The Ninth International Symposium on Operations Research and Its Applications (ISORA'10)
Chengdu-Jiuzhaigou, China, August 19–23, 2010

Callable Russian Options with the Finite Maturity

Atsuo Suzuki[1,*] Katsushige Sawaki[2,†]

[1]Faculty of Urban Science, Meijo University, 4-3-3 Nijigaoka, Kani, Gifu 509-0261, Japan
[2]Nanzan Business School, Nanzan University, 18 Yamazato, Nagoya, Aichi 466-8673, Japan

Abstract We consider callable Russian options with the finite maturity. Callable Russian option is a contract that the seller and the buyer have the rights to cancel and to exercise it at any time, respectively. We discuss the pricing model of callable Russian options when the stock pays dividends continuously. We show that the pricing model can be formulated as a coupled optimal stopping problem which is analyzed as Dynkin game.

Keywords callable Russian option; optimal region; optimal stopping; first hitting times

1 Introduction

Russian option was introduced by Shepp and Shiryaev [8], [9] and is one of perpetual American lookback options. Russian option with the finite maturity was studied by Duistermaat, Kyprianou and van Schaikb [1], Ekström [2] and Peskir [7].

Russian option is the contract that only the buyer has the right to exercise it. On the other hand, callable Russian option is the contract that the seller and the buyer have both the rights to cancel and to exercise it at any time, respectively. This option value is represented as coupled optimal stopping problem for the seller and the buyer. See Cvitanic and Karatzas [3] and Kifer [4]. In the case where there is no dividend and the dividend is positive, Kyprianou [6] and Suzuki and Sawaki [10] derived the value function and its optimal boundaries, respectively. Moreover, Kunita and Seko [5] studied the value function of the game call options and their optimal regions.

In this paper, we study the value function of callable Russian options and their optimal regions. The paper is organized as follows. In Section 2 we introduce a pricing model of callable Russian options with the finite maturity by means of a coupled optimal stopping problem given by Kifer [4]. Section 3 gives the main theorem.

2 Model

We consider the Black-Scholes model. Let B_t be the riskless asset price at time t defined by $B_t = e^{rt}$, where r is a positive constant. Let S_t be the risky asset price at time t determined by

$$dS_t = (r-d)S_t dt + \kappa S_t dW_t,\ S_0 \in \mathbf{R}^+, \tag{1}$$

*atsuo@urban.meijo-u.ac.jp
†sawaki@nanzan-u.ac.jp

where d and κ are nonnegative and positive constants, respectively, d is called dividend rate and $\tilde{W}_t$ is a standard Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$.

We define another probability measure $\tilde{P}$ by

$$\frac{d\tilde{P}}{dP} = \exp\left(\kappa \tilde{W}_t - \frac{1}{2}\kappa^2 t\right).$$

Then, $\tilde{W}_t \equiv W_t - \kappa t$ is standard Brownian motion with respect to $\tilde{P}$ and S_t is represented by

$$S_t = S_0 \exp\left\{\left(r - d + \frac{1}{2}\kappa^2\right)t + \kappa \tilde{W}_t\right\}.$$

We set

$$\Psi_t(x) \equiv \max(S_0 x, \sup_{0 \le u \le t} S_u)/S_t, \ x \ge 1.$$

Let σ be a cancel time for the seller and τ be an exercise time for the buyer. Then the value function $V(x,t)$ is defined by

$$V(x,s) = \inf_{\sigma \in \mathscr{T}_{s,T}} \sup_{\tau \in \mathscr{T}_{s,T}} J_s(\sigma, \tau, x), \tag{2}$$

where

$$J_s(\sigma, \tau, x) = \tilde{E}[e^{-\alpha(\sigma \wedge \tau - s)}\{(\Psi_\sigma(x) + \delta)1_{\{\sigma < \tau\}} + \Psi_\tau(x)1_{\{\tau \le \sigma\}}\}], \ \alpha > 0$$

and $\mathscr{T}_{s,T}$ is the set of all stopping times in the interval $[s,T]$. The infimum and supremum are taken over all stopping times σ and τ, respectively. The value function $V(x,s)$ satisfies the inequalities

$$x \le V(x,s) \le x + \delta.$$

We define the sets A, B and C by

$$\begin{aligned} A &= \{(x,s) \times [0,T) \in \mathbf{R}^+; V(x,s) = x + \delta\}, \\ B &= \{(x,s) \times [0,T) \in \mathbf{R}^+; V(x,s) = x\}. \\ C &= \{(x,s) \times [0,T) \in \mathbf{R}^+; x < V(x,s) < x + \delta\}. \end{aligned}$$

These sets are the subsets of real positive numbers. The set A and B are called the seller's cancellation region and the buyer's exercise region, respectively.

