consider the case (ii). Since F_k contains e, we have $e \notin I[k,l]$ and $I[k,l] \subseteq I[i,l]$. Since $i, j \in A$, we have $e \notin I[i, j]$. Thus, (b) holds. Since F_k is a subtree, it is obvious that $I[i, j] = (I[i, j] \cap I[i, k]) \cup (I[i, j] \cap EF_k) \cup (I[i, j] \cap I[k, j])$. Clearly, $I[i, k] \subseteq I[i,l]$. Since F_k is an oriented path containing e, $I[i, j] \cap EF_k \subseteq I[i,l]$. Therefore, $I[i, j] \subseteq I[i,l] \cup I[k, j]$, and hence (a) holds. In a similar way, (iii) is shown. The cases (i) and (iv) are easily seen.

Let e be an edge whose corresponding partial cut (A_e, B_e) is a cut, and let $e' \in E\Gamma$ be the edge with $(A_{e'}, B_{e'}) = (B_e, A_e)$. Since we may assume that e and e' share their heads or tails, a similar argument as above shows that $b^d_{A_e,B_e} \geq \alpha(e) + \alpha(e')$.

Equalities hold by choosing appropriate elements i, j, k, l. □

Now, we apply the polyhedral split decomposition to a subpath distance d with an oriented-tree realization $(\Gamma, \alpha; \{F_v\}_{v\in V})$. The edges in Γ are classified as follows. Let $P(d)$ be the set of edges $e \in E\Gamma$ that induce proper partial cuts, and let $C(d)$ be the set of edges $e \in E\Gamma$ that correspond to cuts. Recall that (by the technical assumption) for an edge $e \in C(d)$, the edge e' with $(A_{e'}, B_{e'}) = (B_e, A_e)$ also belongs to $C(d)$. We denote by $\vec{C}(d)$ be a maximal subset of $C(d)$ such that either e or e' with $(A_e, B_e) = (B_{e'}, A_{e'})$ belongs to $\vec{C}(d)$. Note that, for a cut (A, B), $l_{H_{A,B}}$ can be identified with $l_{H_{B,A}}$ on cone$X$. As a result, we obtain the following:

$$\begin{aligned}
\overline{-d} &= \sum_{e\in P(d)} b^d_{A_e,B_e} l_{H_{A_e,B_e}} + \sum_{e\in \vec{C}(d)} b^d_{A_e,B_e} l_{H_{A_e,B_e}} + g \\
&= \sum_{e\in P(d)} \alpha(e) l_{H_{A_e,B_e}} + \sum_{e\in \vec{C}(d)} \alpha(e) l_{H_{A_e,B_e}} + \sum_{e\in \vec{C}(d)} (b^d_{A_e,B_e} - \alpha(e)) l_{H_{B_e,A_e}} + g \\
&= \sum_{e\in P(d)} \alpha(e) h_{A_e,B_e} + \sum_{e\in \vec{C}(d)} \alpha(e) h_{A_e,B_e} + \sum_{e\in C(d)\setminus \vec{C}(d)} \alpha(e) h_{A_e,B_e} + g' \qquad (4)
\end{aligned}$$

where g, g' are polyhedral convex functions on cone X and

$$h_{A,B} = \frac{|\langle \mathbf{1}_{A^c} - \mathbf{1}_{B^r}, \cdot \rangle|}{2} + \frac{\langle \mathbf{1}_{A^c} + \mathbf{1}_{B^r}, \cdot \rangle}{2}.$$

It is easy to see that $h_{A,B}(-\mathbf{1}_{uv}) = -1$ if $u \in A$ and $v \in B$; $h_{A,B}(-\mathbf{1}_{uv}) = 0$ otherwise. In fact, $h_{A,B}$ is the extension of the (negative) partial cut distance $-\delta_{A,B}$ (as a discrete function on X). Since d is represented as in (2), g' must vanish from (4). Therefore, we obtain the following as a main result of this paper.

Theorem 7. *Let d be a subpath distance on V having an oriented-tree realization $(\Gamma, \alpha; \{F_v\}_{v \in V})$. Then, we have*

$$\overline{-d} = \sum_{e \in E\Gamma} \alpha(e) h_{A_e, B_e}. \tag{5}$$

According to the duality described in [3], the decomposition of $\overline{-d}$ in (5) is equivalent to that of P_d as follows:

$$P_d = \sum_{e \in E\Gamma} \alpha(e) \{[-\mathbf{1}_{A_e^c} + \mathbf{1}_{B_e^r}, \mathbf{1}_{A_e^c} - \mathbf{1}_{B_e^r}]/2 + (\mathbf{1}_{A_e^c} + \mathbf{1}_{B_e^r})/2\} + (\mathbf{1}, -\mathbf{1})\mathbf{R} + \mathbf{R}_+^{V^{cr}},$$

where "+" means a Minkowski sum. Therefore, if d is a subpath distance, then P_d is a Minkowski sum of a zonotope, the linear space $(\mathbf{1}, -\mathbf{1})\mathbf{R}$, and the nonnegative orthant $\mathbf{R}_+^{V^{cr}}$. Recall that Q_d is the set of minimal points of P_d.

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