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A Resource Allocation Mode Based on DEA Models and Elasticity Analysis*

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Abstract This paper proposes a new resource allocation mode based on DEA models and elasticity analysis for a certain kind of resource allocation problems. It takes into account both the relative efficiency and the returns-to-scale of the decision making units. The decision maker can adopt it to comprehensively evaluate the departments' production capacity and their potential production capacity. At last, this paper applies this mode into the single input and single output case. The results show that it can reflect the returns-to-scale more precisely than before.

Keywords Data Envelopment Analysis (DEA); resource allocation problem; returns-to-scale; elasticity

1 Introduction

The resource allocation problem has a great practical applied value. The difficulty of this problem is: how to evaluate the departments involved in the allocation, and how to determine their allocation weights. Recently, using DEA models to solve this problem has become a new research area. That is because DEA is a synthetic method measuring the relative efficiency of homogeneous production departments (referred to as decision making units, DMU) (see [1][4][5]). Moreover, the inputs and outputs weights vector obtained by DEA models is the Pareto solution for multiobjective programming, and it satisfies the Nash equilibrium condition as well(see [7]). In paper [8] and [9], they clearly define the extra resource allocation problem, and propose that the decision maker should take into account both the efficiency and the scale. Subsequently, paper [2] and [3] suggest the decision maker should consider the returns-to-scale also when dealing with this problem. Returns-to-scale (RTS) is a concept in economics, which reflects the potential production capacity when inputs are increased (see [4][5]). It has 4 types: the increase returns-to-scale (IRS), the constant returns-to-scale (CRS), the decrease returns-to-scale (DRS), and the Congestion. Thus the previous way of solving the resource allocation problem is: calculate the efficiency and the type of returns-to-scale from the DEA models for each DMU, then figure out all DMUs' allocation weights. However, this resource allocation mode based on the 4 types of RTS has some shortcomings, which will be illustrated

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No.	DMU	Input	Output	θ_{CCR}^{Input}	RTS	ρ_i	ω_i
1	A	3	1	0.3333333	IRS	0.9	0.044735
2	B	3.3	1.2	0.3636364	IRS	0.9	0.048802
3	C	3.5	2.5	0.7142857	IRS	0.9	0.095861
4	D	4	2.4	0.6	IRS	0.9	0.080523
5	E	4.5	2.7	0.6	IRS	0.9	0.080523
6	F	4.5	3.6	0.8	IRS	0.9	0.107364
7	G	5	5	1	CRS	1	0.120785
8	H	5.5	3.3	0.6	CRS	1	0.072471
9	I	5.5	4.4	0.8	CRS	1	0.096628
10	J	6	6	1	CRS	1	0.120785
11	K	7	4.2	0.6	DRS	0.7	0.056366
12	L	7	5.6	0.8	DRS	0.7	0.075155
13	M	8	7	0.875	DRS	0	0
14	N	8.5	6	0.7058824	Congestion	0	0
15	O	9	7	0.7777778	Congestion	0	0

Table 1

by the instance in section 2. Therefore, we bring in the elasticity, a concept in economics, to precisely characterize the RTS, and propose a new resource allocation mode based on DEA models and elasticity analysis.

2 Background

A factory has 15 departments: A, B, C,. . . ,O. These departments use the same kind but different amount raw material to produce product, as can be seen in table 1. Here the Input indicates the amount of raw material each department used (taken 1000 kg as a unit), and the Output indicates the number of product each department produced (taken 100 pieces as a unit). If the factory buy extra 100,000 kg raw material, how should the decision maker allocate it among these 15 departments? This problem with 15 DMUs is a single input and single output case.

Furthermore, it can be reduced to such kind of resource allocation problem: suppose there are some extra resource which can be given to all or only a part of DMUs, and if we want the allocation to be most beneficial to the whole system, how the extra resource should be distributed (see [8]).

Dealing with this problems, paper [2] and [3] point out that it should consider both the departments' production capacity (i.e. the efficiency) and their potential production capacity (i.e. the returns-to-sale). And the allocation weights are obtained based on the 4 types of the RTS(see [2]). Next we will explain it is unsatisfactory.