Let σ_A^x and τ_B^x be the first hitting times of the process $\Psi_t(x)$ to the set A and B, respectively, i.e.,

$$\begin{aligned} \sigma_A^x &= \inf\{t > 0 \mid \Psi_t(x) \in A\} \wedge T \\ \tau_B^x &= \inf\{t > 0 \mid \Psi_t(x) \in B\} \wedge T. \end{aligned}$$

For any $x > 0$, $\hat{\sigma}_s^x \equiv \sigma_A^x$ and $\hat{\tau}_s^x \equiv \tau_B^x$ attain the infimum and the supremum. Therefore, we have

$$V(x,s) = J_s(\hat{\sigma}_s^x, \hat{\tau}_s^x, x).$$

When the sets A and B are empty, we understand that $\hat{\sigma}_s^x = T$ and $\hat{\tau}_s^x = T$.

3 Main Theorem

In this section, we give the main theorem. In order to prove it, we need the following lemmas.

Lemma 1.
The value function is nondecreasing in x for any s. and is Lipschitz continuous in x for any s. And it holds

$$0 \leq \frac{\partial V(x,s)}{\partial x} \leq 1. \tag{3}$$

Proof. Replacing the optimal stopping times $\hat{\sigma}_s^x$ and $\hat{\tau}_s^y$ from the nonoptimal stopping times $\hat{\sigma}_s^y$ and $\hat{\tau}_s^x$, we have

$$\begin{aligned} V(y,s) &\geq J_s(\hat{\sigma}_s^y, \hat{\tau}_s^x, y) \\ V(x,s) &\leq J_s(\hat{\sigma}_s^y, \hat{\tau}_s^x, x), \end{aligned}$$

respectively. Note that $z_1^+ - z_2^+ \leq (z_1 - z_2)^+$. For any $x > y$, we have

$$\begin{aligned} 0 \leq V(x,s) - V(y,s) &\leq J_s(\hat{\sigma}_s^y, \hat{\tau}_s^x, x) - J_s(\hat{\sigma}_s^y, \hat{\tau}_s^x, y) \\ &= \hat{E}[e^{-\alpha(\hat{\sigma}_s^y \wedge \hat{\tau}_s^x)}(\Psi_{\hat{\sigma}_s^y \wedge \hat{\tau}_s^x}(x) - \Psi_{\hat{\sigma}_s^y \wedge \hat{\tau}_s^x}(y))] \\ &= \hat{E}[e^{-\alpha(\hat{\sigma}_s^y \wedge \hat{\tau}_s^x)} H^{-1}(s, \hat{\sigma}_s^y \wedge \hat{\tau}_s^x)((x - \sup H(s,u))^+ \\ &\quad -(y - \sup H(s,u)^+)] \\ &\leq (x-y)\hat{E}[e^{-\alpha(\hat{\sigma}_s^y \wedge \hat{\tau}_s^x)} H^{-1}(s, \hat{\sigma}^y \wedge \hat{\tau}^x)], \end{aligned}$$

where

$$H(s,t) = \exp\left\{ \left(r - d + \frac{1}{2}\kappa^2 \right)(t-s) + \kappa(\tilde{W}_t - \tilde{W}_s) \right\}.$$

Since the above expectation is less than 1, we have

$$0 \leq V(x,s) - V(y,s) \leq x - y.$$

This means that $V(x,s)$ is Lipschitz continuous in x and it holds (3). □

Lemma 2.
Let $V^*(x,s)$ be the value function of Russian option with the finite maturity and let $\delta^* = V^*(1,s) - 1$. If $\delta > \delta^*$, the seller never cancels. Therefore callable Russian options are reduced to Russian options with the finite maturity.

Proof. We set $U(x) = V^*(x,s) - x - \delta$. $h'(x) = V^{*\prime}(x,s) - 1 < 0$. Because we know $h(1) = V^*(1,s) - 1 - \delta = \delta^* - \delta < 0$ by the condition $\delta \geq \delta^*$, we have $h(x) < 0$, *i.e.*, $V^*(x,s) < x + \delta$ holds. By using the relation $V(x,s) \leq V^*(x,s)$ we obtain $V(x,s) < x + \delta$, *i.e.*, it is optimal for the seller not to cancel. Therefore the seller never cancels the contract for $\delta \geq \delta^*$. □

Remark 3.
Since $\Psi_t(x) \geq \Psi_0(x) = x \geq 1$, it follows that the seller's optimal cancellation region A is a point $\{1\}$.

Lemma 4.
The value function $V(x,s)$ is convex in x.

