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Cone Quasi-convexity of Set-Valued Mappings in Topological Vector Spaces

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Abstract In this paper, we summarize several different definitions of cone quasiconvex set-valued mappings proposed in recent papers and study the relationships among them. The conclusions show that some of these definitions of cone quasiconvex set-valued mappings are equal, however some are different. Furthermore, a criteria for cone quasiconvexity of set-valued mappings is obtained under lower semicontinuity.

Keywords Cone quasiconvexity; Set-valued mappings; Denseness; Lower semicontinuity

1 Introduction

The research on convexity and generalized convexity is one of the most important aspects of mathematical programming, and the subjects discussed constitute one of the current trends in this area of problems. Numerous generalizations of convex functions have been derived which have proved to be useful for extending optimality conditions and connectedness of solution sets to larger classes of optimization problems. Quasi-convexity is a kind of important generalized convexity. Recently, set-valued mappings optimization problems have attracted a great of attention. In [1-5], various cone quasiconvexity of set-valued mappings have been proposed to express these optimality conditions and connectedness of solution sets, which are extensions of quasiconvexity of real-valued functions in $\mathfrak{R}^n$ [6].

In the present paper, we investigate relationships among these definitions of cone quasiconvex set-valued mappings and give a sufficient condition for cone quasiconvexity of set-valued mappings.

Throughout this paper, let X be a topological vector space, Y be an ordered topological vector space partially ordered by a pointed closed convex $\wedge \subset Y$, $A \subset X$ be a nonempty convex set and $F : A \to 2^Y$ be a set-valued mappings.

2 Relationships among different cone quasi-convex set-valued mappings

In this section, we investigate the relationships among different cone quasiconvex set-valued mappings introduced in [1-5].

Definition 2.1.([1]) F is said to be S-$\wedge$-quasiconvex, if any $x_1, x_2 \in A$, $y \in Y$, $F(x_1) \cap (y-\wedge) \neq \emptyset$ and $F(x_2) \cap (y-\wedge) \neq \emptyset$ imply $F(\lambda x_1 + (1-\lambda)x_2) \cap (y-\wedge) \neq \emptyset$ for any $\lambda \in [0,1]$.

Theorem 2.1. F is S-$\wedge$-quasiconvex if and only if $F^{-1}(y-\wedge) = \{x \in A | F(x) \cap (y-\wedge) \neq \emptyset\}$ is convex for any $y \in Y$.

Proof. Suppose that F is S-$\wedge$-quasiconvex. Let $y \in Y, x_1, x_2 \in F^{-1}(y-\wedge)$. Then,

$$x_1, x_2 \in A, F(x_1) \cap (y-\wedge) \neq \emptyset \text{ and } F(x_2) \cap (y-\wedge) \neq \emptyset.$$

Since A is convex set and F is S-$\wedge$-quasiconvex, we have

$$\lambda x_1 + (1-\lambda)x_2 \in A \text{ and } F(\lambda x_1 + (1-\lambda)x_2) \cap (y-\wedge) \neq \emptyset \quad \text{for any } \lambda \in [0,1].$$

That is, $\lambda x_1 + (1-\lambda)x_2 \in F^{-1}(y-\wedge)$ for any $\lambda \in [0,1]$. Hence, $F^{-1}(y-\wedge)$ is convex.

Conversely, assume that $F^{-1}(y-\wedge)$ is convex for any $y \in Y$. Let $x_1, x_2 \in A$, $y \in Y$, $F(x_1) \cap (y-\wedge) \neq \emptyset$ and $F(x_2) \cap (y-\wedge) \neq \emptyset$. Then,

$$x_1, x_2 \in F^{-1}(y-\wedge).$$

Thus, for any $\lambda \in [0,1]$, we get

$$\lambda x_1 + (1-\lambda)x_2 \in F^{-1}(y-\wedge),$$

and it follows that

$$F(\lambda x_1 + (1-\lambda)x_2) \cap (y-\wedge) \neq \emptyset, \quad \text{for any } \lambda \in [0,1].$$

This means, by Definition 2.1, F is S-$\wedge$-quasiconvex.

Definition 2.2.([3]) F is J-$\wedge$-quasiconvex , if for any $y \in Y$ the level set $L_F(y) = \{x \in A | y \in F(x) + \wedge\}$ is convex.

By Theorem 2.1, we obtain the following result

Theorem 2.2. F is S-$\wedge$-quasiconvex if and only if F is J-$\wedge$-quasiconvex.

Definition 2.3.([4]) F is said to be L-$\wedge$-quasiconvex, if any $x_1, x_2 \in A$, $F(x_1) \subset F(x_2) + \wedge$ imply $F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) + \wedge$ for any $\lambda \in [0,1]$.