According to the real-time water resource allocation mechanism in [2], we use DEA models to calculate these 15 DMUs' efficiency (here we use the CCR-efficiency) and the types of RTS. Subsequently the production possibility set can be divided into these 4 type (see figure 1), which are assigned the weights as follows:

$i\varepsilon IRS$	$i \in CRS$	$i \in DRS$	$i \in Congestion$
1	0.9	0.7	0

Table 2

In the end, compute the resource allocation weights for each DMU: $\omega_i = \frac{\theta_i \rho_i}{\Sigma \theta_i \rho_i}$, i=1, ..., 15. The results can be seen in table 1.

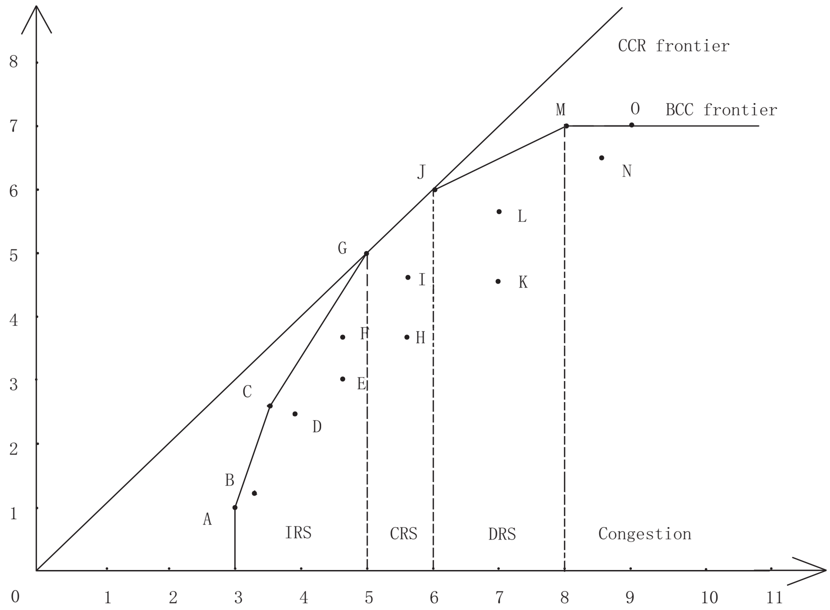

Figure 1

However, the allocation weights are undesirable. Take the DMU D and the DMU E for example, their CCR efficiency are 0.6, and they are both IRS. Thus they are allocated the same amount of resource. As can be seen in figure 1, it is obvious that DMU D has more potential production capacity than DMU E, so DMU D should be allocate more resource than DMU E. The short of this allocation mechanism is that it roughly assigns the same weights to the DMUs which are in the same region of RTS.

3 A New Resource Allocation Mode

For the efficient DMUs, i.e. the DMUs which are on the production frontier, we can easily determine their returns-to-scale (see [4][5]). For the inefficient DMUs, i.e. the DMUs which are enveloped in the production set by the production frontier, we project them onto the production frontier, and use their projections to decide their returns-to-scale (see [4][5]). Therefore, we study the property of projection first.

Theorem 1.
In both the CCR Output-oriented model and the BCC Output-oriented model, if

$DMU_1(X_1,Y_1)$ and $DMU_2(X_2,Y_2)$ have $X_1 = X_2$ $Y_2 = kY_1$ (k is a positive real number), then their efficiency θ_1 and θ_2 such that $\theta_2 = \theta_1/k$.

Proof : We will only consider the case in BCC model, the case in CCR model can be proved in the same way. Use the BCC model to calculate the optimal solutions of $DMU_1(X_1,Y_1)$ and $DMU_2(X_2,Y_2)$. Let them be $(\theta_1^*|\lambda_1^*, s_1^{-*}, s_1^{+*})$ and $(\theta_2^*|\lambda_2^*, s_2^{-*}, s_2^{+*})$. Then bring them back to BCC model:

$$\begin{aligned} &max \quad \theta_1^* \\ s.t.\quad & X\lambda_1^* = X_1 - s_1^{-*} \\ & Y\lambda_1^* = \theta_1^* Y_1 + s_1^{+*} \\ & e\lambda = 1 \\ & \lambda \geq 0; \end{aligned} \qquad [1] \qquad\qquad \begin{aligned} &max \quad \theta_2^* \\ s.t.\quad & X\lambda_2^* = X_2 - s_2^{-*} \\ & Y\lambda_2^* = \theta_2^* Y_2 + s_2^{+*} \\ & e\lambda = 1 \\ & \lambda \geq 0; \end{aligned} \qquad [2]$$