Proof. The function V satisfies

$$\frac{1}{2}\kappa^2 x^2 \frac{\partial^2 V}{\partial x^2} = -\frac{\partial V}{\partial s} - (r-d)x\frac{\partial V}{\partial x} + \alpha V.$$

If $r \le d$, we get $\frac{\partial^2 V}{\partial x^2} > 0$. Next assume that $r > d$. We consider function $\tilde{V}(x) = V(-x)$ for $x < 0$. Then,

$$\frac{1}{2}\kappa^2 x^2 \frac{\partial^2 \tilde{V}}{\partial x^2} - (r-d)x\frac{\partial \tilde{V}}{\partial x} - \alpha\tilde{V} = \frac{1}{2}\kappa^2 x^2 \frac{\partial^2 V}{\partial x^2} + (r-d)x\frac{\partial V}{\partial x} - \alpha V = 0.$$

Since we find that $\frac{\partial^2 \tilde{V}}{\partial x^2} > 0$ from the above equation, $\tilde{V}$ is a convex function. It follows from this fact that V is a convex function. □

Lemma 5.
Suppose $d = 0$. The the first derivative $\frac{\partial V}{\partial x}(x,s)$ is strictly increasing.

From the above lemmas, we have the following theorems.

Theorem 6.
Let A and B be the seller's cancellation region and the buyer's exercise region, respectively.

1. *If $\delta < \delta^*$, the seller's cancellation region is $A = \{1\}$.*
2. (a) *If $d = 0$, the buyer's exercise region is empty, i.e., the buyer never exercises.*
   (b) *Suppose $d > 0$. Then the buyer's exercise region is*

$$B = \{x; b(s) \le x < \infty\},$$

   where $(b(s), s \in [0,T))$ is a nonincreasing function.

Theorem 7.
Let $V(x,s)$ be the value function of callable Russian option with the finite maturity defined by (2). Then we have the following.

1. *The function $V(x,s)$ is convex with respect to x for any s and Lipschitz continuous with respect to x for any s.*
2. (a) *Suppose $d = 0$. If $\delta \ge \delta^*$, the value function $V(x,s) = V_E(x,s)$. When $\delta < \delta^*$, we get $V(x,s) < V_E(x,s)$, where $V_E(x,s)$ is the value function of the European call option with the exercise price $K > 0$ and is given by*

$$V_E(x,s) = \tilde{E}[e^{-r(T-s)}(S_t - K)^+ \mid S_0 = x].$$

   (b) *Suppose $d > 0$. If $\delta \ge \delta^*$, we have $V(x,s) = V^*(x,s)$. When $\delta < \delta^*$, we get $V(x,s) < V^*(x,s)$.*
3. *The first derivative $\frac{\partial V}{\partial x}(x,s)$ is increasing and it satisfies*

$$\frac{\partial V}{\partial x}(b(s)-,s) = \frac{\partial V}{\partial x}(b(s)+,s) = 1.$$

Acknowledgements

This work was supported by Grant-in-Aid for Scientific Research(A)(20241037) and Grant-in-Aid for Young Scientists (B)(22710154).

References

[1] Duistermaat, J.J., Kyprianou, A.E. and van Schaikb, K., Finite expiry Russian options, *Stochastic Processes and their Applications*, **115**, 609-638, (2005).

[2] Ekström, E., Russian options with a finite time horizon, *Journal of Applied Probability*, **41**, 313-326, (2004).

[3] Cvitanic, J. and Karatzas, I., Backward stochastic differential equations with reflection and Dynkin games, *The Annals of Probability*, **24**, 2024-2056, (1996).

[4] Kifer, Y., Game options, *Finance and Stochastics*, **4**, 443-463, (2000).

[5] Kunita, H. and Seko, S., Game call options and their exercise regions, *Technical Report of the Nanzan Academic Society*, (2004).

[6] Kyprianou, A.E., Some calculations for Israeli options, *Finance and Stochastics*, **8**, 73–86, (2004).

[7] Peskir, G., The Russian option: Finite horizon, *Finance and Stochastics*, **9**, 251-267, (2005).

[8] Shepp, L.A. and Shiryaev, A.N., The Russian option: reduced regret, *The Annals of Applied Probability*, **3**, 631-640, (1993).

[9] Shepp, L.A. and Shiryaev, A.N., A new look at pricing of the ‘Russian option‘, *Theory of Probability and its Applications*, **39**, 103-119, (1994).

[10] Suzuki, A. and Sawaki, K., Callable Russian options and their optimal boundaries, *Journal of Applied Mathematics and Decision Sciences*, Volume 2009, Article ID 593986, 13 pages, doi:10.1155/2009/593986, (2009).