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Generalized Algorithm of Tchebyshev Scalarization for Set-Valued Maps

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Abstract The aim of this paper is to investigate several inherited properties of convexity for set-valued maps and develop computational procedure based on such inherited properties. In this paper, we introduced two types of characteristic functions by using Tchebyshev scalarization, and defined four types of scalarization functions to characterize the images of set-valued maps.

Keywords Operations Research; Vector Optimization; Multiobjective Programming; Scalarization; Algorithm

1 Introduction

Since it will be very common for parallel translation of function $f(x)$ in the deciding space to non-negative quadrant R_+^p, then as for $i = 1, \cdots, p$, define the point

$$y_i^* = \inf \{ f_i(x) \mid x \in X \}$$

where $y_i^* > -\infty$ and X be a nonempty compact convex subset of a topological vector space. It is the Tchebyshev norm minimum method to minimize norm $| f(x) - \bar{y} |$ by taking this point or the point $\bar{y}$ of $\bar{y} \leqq y^*$ as the criterion point. That is

$$(R_w) \begin{cases} \text{minimize} & \max\limits_{1 \le i \le p} \{ w_i \mid f_i(x) - \bar{y}_i \mid \} \\ \text{subject to} & x \in X = \{ x \in R^n \mid g(x) \le 0 \} \end{cases}$$

where $| f_i(x) - \bar{y}_i |$ is available for assuming $\bar{y}$ of $\bar{y} \leqq y^*$ to be very small. If absolute value is taken off, then Tchebyshev scalarization function can be written as follows

$$\rho_\infty = \max_{1 \le i \le p} w_i (f_i(x) - \bar{y}_i)$$

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For $w \in R_+^p$, Tchebyshev scalarization problem (R_w) can be rewritten as follows

$$(R_w^{'})\begin{cases}\text{minimize} \quad z \\ \text{subject to} \quad w_i(f_i(x) - \overline{y}_i) \le z, \quad i = 1, \cdots, p \\ \qquad\qquad\quad x \in X = \{x \in R^n \mid g(x) \le 0\}\end{cases}$$

where $\max_{1 \le i \le p}\{w_i(f_i(x) - \overline{y}_i)\} \leqq z \Leftrightarrow w_i(f_i(x) - \overline{y}_i) \leqq z$, $i = 1, \cdots, p$.

In this paper, we introduced two types of characteristic functions by using Tchebyshev scalarization, and defined four types of scalarization functions by characteristics of set-valued maps. The aim of this paper consists of two parts: one is concerned with inherited properties of set-valued maps, another is scalarization algorithms for set-valued maps.

Firstly, we presented certain results on inherited properties of convexity and semi- continuity. Convexity and lower semi-continuity of real-valued maps are useful properties for analysis of optimization problems, and they are dual concepts to concavity and upper semi-continuity, respectively. These properties are related to the total ordering of R^n. We consider certain generalizations and modifications of convexity and semi-continuity for set-valued maps in a topological vector space with respect to a cone preorder in the target space for generalizing the classical Fan's inequality [1, 3, 4]. These properties are inherited by special scalarization functions:

$$\inf\{h_C^-(x, y; k) : y \in F(x)\} \tag{1.1}$$

and

$$\sup\{h_C^-(x, y; k) : y \in F(x)\} \tag{1.2}$$

where $h_C^-(x, y; k) = \inf\{t : y \in tk - C(x)\}$, $C(x)$ is a closed convex cone with nonempty interior, x and y are vectors in two topological vector spaces E, Y, and $k \in \operatorname{int} C(x)$. Note that $h_C(x, \cdot; k)$ is positively homogeneous and subadditive for every fixed $x \in X$ and $k \in \operatorname{int} C(x)$. Another function $h_C^+(x, y; k) = -h_C(x, -y; k) = \sup\{t : y \in tk + C(x)\}$ is also employed.

Secondly, we developed computational procedures how to calculate the values of scalarization functions (1.1) and (1.2). In order to find solutions of multi-objective problems, we used some types of scalarization algorithms such as positive linear functions and Tchebyshev scalarization. The function $h_C(x, y; k)$ is regarded as a generalization of the Tchebyshev scalarization. By using the function, we gave four types of characterizations of set-valued maps.

2 Inherited properties of set-valued maps

The aim of this section is to investigate how the property of cone-convexity which is inherited into scalarization functions (1.1) and (1.2) from set-valued maps.

