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An Increasing-Mapping Approach to Integer Programming Based on Lexicographic Ordering and Linear Programming*

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Abstract Integer programming has many applications in economics and management. Applying lexicographic ordering and linear programming, we develop an iterative method for integer programming, which defines an increasing mapping from a finite lattice into itself. Given any polytope, within a finite number of iterations, the method either yields an integer point in the polytope or proves no such point exists. The method is able to determine all integer points in a polytope and can be easily implemented in parallel and extended to convex integer programming.

Keywords Integer Point; Polytope; Integer Programming; Lexicographic Ordering; Linear Programming; Iterative Method; Increasing Mapping

1 Introduction

As a powerful mechanism, integer programming has been extensively applied in economics (Scarf, 1981; 1986) and management (Schrijver, 2003). Let $P = \{x \in R^n \mid Ax + Gw \leq b$ for some $w \in R^p\}$, where A is an $m \times n$ matrix, G is an $m \times p$ matrix, and b is a vector of R^m. We assume without loss of generality that P is bounded and full dimensional. It is well known that determining whether there is an integer point in P is an NP-complete problem (Gary and Johnson, 1979). To solve such a problem, several methods have been developed in the literature, which include the cutting plane method in Gomory (1958), the branch-and-bound method in Land and Doig (1960), the neighborhood method in Scarf (1981, 1986), the basis-reduction method in Lenstra (1983), and the simplicial method in Dang and Maaren (1998). Further developments of some of these methods and their variants can be found in Schrijver (2003) and references therein. These methods play a very important role in the development of integer programming, however, it remains a challenging problem to determine whether there is an integer point in a polytope and thus appeals for more effective and efficient alternatives, which is the driving force behind this research.

Let $N = \{1, 2, \ldots, n\}$. For x and y of R^n, $x \leq_l y$ if either $x = y$ or $x_i = y_i$, $i = 1, 2, \ldots, k-1$, and $x_k < y_k$ for some $k \in N$, and $x \leq y$ if $x_i \leq y_i$ for all $i \in N$, where $\leq_l$ is called a

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lexicographic order on R^n. Then, for any nonempty subset $C \subseteq R^n$, one can see that $\leq_l$ is a binary relation and reflexive, transitive and antisymmetric on C. Thus, $(C, \leq_l)$ is a totally-order set. Let f be a mapping from C into itself. Under the lexicographic ordering, f is an increasing mapping from C into itself if $f(x) \leq_l f(y)$ for all x and y in C with $x \leq_l y$. When C is finite or compact, Tarski's fixed point theorem (Tarski, 1955) asserts that there is a point $x^* \in C$ such that $f(x^*) = x^*$, which is a fixed point of f. A significant feature of Tarski's fixed point theorem is that C can be a finite set. A study of the computational complexity of Tarski's fixed point theorem together with the lexicographic ordering and linear programming leads us to the idea of the method in this paper.

Applying lexicographic ordering and linear programming, we develop an iterative method for integer programming, which defines an increasing mapping from a finite lattice into itself. Given any polytope, within a finite number of iterations, the method either yields an integer point in the polytope or proves no such point exists. The method is able to determine all integer points in a polytope and can be easily implemented in parallel and extended to convex integer programming.

The rest of the paper is organized as follows. Based on lexicographic ordering and linear programming, an iterative method is developed for integer programming in Section 2. For a special class of polytopes, based on componentwise ordering and linear programming, an increasing mapping is constructed and an iterative method is proposed in Section 3.

2 An Iterative Method for Integer Programming Based on Lexicographic Ordering and Linear Programming

We assume that $n \geq 2$. For any real number α and any vector $x = (x_1, x_2, \ldots, x_n)^\top \in R^n$, let $\lfloor \alpha \rfloor$ denote the greatest integer less than or equal to α, $\lceil \alpha \rceil$ the smallest integer greater than or equal to α, $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \ldots, \lfloor x_n \rfloor)^\top$, and $\lceil x \rceil = (\lceil x_1 \rceil, \lceil x_2 \rceil, \ldots, \lceil x_n \rceil)^\top$.