Theorem 2.3. If F is S-$\wedge$-quasiconvex, then F is L-$\wedge$-quasiconvex.

Proof. Let $x_1, x_2 \in A, F(x_1) \subset F(x_2) + \wedge$. Then, for any $y \in F(x_1)$,we obtain

$$F(x_1) \cap (y-\wedge) \neq \emptyset \text{ and } F(x_2) \cap (y-\wedge) \neq \emptyset.$$

That is,

$$x_1, x_2 \in F^{-1}(y-\wedge).$$

Since, from Theorem 2.1, $F^{-1}(y-\wedge)$ is convex, then for any $\lambda \in [0,1]$,

$$\lambda x_1 + (1-\lambda)x_2 \in F^{-1}(y-\wedge),$$

and it follows that

$$F(\lambda x_1 + (1-\lambda)x_2) \cap (y-\wedge) \neq \emptyset, \quad \text{for any } \lambda \in [0,1].$$

Thus,

$$y \in F(\lambda x_1 + (1-\lambda)x_2) + \wedge, \quad \text{for any } \lambda \in [0,1].$$

Since y is an arbitrary points belonging to $F(x_1)$, we have

$$F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) + \wedge,$$

which shows, by Definition 2.3, F is L-$\wedge$-quasiconvex.

Remark 2.1. The converse of Theorem 2.3 is not true. See example 2.1.

Example 2.1. Let $F : \mathbb{R} \to 2^{\mathbb{R}^2}$ be defined by as

$$F(x) = \begin{cases} \{(x,3)^{\mathrm{T}}\} & x \le 0. \\ \{(x,3-x)^{\mathrm{T}}\} & x \ge 0 \text{ and } x \ne 1 \\ \{(4,4)^{\mathrm{T}}\} & x = 1. \end{cases}$$

and $\wedge = \mathbb{R}^2_+$. It is easy to verify that F is is L-$\wedge$-quasiconvex, but F is not S-$\wedge$-quasiconvex.

Definition 2.4. Let $M \subset Y$ and $N \subset M$. N is said to be externally stable if, for each $y \in M \setminus N$, there exists some $\hat{y} \in N$ such that $y \in \hat{y} + \wedge$.

Definition 2.5. Let $y_1, y_2, \dots, y_m \in Y (m \ge 1)$. A point $\tilde{y} \in \bigcap_{i=1}^m (y_i + \wedge)$ is said to be a $\wedge-$bound point of vector set $\{y_1, y_2, \dots, y_m\}$, if there exists no $y \in \bigcap_{i=1}^m (y_i + \wedge)$ such that $\tilde{y} - y \in \wedge \setminus \{0\}$. The set of $\wedge-$bound point of vector set $\{y_1, y_2, \dots, y_m\}$ is denoted by $\wedge - \text{bou}\{y_1, y_2, \dots, y_m\}$.

Definition 2.6.([5]) F is said to be C-$\wedge-$quasiconvex , if for any $x_1, x_2 \in A, y_1 \in F(x_1), y_2 \in F(x_2)$ and $\lambda \in [0,1]$, we have

$$\wedge - \text{bou}\{y_1, y_2\} \subset F(\lambda x_1 + (1-\lambda)x_2) + \wedge.$$

Theorem 2.4. if F is C-$\wedge-$quasiconvex and $\wedge - \text{bou}\{y_1, y_2\}$ is externally stable for any $x_1, x_2 \in A, y_1 \in F(x_1), y_2 \in F(x_2)$, then F is S-$\wedge-$quasiconvex.

Proof. Let $y \in Y, x_1, x_2 \in F^{-1}(y - \wedge)$. Then, $x_1, x_2 \in A$ and there exist $y_1 \in F(x_1), y_2 \in F(x_2)$ such that

$$y \in y_1 + \wedge, y \in y_2 + \wedge.$$

That is,

$$y \in (y_1 + \wedge) \cap (y_2 + \wedge).$$

Case(i): $y \in \wedge - \text{bou}\{y_1, y_2\}$. since F is C-$\wedge-$quasiconvex ,we get

$$y \in F(\lambda x_1 + (1-\lambda)x_2) + \wedge, \quad \text{for any } \lambda \in [0,1].$$

Then, $\lambda x_1 + (1-\lambda)x_2 \in F^{-1}(y - \wedge)$ for any $\lambda \in [0,1]$. Hence, $F^{-1}(y - \wedge)$ is convex.

Case(ii): $y \notin \wedge - \text{bou}\{y_1, y_2\}$. Since $\wedge - \text{bou}\{y_1, y_2\}$ is externally stable, there exists $\hat{y} \in \wedge - \text{bou}\{y_1, y_2\}$ such that $y - \hat{y} \in \wedge \setminus \{0\}$. That is, $y \in \hat{y} + \wedge$.