Let E and Y be topological vector spaces and $F, C : E \to 2^Y$ is two multivalued mappings. Denote $B(x) = (\operatorname{int} C(x) \cap 2S \setminus \overline{S})$ (an open base of $\operatorname{int} C(x)$), where S is a neighborhood of 0 in Y. To avoid confusion for properties of convexity, we consider the constant case of $C(x) = C$ (a convex cone) and its base $B(x) = B$, then function $h_C^-(x, y; k) = h_C^-(y; k) := \inf\{t : y \in tk - C\}$. We observe the following four types of scalarization functions:

$$\varphi_C^F(x;k) = \sup_{y \in F(x)} h_C^-(y;k), \qquad \varphi_C^F(x;k) = \inf_{y \in F(x)} h_C^-(y;k),$$

$$-\varphi_C^{-F}(x;k) = \sup_{y \in F(x)} h_C^+(y;k), \qquad -\varphi_C^{-F}(x;k) = \inf_{y \in F(x)} h_C^+(y;k).$$

The first and fourth functions have symmetric properties and then results for the fourth function $-\varphi_C^{-F}(x;k)$ can be easily proved by those for the first function $\varphi_C^F(x;k)$. Similarly, the results for the third function $-\varphi_C^{-F}(x;k)$ can be deduced by those for the second function $\varphi_C^F(x;k)$. By using these four functions we measure each image of set-valued maps F with respect to its 4-couple of scalars, which can be regarded as standpoints for the evaluation of the image.

Proposition 2.1 *Let arbitrary vector* $k \in \operatorname{int} C(x)$. *Considering the corresponding* $C(x) = C$ *and* $h_C^-(y;k) = \inf\{t : y \in tk - C\}$, *we have*

(i) $h_C^-(y; -c) \le 0$ *for each* $y \in Y$ *and* $c \in C$.

(ii) $h_C^-(\alpha y; k) = \alpha h_C^-(y;k)$ *for each* $y \in Y$ *and* $\alpha > 0$.

(iii) $h_C^-(y_1 + y_2; k) \le h_C^-(y_1;k) + h_C^-(y_2;k)$ *for each* $y_1, y_2 \in Y$.

Proof. *To prove* (iii), *for every* $\varepsilon > 0$ *and* $\forall y_1, y_2 \in Y$ *there exist* $t_i \in R$ *such that for each* $i = 1,2$ $y_i \in t_i k - C$ *and* $t_i < h_C^-(y_i;k) + \varepsilon/2$. *Thereby*

$$t_1 + t_2 < h_C^-(y_1;k) + h_C^-(y_2;k) + \varepsilon. \tag{2.1}$$

For every $y_1, y_2 \in Y$ *there exist* $c_1, c_2 \in C$ *such that* $y_i = t_i k - c_i$, $i = 1,2$. *We have*

$$y_1 + y_1 = (t_1 + t_2)k - (c_1 + c_2). \tag{2.2}$$

Since C *is convex cone,* $c_1 + c_2 \in C$, *and* $y_1 + y_1 \in (t_1 + t_2)k - C$, *the following is obtained*

$$t_1 + t_2 \ge h_C^-(y_1 + y_2; k).$$

By the formula (2.1) *and formula* (2.2), *we have*

$$h_C^-(y_1 + y_2;k) \le h_C^-(y_1;k) + h_C^-(y_2;k) + \varepsilon.$$

Since $\varepsilon > 0$ *is arbitrarily small, we obtain*

$$h_C^-(y_1 + y_2; k) \leq h_C^-(y_1; k) + h_C^-(y_2; k) . \quad \blacksquare$$

(i) and (ii) of Proposition 2.1 can be proved simply and is omitted in the paper.

Definition 2.1 *A multifunction* $F : E \to 2^Y$ *is called C-quasiconvex, if the set* $\{x \in E : F(x) \cap (a - C) \neq \varnothing\}$ *is convex for every* $a \in Y$ *. If* $-F$ *is C-quasiconvex, then* F *is called C-quasiconvex, which is equivalent to* (-*C*)*-quasiconvex mapping*.

Definition 2.2 [4] *A multifunction* $F : E \to 2^Y$ *is called C-properly quasiconvex* (*type*-(v)), *if for every two points* $x_1, x_2 \in X$ *and every* $\lambda \in [0,1]$ *we have either* $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) - C$ *or* $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_2) - C$.

Definition 2.3 [4] *A multifunction* $F : E \to 2^Y$ *is called C-properly quasiconvex* (*type*-(iii)), *if for every two points* $x_1, x_2 \in X$ *and every* $\lambda \in [0,1]$ *we have either* $F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$ *or* $F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$.