Let $x^{\max} = (x_1^{\max}, x_2^{\max}, \ldots, x_n^{\max})^\top$ with $x_j^{\max} = \max_{x \in P} x_j,\ j = 1, 2, \ldots, n$, and $x^{\min} = (x_1^{\min}, x_2^{\min}, \ldots, x_n^{\min})^\top$ with $x_j^{\min} = \min_{x \in P} x_j,\ j = 1, 2, \ldots, n$. Then, $x^{\min} \leq x \leq x^{\max}$ for all $x \in P$. Let $Z^n = \{x = (x_1, x_2, \ldots, x_n)^\top \in R^n \mid x_i \text{ is an integer for all } i \in N\}$ and

$$D(P) = \{x \in Z^n \mid x^l \leq x \leq x^u\},$$

where

$$x^u = \lfloor x^{\max} \rfloor = (\lfloor x_1^{\max} \rfloor, \lfloor x_2^{\max} \rfloor, \ldots, \lfloor x_n^{\max} \rfloor)^\top$$

and

$$x^l = \lfloor x^{\min} \rfloor = (\lfloor x_1^{\min} \rfloor, \lfloor x_2^{\min} \rfloor, \ldots, \lfloor x_n^{\min} \rfloor)^\top.$$

Since $x^{\min} \leq x \leq x^{\max}$ for all $x \in P$, hence, $x \in D(P)$ for all $x \in P \cap Z^n$. We assume without loss of generality that

$$x_1^u - x_1^l \leq x_2^u - x_2^l \leq \cdots \leq x_n^u - x_n^l$$

and that $x^l < x^{\min}$ (If $x_i^l = x_i^{\min}$ for some $i \in N$, let $x_i^l = x_i^{\min} - 1$).

For $y \in R^n$ and $k \in N$, let

$$P(y, k) = \{x \in P \mid x_i = y_i,\ i = 1, 2, \ldots, k\}.$$

Definition 1 (An Iterative Method).
For $y \in D(P)$,

$$h(y) = (h_1(y), h_2(y), \ldots, h_n(y))^\top \in D(P)$$

is given as follows:

Step 0 (Initialization): *If $y_1 = x_1^l$, let $h(y) = x^l$ and* **Stop***; else, let $y^0 = y$, $k = 2$, and $q = 0$, and go to* **Step 1**.
Step 1: *If either $k > n$ or $k \le 1$, let $h(y) = y^q$ and* **Stop***; else, go to* **Step 2**.
Step 2: *If $y^q \in P$, let $h(y) = y^q$ and* **Stop**. *Otherwise, if $P(y^q, k-1) \neq \emptyset$, go to* **Step 3***; else, go to* **Step 6**.
Step 3: *Solve the linear program*

$$\begin{aligned} \min \quad & x_k - v_k \\ \textit{subject to} \quad & x \in P(y^q, k-1) \textit{ and } v \in P(y^q, k-1), \end{aligned}$$

to obtain its optimal solution (x^*, v^*). Let

$$d_k^{\min}(y^q) = x_k^* \textit{ and } d_k^{\max}(y^q) = v_k^*.$$

If $y_k^q \ge \lceil d_k^{\min}(y^q) \rceil$, go to **Step 4***; else, go to* **Step 5**.
Step 4: *If $\lfloor d_k^{\max}(y^q) \rfloor < \lceil d_k^{\min}(y^q) \rceil$, go to* **Step 5***; else, proceed as follows: If $y_k^q > \lfloor d_k^{\max}(y^q) \rfloor$, let $y^{q+1} = (y_1^{q+1}, y_2^{q+1}, \ldots, y_n^{q+1})^\top$ with*

$$y_i^{q+1} = \begin{cases} y_i^q & \textit{if } 1 \le i \le k-1, \\ \lfloor d_k^{\max}(y^q) \rfloor & \textit{if } i = k, \\ x_i^u & \textit{if } k+1 \le i \le n, \end{cases}$$