Since F is C-$\wedge-$quasiconvex, we have

$$\hat{y} \in F(\lambda x_1 + (1-\lambda)x_2) + \wedge, \quad \text{for any } \lambda \in [0,1].$$

Therefore,

$$y \in \hat{y}+\wedge \subset F(\lambda x_1+(1-\lambda)x_2)+\wedge+\wedge = F(\lambda x_1+(1-\lambda)x_2)+\wedge, \quad \text{for any } \lambda \in [0,1].$$

That is, $\lambda x_1 + (1-\lambda)x_2 \in F^{-1}(y-\wedge)$ for any $\lambda \in [0,1]$. Hence, $F^{-1}(y-\wedge)$ is convex.

From Theorem 2.1, F is S-$\wedge$−quasiconvex.

Theorem 2.5. if F is S-$\wedge$−quasiconvex , then F is C-$\wedge$−quasiconvex.

Proof. Take any $x_1,x_2 \in A, y_1 \in F(x_1), y_2 \in F(x_2), y \in \wedge - \text{bou}\{y_1,y_2\}$. Then, we have

$$y \in (y_1+\wedge)\cap(y_2+\wedge),$$

and it follows that

$$y \in F(x_1)+\wedge, y \in F(x_2)+\wedge.$$

That is,

$$x_1,x_2 \in F^{-1}(y-\wedge).$$

Since F is S-$\wedge$−quasiconvex, then for any $\lambda \in [0,1]$,

$$\lambda x_1 + (1-\lambda)x_2 \in F^{-1}(y-\wedge),$$

and it follows that

$$F(\lambda x_1 + (1-\lambda)x_2)\cap(y-\wedge) \neq \emptyset, \quad \text{for any } \lambda \in [0,1].$$

That is,

$$y \in F(\lambda x_1 + (1-\lambda)x_2)+\wedge, \quad \text{for any } \lambda \in [0,1].$$

Therefore,

$$\wedge - \text{bou}\{y_1,y_2\} \subset F(\lambda x_1 + (1-\lambda)x_2)+\wedge, \quad \text{for any } \lambda \in [0,1].$$

This means, by Definition 2.6, F is C-$\wedge$−quasiconvex.

3 Criteria for cone quasiconvexity of set-valued mappings

Definition 3.1.([7]) F is called to be lower semicontinuous (l.s.c in brief) at $\hat{x} \in A$, if for any open set V satisfying $F(\hat{x})\cap V \neq \emptyset$, there exists a neighborhood U of $\hat{x}$ such that $F(x)\cap V \neq \emptyset$, $\forall x \in U \cap A$.

F is said to be l.s.c on A, if F is l.s.c at every point $x \in A$.

Lemma 3.1. If there exists $\alpha \in (0,1)$, such that $F(x_1)\cap(y-\wedge) \neq \emptyset$ and $F(x_2)\cap(y-\wedge) \neq \emptyset$ imply $F(\alpha x_1 + (1-\alpha)x_2)\cap(y-\wedge) \neq \emptyset$, for any $x_1,x_2 \in A, y \in Y$, then the set $K = \{\lambda \in [0,1] | F(x_1)\cap(y-\wedge) \neq \emptyset$ and $F(x_2)\cap(y-\wedge) \neq \emptyset$ imply $F(\lambda x_1 + (1-\lambda)x_2)\cap(y-\wedge) \neq \emptyset$, for any $x_1,x_2 \in A, y \in Y\}$ is dense in $[0,1]$.

Proof. By the assumption, we get $\alpha \in K$, then $K \neq \emptyset$.

We proceed by contradiction. Suppose that there exist $\lambda_0 \in (0,1)\setminus K$ and a neighborhood U of λ_0 such that $U \cap K = \emptyset$. Let $\lambda_1 = \inf\{\lambda \in A | \lambda \geq \lambda_0\}$ and $\lambda_2 = \sup\{\lambda \in A | \lambda \leq \lambda_0\}$, then $0 \leq \lambda_2 \leq \lambda_1 \leq 1$.