Definition 2.4 *A multifunction* $F : E \to 2^Y$ *is called C-naturally quasiconvex, if for every two points* $x_1, x_2 \in X$ *and every* $\lambda \in [0,1]$ *we have* $F(\lambda x_1 + (1-\lambda)x_2) \subset \lambda F(x_1) + (1-\lambda)F(x_2) - C$.

If $-F$ is *C*-properly quasiconvex (type-(v)), then F is called *C*-properly quasiconvex (type-(v)), which is equivalent to (-*C*)-properly quasiconvex mapping (type-(v)). If $-F$ is *C*-naturally quasiconvex (type-(v)), then F is called *C*-naturally quasiconvex (type-(v)), which is equivalent to (-*C*)-naturally quasiconvex mapping (type-(v)).

Theorem 2.1 (inherited convexity 1)

(i) *If the multifunction* $F : E \to 2^Y$ *is C-properly quasiconvex* (*type*-(v)), *then the function* $\varphi_C^F(x;k) = \sup_{y \in F(x)} h_C^-(y;k)$ *is quasiconvex.*

(ii) *If the multifunction* $F : E \to 2^Y$ *is C-properly quasiconvex* (*type*-(iii)), *then the function* $\varphi_C^F(x;k) = \sup_{y \in F(x)} h_C^-(y;k)$ *is quasiconvex.*

(iii) *If the multifunction* $F : E \to 2^Y$ *is C-properly quasiconvex* (*type*-(v)), *then the function* $\varphi_C^F(x;k) = \inf_{y \in F(x)} h_C^-(y;k)$ *is quasiconvex.*

(iv) *If the multifunction* $F : E \to 2^Y$ *is C-properly quasiconvex* (*type*-(iii)), *then the function* $\varphi_C^F(x;k) = \inf_{y \in F(x)} h_C^-(y;k)$ *is quasiconvex.*

Proof. *To prove* (i) *by Definition 2, for every* $x_1, x_2 \in X$ *and* $\lambda \in [0,1]$, *we have either* $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) - C$ *or* $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_2) - C$. *Assume that* $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) - C$, *then we have*

$$\begin{aligned}\varphi_C^F(\lambda x_1+(1-\lambda)x_2;k) &= \sup\{h_C^-(y;k) \mid y\in F(\lambda x_1+(1-\lambda)x_2)\}\\ &\le \sup\{h_C^-(y;k)\mid y\in F(x_1)-C\}\\ &= \sup_{y\in F(x_1),c\in C} h_C^-(y-c;k)\\ &\le \sup_{y\in F(x_1),c\in C}(h_C^-(y;k)+h_C^-(-c;k)) \quad (\textit{by (iii) of Proposition 2.1})\\ &\le \sup_{y\in F(x_1)} h_C^-(y;k)\\ &= \varphi_C^F(x_1;k)\\ &\le \max\{\varphi_C^F(x_1;k),\varphi_C^F(x_2;k)\}. \qquad \blacksquare\end{aligned}$$

Analogously, we can prove the case of $F(\lambda x_1+(1-\lambda)x_2)\subset F(x_2)-C$.

To prove (iii), *we assume that for every* $x_1,x_2\in X$ *and* $\lambda\in[0,1]$, F *satisfies either* $F(x_1)\subset F(\lambda x_1+(1-\lambda)x_2)+C$ *or* $F(x_2)\subset F(\lambda x_1+(1-\lambda)x_2)+C$. *Assume that* $F(x_1)\subset F(\lambda x_1+(1-\lambda)x_2)+C$. *Then we have*

$$\begin{aligned}\varphi_C^F(\lambda x_1+(1-\lambda)x_2;k) &= \inf\{h_C^-(y;k) \mid y\in F(\lambda x_1+(1-\lambda)x_2)\}\\ &\ge \inf\{h_C^-(y;k)\mid y\in F(x_1)+C\}\\ &= \inf_{y\in F(x_1),c\in C} h_C^-(y+c;k)\\ &\ge \inf_{y\in F(x_1),c\in C}(h_C^-(y;k)-h_C^-(-c;k)) \quad (\textit{by (iii) of Proposition 2.1})\\ &\ge \inf_{y\in F(x_1)} h_C^-(y;k)\\ &= \varphi_C^F(x_1;k)\\ &\ge \min\{\varphi_C^F(x_1;k),\varphi_C^F(x_2;k)\}. \qquad \blacksquare\end{aligned}$$

Similarly, we can prove the case of $F(x_2)\subset F(\lambda x_1+(1-\lambda)x_2)+C$. (ii) and (iv) can be proved in the same way and is omitted in the paper.