$i = 1, 2, \ldots, n$, and $q = q+1$. Let $k = k+1$ and go to **Step 1**.
Step 5: *If $y_{k-1}^q \le x_{k-1}^l + 1$, go to* **Step 6***; else, let $y^{q+1} = (y_1^{q+1}, y_2^{q+1}, \ldots, y_n^{q+1})^\top$ with*

$$y_i^{q+1} = \begin{cases} y_i^q & \textit{if } 1 \le i \le k-2, \\ y_{k-1}^q - 1 & \textit{if } i = k-1, \\ x_i^u & \textit{if } k \le i \le n, \end{cases}$$

$i = 1, 2, \ldots, n$, and $q = q+1$, and go to **Step 2**.
Step 6: *Let $y^{q+1} = (y_1^{q+1}, y_2^{q+1}, \ldots, y_n^{q+1})^\top$ with*

$$y_i^{q+1} = \begin{cases} y_i^q & \textit{if } 1 \le i \le k-2, \\ x_i^l & \textit{if } k-1 \le i \le n, \end{cases}$$

$i = 1, 2, \ldots, n$, $q = q+1$, and $k = k-1$. Go to **Step 1**.

Theorem 1.
For any given $y \in D(P)$,

- *if there is no integer point $z^0 \in P$ such that $z^0 \leq_l y$, then $h(y) = x^l$;*
- *if there is some integer point $z^0 \in P$ such that $z^0 \leq_l y$, then $z^0 \leq_l h(y) \in P$.*

This theorem also shows that the method can be easily implemented in parallel. As a corollary of Theorem 1, we obtain that

Corollary 1.
Either $h(x^u) \in P$ or $h(x^u) = x^l$, which show that, starting from x^u, within a finite number of iterations, the iterative method either yields an integer point in P or proves no such point exists. Furthermore, under the lexicographic ordering, h is an increasing mapping from $D(P)$ into $(Z^n \cap P) \cup \{x^l\}$.

As follows, we show how to apply the method to compute all integer points in P.

Step 0: Use the method starting from x^u to compute an integer point in P. If no integer point has been found, **Stop**. Otherwise, let s^1 be the solution found by the method, $k = 1$, and $q = n$, and go to **Step 1**.
Step 1: If $s_q^k > x_q^l$, let $y^0 = (y_1^0, y_2^0, \ldots, y_n^0)^\top$ with

$$y_i^0 = \begin{cases} s_i^k - 1 & \text{if } i = q, \\ s_i^k & \text{otherwise,} \end{cases}$$

$i = 1, 2, \ldots, n$, and go to **Step 3**. Otherwise, let $q = q - 1$ and go to **Step 2**.
Step 2: If $q < 1$, **Stop**. Otherwise, go to **Step 1**.
Step 3: If $y^0 \in P$, let $s^{k+1} = y^0$ and $k = k + 1$, and go to **Step 1**. Otherwise, go to **Step 4**.
Step 4: Use the method starting from y^0 to compute an integer point in P. If no integer point has been found, **Stop**. Otherwise, let s^{k+1} be the solution found by the method, $k = k + 1$, and $q = n$, and go to **Step 1**.

The efficiency of the method depends on the shape of P, which should be as “round" as possible. The basis reduction in (Lenstra et al., 1982) could be applied for this purpose.

3 An Increasing Mapping Based on Componentwise Ordering and Linear Programming for a Class of Polytopes

In this section, we consider a class of polytopes given by $P = \{x \in R^n \mid Ax \leq b\}$, where A is an $m \times n$ matrix satisfying that each row of A has at most one positive entry and b is a vector of R^m. The problem is NP-complete (Lagarias, 1985), though it is a special case of a general integer program. As a result of the property of A, one can easily obtain that

Lemma 1.
If $x^1 = (x_1^1, x_2^1, \ldots, x_n^1)^\top \in P$ and $x^2 = (x_1^2, x_2^2, \ldots, x_n^2)^\top \in P$,
then $\bar{x} = \max(x^1, x^2) = (\max\{x_1^1, x_1^2\}, \max\{x_2^1, x_2^2\}, \ldots, \max\{x_n^1, x_n^2\})^\top \in P$.