Since $\alpha, 1-\alpha \in (0,1)$, there exist $u_1, u_2 \in K, u_1 \geq \lambda_1, u_2 \leq \lambda_2$ such that

$$(u_1 - u_2)\max\{\alpha, 1-\alpha\} \leq \lambda_1 - \lambda_2.$$

Define $\bar{\lambda} = \alpha u_1 + (1-\alpha)u_2$, we can claim that $\bar{\lambda} \in K$. In fact, take $x_1, x_2 \in A, y \in Y$ satisfying $F(x_1) \cap (y - \wedge) \neq \emptyset$ and $F(x_2) \cap (y - \wedge) \neq \emptyset$. Since $u_1, u_2 \in K$, there hold that

$$F(u_1x_1 + (1-u_1)x_2) \cap (y - \wedge) \neq \emptyset \text{ and } F(u_2x_1 + (1-u_2)x_2) \cap (y - \wedge) \neq \emptyset.$$

By virtue of the definition of $\bar{\lambda}$, we have

$$\bar{\lambda}x_1 + (1-\bar{\lambda})x_2 = \alpha(u_1x_1 + (1-u_1)x_2) + (1-\alpha)(u_2x_1 + (1-u_2)x_2).$$

Since $\alpha \in K$, we obtain

$$F(\bar{\lambda}x_1 + (1-\bar{\lambda})x_2) \cap (y - \wedge) \neq \emptyset,$$

which implies $\bar{\lambda} \in K$.

Case(i): $\bar{\lambda} \geq \lambda_0$. Since $\bar{\lambda} - u_2 = \alpha(u_1 - u_2) < \lambda_1 - \lambda_2$,we have $\bar{\lambda} < \lambda_1$ contradicting the definition of λ_1.

Case(ii): If $\bar{\lambda} \leq \lambda_0$. Since $u_1 - \bar{\lambda} = (1-\alpha)(u_1 - u_2) < \lambda_1 - \lambda_2$, we obtain $\bar{\lambda} > \lambda_2$ contradicting the definition of λ_2.

Theorem 3.2. If F is l.s.c on A and there exists $\alpha \in (0,1)$, such that $F(x_1) \cap (y - \wedge) \neq \emptyset$ and $F(x_2) \cap (y - \wedge) \neq \emptyset$ imply $F(\alpha x_1 + (1-\alpha)x_2) \cap (y - \wedge) \neq \emptyset$, for any $x_1, x_2 \in A, y \in Y$, then F is S-$\wedge-$quasiconvex.

Proof. Suppose that F is not S-$\wedge-$quasiconvex. Then there exist $\bar{x_1}, \bar{x_2} \in A, \bar{y} \in Y$ and $\bar{\lambda} \in (0,1)$ such that

$$F(\bar{x_1}) \cap (\bar{y} - \wedge) \neq \emptyset, F(\bar{x_2}) \cap (\bar{y} - \wedge) \neq \emptyset \text{ and } F(\bar{\lambda}\bar{x_1} + (1-\bar{\lambda})\bar{x_2}) \cap (\bar{y} - \wedge) = \emptyset.$$

Let $z = \bar{\lambda}\bar{x_1} + (1-\bar{\lambda})\bar{x_2}$,

$K = \{\lambda \in [0,1] | F(x_1) \cap (y - \wedge) \neq \emptyset$ and $F(x_2) \cap (y - \wedge) \neq \emptyset$ imply $F(\lambda x_1 + (1-\lambda)x_2) \cap (y - \wedge) \neq \emptyset$, for any $x_1, x_2 \in A, y \in Y\}$.

From Lemma 3.1, there exists a sequence $\{\lambda_n\} \subset K$ such that $\lambda_n \leq \bar{\lambda}$ and $\lambda_n \to \bar{\lambda}$.

Denote $\bar{x_2}^n = \frac{\bar{\lambda} - \lambda_n}{1-\lambda_n}\bar{x_1} + (1 - \frac{\bar{\lambda} - \lambda_n}{1-\lambda_n})\bar{x_2}$. Thus, $\bar{x_2}^n \to \bar{x_2}$ and $\lambda_n\bar{x_1} + (1-\lambda_n)\bar{x_2}^n = z$. Since A is a convex set and $0 < \frac{\bar{\lambda} - \lambda_n}{1-\lambda_n} < 1$, it follows that $\bar{x_2}^n \in A$.

Since F is l.s.c on A, $F(\bar{x_2}) \cap (\bar{y} - \wedge) \neq \emptyset$ and $\bar{x_2}^n \to \bar{x_2}$, there exists an N such that

$$F(\bar{x_2}^n) \cap (\bar{y} - \wedge) \neq \emptyset, \qquad \text{for any } n > N.$$

Therefore, from $\lambda_n \in K$, we have

$$F(\lambda_n\bar{x_1} + (1-\lambda_n)\bar{x_2}^n) \cap (\bar{y} - \wedge) \neq \emptyset.$$

That is,

$$F(z) = F(\bar{\lambda}\bar{x_1} + (1-\bar{\lambda})\bar{x_2}) \cap (\bar{y} - \wedge) \neq \emptyset.$$

which contradicts $F(\bar{\lambda}\bar{x_1} + (1-\bar{\lambda})\bar{x_2}) \cap (\bar{y} - \wedge) = \emptyset$.

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