Theorem 2.2 (inherited convexity 2) *If the multifunction* $F:E\to 2^Y$ *is C-quasiconvex, then for every* $k\in B$ *the function* $\varphi_C^F(x;k)=\inf_{y\in F(x)} h_C^-(y;k)$ *is quasiconvex.*

Proof. *By the definition of* $\varphi_C^F(x;k)$, *for every* $\varepsilon>0$ *and* $x_1,x_2\in X$ *there exist* $z_i\in F(x_i)$ *and* $t_i\in R$ *such that for each* $i=1,2$ $z_i-t_ik\in -C$ *and* $t_i<\varphi_C^F(x_i;k)+\varepsilon$. *Since* $t_1k-C\subseteq t_2k-C$ *for* $t_1\le t_2$, *we have*

$$z_i\in t_ik-C\subseteq \max\{t_1,t_2\}k-C.$$

Hence, by the C-quasiconvex of F, *for every* $x_1,x_2\in X$ *and* $\lambda\in[0,1]$ *there* *exists* $y \in F(\lambda x_1 + (1-\lambda)x_2)$ *such that* $y \in \max\{t_1, t_2\}k - C$. *We have*

$$h_C(y;k) \le \max\{t_1, t_2\}k - C$$
$$< \max\{\varphi_C^F(x_1;k), \varphi_C^F(x_2;k)\} + \varepsilon .$$

Therefore, we have

$$\varphi_C^F(\lambda x_1 + (1-\lambda)x_2;k) = \inf\{h_C^-(y;k) \mid y \in F(\lambda x_1 + (1-\lambda)x_2)\}$$

and since $\varepsilon > 0$ *is arbitrarily small, we obtain*

$$\varphi_C^F(\lambda x_1 + (1-\lambda)x_2;k) \le \max\{\varphi_C^F(x_1;k), \varphi_C^F(x_2;k)\} .$$ ■

Theorem 2.3 (inherited semicontinuity 1) ([1]) *Suppose that multifunction* $W : X \to 2^Y$ *is defined as* $W(x) = Y \setminus \operatorname{int} C(x)$ *and has a closed graph. If the multifunction* F *is* $(-C(x))$*-upper semicontinuous at* x *for each* $x \in X$ *, then the function* $f_1(x) = \inf_{k\in B} \varphi_C^F(x;k) = \inf_{k\in B(x)} \sup_{y\in F(x)} h_C^-(x,y;k)$ *is upper semicontinuous. If the mapping* C *is constant value, then* $f_1(x)$ *is upper semicontinuou.*

Theorem 2.4 (inherited semicontinuity 2) ([1]) *Suppose that multifunction* $W : X \to 2^Y$ *defined as* $W(x) = Y \setminus \operatorname{int} C(x)$ *and has a closed graph. If the multifunction* F *is* $(-C(x))$*-lower semicontinuous at* x *for each* $x \in X$ *, then the function* $f_2(x) = \inf_{k\in B} \varphi_C^F(x;k) = \inf_{k\in B(x)} \inf_{y\in F(x)} h_C^-(x,y;k)$ *is upper semicontinuous. If the mapping* C *is constant value, then* $f_2(x)$ *is upper semicontinuous.*

3 Scalarization algorithms of set-valued maps

In this paper, the notation $h_C^+(y;k) = \sup\{t : y \in tk + C\}$ is used as another scalarization function. Assume that $x \in X$ is a fixed, set-valued map $F(x)$ is fixed, and the set-valued map $F(x)$ is convex combination of finite vectors. In this case, we characterize set-valued map $F(x)$ by using small quantity of parameters based on heritability of convexity. In this paper, we consider of using convex polyhedron consisted by extreme points of $y_1, y_2, \cdots, y_n$ to get Pareto solution, namely Pareto solution of Pareto side of convex combination of $co\{y_1, y_2, \cdots, y_n\}$. Therefore, we construct the following four types of characterization of set-valued maps by using scalarization functions $h_C^-(y;k)$ and $h_C^+(y;k)$:

$$\varphi_C^F(x;k) = \sup_{y\in F(x)} h_C^-(y;k) = \max_i \max_j \{y_i^j / k^j\} ,$$

$$\varphi_C^F(x;k) = \inf_{y\in F(x)} h_C^-(y;k) \le \min_i \max_j \{y_i^j / k^j\} ,$$

$$-\varphi_C^{-F}(x;k) = \sup_{y\in F(x)} h_C^-(y;k) \ge \max_i \min_j \{y_i^j / k^j\} ,$$

$$-\varphi_C^{-F}(x;k)=\inf_{y\in F(x)} h_C^-(y;k)=\min_i \min_j\{y_i^j/k^j\}.$$

where $y_i\in F(x)$, $i=1,2,\cdots,n$, $j=1,2,\cdots,p$, $h_C^+(y;k)=\sup\{t:y\in tk+C\}$.