From mode [l], we can get $Y\lambda_1^* - s_1^{+*} = \theta_1^* Y_1 = \theta_1^*(Y_2/k) = (\theta_1^*/k)Y_2$, thus $(\theta_1^*/k|\lambda_1^*, s_1^{-*}, s_1^{+*})$ is a feasible solution of model [2]. $(\theta_2^*|\lambda_2^*, s_2^{-*}, s_2^{+*})$ is defined as an optimal solution of model [2], so $\theta_2^* \geq \theta_1^*/k$. From model [2], we can get $Y\lambda_2^* - s_2^{+*} = \theta_2^* Y_2 = \theta_2^*(kY_1) = (k\theta_2^*)Y_1$, thus $(k\theta_2^*|\lambda_2^*, s_2^{-*}, s_2^{+*})$ is a feasible solution of model [1]. $(\theta_1^*|\lambda_1^*, s_1^{-*}, s_1^{+*})$ is defined as an optimal solution of model [1], so $\theta_1^* \geq k\theta_2^*$. Then we have $\theta_2^* = \theta_1^*/k$. □

Theorem 2.
In both the CCR Output-oriented model and the BCC Output-oriented model, if $DMU_1(X_1,Y_1)$ and $DMU_2(X_2,Y_2)$ have $X_1 = X_2$ $Y_2 = kY_1$ (k is a positive real number), then their projections $(\hat{X}_1, \hat{Y}_1)$ and $(\hat{X}_2, \hat{Y}_2)$ such that $(\hat{X}_1, \hat{Y}_1) = (\hat{X}_2, \hat{Y}_2)$.

Proof : We will only consider the case in BCC model, the case in CCR model can be proved in the same way. Use the BCC model to calculate the optimal solutions of $DMU_1(X_1,Y_1)$ and $DMU_2(X_2,Y_2)$. Let them be $(\theta_1^*|\lambda_1^*, s_1^{-*}, s_1^{+*})$ and $(\theta_2^*|\lambda_2^*, s_2^{-*}, s_2^{+*})$. According to the theorem 1, we have $\theta_2^* = \theta_1^*/k$. By the proof of theorem 1, we know that $(\theta_1^*/k|\lambda_1^*, s_1^{-*}, s_1^{+*})$ is a feasible solution of model [2], so $(\theta_2^*|\lambda_1^*, s_1^*, s_1^*)$ is also an optimal solution of model (2). The projection of DMU_1 $(\hat{X}_1, \hat{Y}_1)$ have $\hat{X}_1 = X_1 - s_1^* = X\lambda_1^*$ and $\hat{Y}_1 = \theta_1^* Y_1 + s_1^{+*} = Y\lambda_1^*$. The projection of DMU_2 $(\hat{X}_2, \hat{Y}_2)$ have $\hat{X}_2 = X_2 - s_1^* = X\lambda_1^*$ and $\hat{Y}_2 = \theta_2^* Y_2 + s_1^{+*} = Y\lambda_1^*$. Then $(\hat{X}_1, \hat{Y}_1) = (\hat{X}_2, \hat{Y}_2)$. □

Theorem 3.
In both the CCR Input-oriented model and the BCC Input-oriented model, if $DMU_1(X_1,Y_1)$ and $DMU_2(X_2,Y_2)$ have $X_2 = kX_1$ $Y_1 = Y_2$(k is a positive real number), then their efficiency have $\theta_1 = k\theta_2$, and their projections have $(\hat{X}_1, \hat{Y}_1) = (\hat{X}_2, \hat{Y}_2)$.

Proof :it can be proved in the same way as the proof of theorem 1. □

Corollary: In both the CCR Output-oriented model and the BCC Output-oriented model, when it is in the case of single input and single output, if there are two DMUs whose inputs are same, then they will be projected to the same point on the frontier. That is their projections are same, and their returns-to-scale are same as well.