Given $e = (1,1,\ldots,1)^\top \in R^n$, Lemma 1 implies that $\max_{x\in P} e^\top x$ has a unique solution, which is denoted by $x^{\max} = (x_1^{\max}, x_2^{\max}, \ldots, x_n^{\max})^\top$. Let $x^{\min} = (x_1^{\min}, x_2^{\min}, \ldots, x_n^{\min})^\top$, where $x_j^{\min} = \min_{x\in P} x_j$, $j = 1,2,\ldots,n$. Thus, $x^{\min} \le x \le x^{\max}$ for all $x \in P$.

Let

$$C(P) = \{x \in R^n \mid x^l \le x \le x^u\},$$

where

$$x^u = \lfloor x^{\max} \rfloor = (\lfloor x_1^{\max} \rfloor, \lfloor x_2^{\max} \rfloor, \ldots, \lfloor x_n^{\max} \rfloor)^\top$$

and

$$x^l = \lfloor x^{\min} \rfloor = (\lfloor x_1^{\min} \rfloor, \lfloor x_2^{\min} \rfloor, \ldots, \lfloor x_n^{\min} \rfloor)^\top.$$

Then, $x \in C(P)$ for all integer points $x \in P$. We assume without loss of generality that $x^l \notin P$.

For $x \in R^n$, we define $f(x) = \lfloor d(x) \rfloor$ with

$$d(x) = \begin{cases} x^{\min} & \text{if } P(x) = \emptyset, \\ \\ \operatorname{argmax}_{y \in P(x)} e^\top y & \text{otherwise,} \end{cases}$$

where $P(x) = \{y \in P \mid y \le x\}$. From Lemma 1, one can see that $d(x)$ is well defined.

Lemma 2.
f is an increasing mapping from R^n into $C(P) \cap Z^n$. Moreover, $f(x^*) = x^*$ with $x^* \ne x^l$ if and only if x^* is an integer point in P.

Applying the increasing mapping f, we obtain an iterative method for determining whether there is an integer point in P, which is as follows.

Step 0: Let $x^0 = x^u$ and $k = 0$. Go to **Step 1**.
Step 1: If $x^k \in P$, the method terminates. Otherwise, go to **Step 2**.
Step 2: If $P(x^k) = \emptyset$ or $x^k = x^l$, the method terminates and there is no integer point in P. Otherwise, proceed as follows: Solve the linear program

$$\begin{aligned} \max \quad & e^\top y \\ \text{subject to} \quad & Ay \le b \text{ and } y \le x^k, \end{aligned}$$

to obtain its unique solution y^k. Let $x^{k+1} = \lfloor y^k \rfloor$ and $k = k+1$. Go to **Step 1**.

Theorem 2.
Within a finite number of iterations, the method either yields an integer point in P or proves no such point exists.

References

[1] C. Dang and H. van Maaren (1998). A simplicial approach to the determination of an integer point in a simplex, Mathematics of Operations Research 23: 403-415.

[2] G.B. Dantzig (1963). Linear Programming and Extensions, Princeton University Press, Princeton, New Jersey.

[3] M.R. Garey and D.S. Johnson (1979). Computers and Intractability: A Guide to the Theory of NP-completeness, W.H. Freeman, San Francisco.

[4] R.E. Gomory (1958). Outline of an algorithm for integer solution to linear programs, Bulletin American Mathematical Soceity 64: 275-278.

[5] J.C. Lagarias (1985). The computational complexity of simultaneous Diophantine approximation problems, SIAM Journal on Computing 14: 196-209.

[6] A.H. Land and A.G. Doig (1960). An automatic method for solving discrete programming problems, Econometrica 28: 497-520.

[7] H.W. Lenstra, Jr. (1983). Integer programming with a fixed number of variables, Mathematics of Operations Research 8: 538-548.

[8] A.K. Lenstra, H.W. Lenstra, Jr., L. Lovász (1982). Factoring polynomials with rational coefficients, Mathematische Annalen 261: 515-534.

[9] H.E. Scarf (1981). Production sets with indivisibilities - Part I: generalities, Econometrica 49: 1-32.

[10] H.E. Scarf (1986). Neighborhood systems for production sets with indivisibilities, Econometrica 54: 507-532.

[11] A. Schrijver (2003). Combinatorial Optimization, Springer, Berlin.

[12] A. Tarski (1955). A lattice-theoretical fixpoint theorem and its applications, Pacific Journal of Mathematics 5: 285-308.