In this paper, for $\varphi_C^F(x;k)=\inf_{y\in F(x)} h_C^-(y;k)$, we developed the algorithms of characterization of set-valued map and Pareto solution of Pareto side base.

3.1 Characterization of set-valued maps

On the condition of $\{kt\mid t>0\}\cap co\{y_1,y_2,\cdots,y_n\}=\varnothing$, we characterize set-valued map according to following algorithm.

Step 1: Chose $k\in B$, $y_j:=y\in\{y_1,y_2,\cdots,y_n\}$, and $j=1$.

Step 2: If $j>n$, stop the calculation. If $j\le n$, calculate t_j of extreme point y_j according to $\varphi_C^F(x;k)=\inf_{y\in F(x)} h_C^-(y;k)$ and continue to Step 3.

Step 3: If $j=1$, then $t^*:=t_j$. If $j>1$, then

$$t^*:=\begin{cases} t^*, & t_j>t^* \\ t_j, & t_j\le t^* \end{cases}$$

and return to Step 2 and $j:=j+1$.

3.2 Calculation of Pareto solution

On the condition of $\{kt\mid t>0\}\cap co\{y_1,y_2,\cdots,y_n\}\neq\varnothing$, we calculate Pareto solution according to following algorithm.

Step 1: Choose $k\in B$ and $j=1$.

Step 2: If $p=2$, continue to Step 2.1. If $p>2$, continue to Step 2.2.

Step 2.1: If $j>n!/2!(n-2)!$, continue to Step3, otherwise choose extreme points y_u and y_v ($u,v\in\{1,2\},u\neq v$), calculate α_j and t_j according to following algorithm

$$\begin{cases} \alpha_j y_u^1+(1-\alpha_j)y_v^1=t_jk^1 \\ \alpha_j y_u^2+(1-\alpha_j)y_v^2=t_jk^2 \end{cases}$$

and continue to Step 2.3.

Step 2.2: If $j>n!/3!(n-3)!$, continue to Step3, otherwise choose extreme points y_u, y_v and y_q ($u,v,q\in\{1,2,\cdots,p\},u\neq v\neq q$), calculate α_j, β_j and t_j according to following algorithm

$$\begin{cases} \beta_j y_u^1 + (1-\beta_j) y_v^1 = \overline{y}^1 \\ \cdots \quad \cdots \quad \cdots \\ \beta_j y_u^p + (1-\beta_j) y_v^p = \overline{y}^p \\ \alpha_j \overline{y}^1 + (1-\alpha_j) y_q^1 = t_j k^1 \\ \cdots \quad \cdots \quad \cdots \\ \alpha_j \overline{y}^p + (1-\alpha_j) y_q^p = t_j k^p \end{cases}$$

and continue to Step 2.3.

Step 2.3: When $p = 2$, if $0 \le \alpha_j \le 1$ calculate t^* according to following algorithm and $j := j+1$. And return to Step2.1, otherwise return to Step2.2. When $p = 3$, if $0 \le \alpha_j \le 1$ and $0 \le \beta_j \le 1$ calculate t^* according to following algorithm, and $j := j+1$, otherwise return to Step2.2.

$$t^* := \begin{cases} t^*, & t_j > t^* \\ t_j, & t_j \le t^* \end{cases}.$$

Step 3: Calculate value of y^* corresponding to t^*. Stop the calculation when y^* is the desired solution, otherwise return to Step 1 and revise k.

This method of calculating maximum/minimum solution of scalar function is one dialogue-based method for decision-makers to get Acceptance Solution. Whether the solution is accepted or not depends on whether the solution satisfies judgment value benchmark of decision-makers. There is no very clear quantitative relation between target function value of the solution and k. At the beginning, let $k = 1$ and try to obtain the expected solution. Otherwise correct k value toward getting expected solution.

4 Conclusions

The paper studies the basic theory of multi-objective programming problems and scalarization method of set-valued maps.

Scalarization of value range of functions in feasible domain was discussed. The scalarization function of Tchebyshev was generalized and described in the case of $C = R_+^p$. The scalarization functions of set-valued maps were investigated. The algorithms of characterization of set-valued map and Pareto solution of Pareto side were developed. This method suitable of non-convex set partly.

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