For the instance in section 2, according to the above corollary, the inefficient DMUs will be projected onto the BCC frontier in the direction of Y-axis (see figure 2). There are two rays under the CCR frontier. The points on the same ray have the same CCR efficiency (see[4][5]). Because it is easier for show in the figure, we use the CCR efficiency (not BCC efficiency) to reflect these DMUs' production capacity. For example, DMU D and DMU E have the same CCR efficiency, and they are both IRS. But it is obvious that DMU D has more growth potential than DMU E, so DMU D should be allocate more resource. Therefore, we have to finely decide the degree of the returns-to-scale for each DMU.

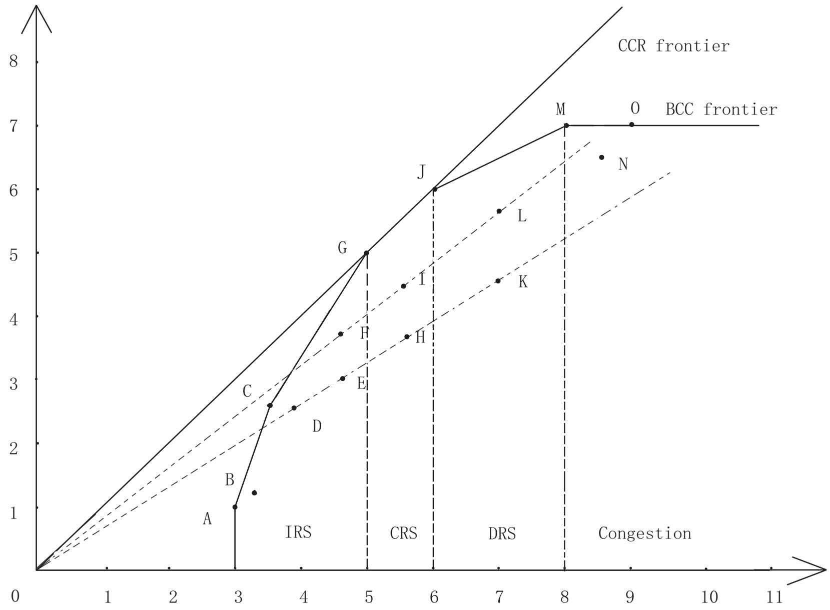

Figure 2

We resort to the elasticity, a economic concept, to characterize the returns-to-scale. Here the elasticity is the ratio of the output's change to the input's change. Suppose there are n DMUs to be allocated, for a efficient DMU_i, its elasticity is: $E_i = \lim_{\Delta \to 0} \frac{\Delta f(x)/f(x)}{\Delta x/x} = \frac{\partial f(x)}{\partial x_i} \cdot \frac{x_i}{f(x)}$.

The elasticity can reflect the returns-to-scale as well: if $E_i > 1$, then DMU_i is IRS; if $E_i = 1$, then DMU_i is CRS, if $0 < E_i < 1$, then DMU_i is DRS; if $E_i = 0$, then DMU_i is Congestion.

When we calculate elasticity, we adopt the BCC model. That is because BCC frontier can be seen as an approximation of the production frontier (see [6]). Because of the in resource allocation problem, we use the right derivative of BCC frontier. For the efficient DMUs, we get their elasticity by above formula. For the inefficient DMUs, we can calculate the elasticity of their projections. Therefore, for DMU_i, the steps of determining its allocation weights are: Firstly, use the CCR model(or BCC model) to compute its efficiency, denote as θ_i. Secondly, if it is BCC-efficienct, we compute its elasticity according to the above formula; else if it is BCC-inefficienct, we compute its projection's elasticity.

Denote it as E_i. Thirdly, compute its allocation weight:$\bar{\omega}_i = \frac{\theta_i E_i}{\sum \theta_i E_i}$.

4 Numerical Example

In this section, we recalculate the instance of section 2 and get a new set of allocation weights $\bar{\omega}_i$, i=1,...,n, according to our resource allocation mode based on DEA models and elasticity. The result can be seen in table 3. This time the weights of DMU D and DMU E are not same. The weight of DMU D is larger than the weight of DMU E, so DMU D can be allocated more resource. By comparison of these two sets of allocation weight, it can be seen that the new one is more reasonable than before.

DMU	θ_{CCR}^{Input}	θ_{BCC}^{Input}	θ_{BCC}^{Output}	$Projections_{BCC}^{Output}$	E_i	ω_i	$\bar{\omega}_i$
A	0.3333333	1	1		9	0.044735	0.215281
B	0.3636364	0.929292	1.583333	(3.3 , 1.9)	5.2105263	0.048802	0.135967
C	0.7142857	1	1		2.3333333	0.095861	0.119601
D	0.6	0.866667	1.388889	(4 , 3.3333)	2	0.080523	0.086113
E	0.6	0.804444	1.54321	(4.5 , 4.1667)	1.8	0.080523	0.077501
F	0.8	0.924444	1.157407	(4.5 , 4.1667)	1.8	0.107364	0.103335
G	1	1	1		1	0.120785	0.07176
H	0.6	0.723636	1.666667	(5.5 , 5.5)	1	0.072471	0.043056
I	0.8	0.843636	1.25	(5.5 , 5.5)	1	0.096628	0.057408
J	1	1	1		0.5	0.120785	0.03588
K	0.6	0.645714	1.547619	(7 , 6.5)	0.5384615	0.056366	0.023184
L	0.8	0.8	1.160714	(7 , 6.5)	0.5384615	0.075155	0.030912
M	0.875	1	1		0	0	0
N	0.7058824	0.705882	1.166667	(8 , 7)	0	0	0
O	0.7777778	0.888889	1	(8 , 7)	0	0	0

Table 3

5 Conclusion

Recently the resource allocation problem has become a hot topic in the management and decision-making field. Paper [2] and [3] suggest that the decision-maker should consider both the production efficiency and the returns-to-scale when dealing with this problem. This paper proposes a new resource allocation mode based on DEA models and elasticity analysis. First calculate the efficiency from the DEA mode. Then compute their elasticity. Finally compute the allocation weights. As can be seen in the instance, the allocation weights get from this mode is more reasonable than before. However, we just apply this mode in single input and single output case. Next we will try to apply it in multi-input and multi-output case, so as to provide a common mode of resource allocation problem.

The essential of the weights given by this resource allocation mode is the evaluation of these departments' production capacity and their potential production capacity. Therefore, besides the resource allocation problem, we can apply it to some other management problems, such as sorting, evaluation and choice, etc. In these cases, the weights can be seen as a marking for the production status of these departments. By reference of these weights, the decision-maker can make a better decision.

References

[1] A.Charnes, W. W. Cooper, and E. Rhodes, Measuring the efficiency of decision making units, European Journal of Operational Research, 1978, 2(6): 429-444.

[2] X. Y. Li and J. C. Cui, Real-time water resource allocation: methodology and mechanism. IEEM 2009 Proceedings.

[3] X. Y. Li and J. C. Cui, A comprehensive DEA approach for the resource allocation problem based on scale economies classification, Journal of Systems Science and Complexity, 2008, 21: 540-557

[4] W. W. Cooper, L. M. Seiford, and Kaoru Tone, Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References, and DEA-Solver Software, Kluwer Academic Publishers, Norwell, MA, 2000.

[5] Quanling Wei, Data Envelopment Analysis(in Chinese), Science Press, 2004

[6] R. D. Banker, A. Charnes, and W. W. Cooper, Some models for estimating technical and scale ineffciencies in data envelopment analysis, Management Science, 1984, 30(9): 1079-1092.

[7] X. Y. Li and J. C. Cui, n-person game model with payoff formed by relative efficiency score, OR TRANSECTIONS (in Chinese), 2007,11: 183-189.

[8] X. Y. Li and J. C. Cui, Arithmetic of extra resource allocation based on DEA method, Journal of Systems Science and Complexity (in Chinese), 2007, 22(1): 57-61

[9] X. Y. Li and J. C. Cui, Extra resource allocation problem with the most compromise common weights based on DEA, Lecture Notes in Operational Research, ORSC and APORC, 2007, 7: 397-404.