The Ninth International Symposium on Operations Research and Its Applications (ISORA'10)
Chengdu-Jiuzhaigou, China, August 19–23, 2010

Three Kinds of Replacement Models Combined with Additive and Independent Damages

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Abstract In some practical situations, most systems would fail roughly with time by both causes of additive and independent damages. From such viewpoints, this paper considers three combined replacement policies with two kinds of damages: The unit is replaced at a planned time or when the total additive damage exceeds a failure level, whichever occurs first, and undergoes minimal repair when independent damage occurs. First, a standard cumulative damage model where the unit suffers some damage due to shocks and the total damage is additive is considered. Second, the total damage is measured only at periodic times. Third, the total damage approximately increases linearly with time t. Using the theory of cumulative processes, expected cost rates are obtained, and optimal policies which minimize them are derived analytically.

Keywords Age Replacement; Periodic Times; Additive Damage; Independent Damage

1 Introduction

Most systems might fail due to the total damage stored within them by shocks such as jolt, stress, or environment change. This is well-known as the *cumulative damage model*: A unit is subjected to shocks and suffers some damage due to shocks. The total damage is additive, and the unit fails when it has exceeded a failure level. The reliability properties and optimal maintenance policies for various damage models were summarized sufficiently [1]. On the other hand, the total damage is not additive, and the unit fails when the damage due to some shock has exceeded a failure level. This is called the *independent damage model*, and its typical examples are the fracture of brittle materials such as glasses [2], and semiconductor parts which have failed by some overcurrent or fault voltage. Most units would fail roughly with time by both causes of additive and independent damages.

First, we take up a standard cumulative damage model where the unit suffers some damage due to shocks and the total damage is additive. However, it might be impossible to estimate and know occurrences of shocks and the total damage every at each shock. Second, the total damage is measured only at periodic times. Third, the total damage approximately increases linearly with time.

This paper considers age replacement policies that are combined with additive and independent damage, in which the unit is replaced at a planned time or when the total damage exceeds a failure level, whichever occurs first, and undergoes minimal repair when independent damage occurs. Expected cost rates of three kinds of models are obtained by using the techniques of cumulative processes [3] and reliability theory [4], and optimal policies which minimize them are derived analytically.

The damage model can be applied to the garbage collection model [5] of database systems by replacing *shock* by *update* and *damage* by *garbage*, and the backup model [6] by replacing *shock* by *update* and *damage* by *dumped file*. This could be also applied to a reorganization model of structural database by replacing *shock* by *update*, and *damage 1* by *structural deterioration* and *damage 2* by *split data deterioration* [7, 8].

2 Standard Model

Suppose that shocks occur at a renewal process with a general distribution $F(t)$ with finite mean $1/\lambda$ and a density function $f(t) \equiv F'(t)$. An amount W_j of damage due to the jth shock has an identical distribution $G(x) \equiv \Pr\{W_j \le x\}$ with finite mean μ, and the total damage is additive. We call it *damage 1*. It is assumed that the unit fails when the total damage exceeds a failure level K $(0 < K < \infty)$ at some shock.

Suppose that the unit is replaced at time T $(0 < T \le \infty)$ or at failure, whichever occurs first. Then, the expected cost rate is, from [1, p.42],

$$\widetilde{C}_1(T) = \frac{c_K - (c_K - c_T)\sum_{j=0}^{\infty}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K)}{\sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t}, \tag{1}$$

where $\phi^{(j)}(x)$ $(j = 1, 2, \cdots)$ denotes the j-fold Stieltjes convolution of any function $\phi(x)$ with itself and $\phi^{(0)}(x) \equiv 1$ for $t \ge 0$, c_K = replacement cost at failure and c_T = replacement cost at time T, where $c_K > c_T$.

Next, suppose that another *damage 2* occurs at a nonhomogeneous Poisson process with an intensity function $h(t)$ and a mean-value function $H(t) \equiv \int_0^t h(u)\mathrm{d}u$, *i.e.*, the probability of j occurrences of damage 2 during $(0, t]$ is $p_j(t) = \{[H(t)]^j / j!\}\mathrm{e}^{-H(t)}$ $(j = 0, 1, 2, \cdots)$.

It is assumed that damage 2 occurs independently of damage 1, and also its damage is not additive which is called *independent damage* [1, p.21]. That is, when damage 2 occurs, the unit undergoes only minimal repair. Thus, the expected number of occurrences of damage 2, *i.e.*, minimal repair, before the replacement is

$$\begin{aligned} &H(T)\sum_{j=0}^{\infty}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K) \\ &+ \sum_{j=0}^{\infty}[G^{(j)}(K) - G^{(j+1)}(K)]\int_0^T H(t)\mathrm{d}F^{(j+1)}(t) \\ &= \sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}H(t). \end{aligned} \tag{2}$$

Therefore, adding the minimal repair cost to $\widetilde{C}_1(T)$ in (1),

$$C_1(T) = \frac{\begin{array}{c} c_K - (c_K - c_T)\sum_{j=0}^{\infty} G^{(j)}(K)[F^{(j)}(T) - F^{(j+1)}(T)] \\ +c_M \sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}H(t) \end{array}}{\sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t}, \tag{3}$$

where c_M = minimal repair cost for damage 2. Clearly, $C_1(0) = \infty$, and

$$C_1(\infty) = \frac{c_K + c_M \sum_{j=0}^{\infty} G^{(j)}(K)\int_0^{\infty}[F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}H(t)}{[1+M(K)]/\lambda},$$

where $M(K) \equiv \sum_{j=1}^{\infty} G^{(j)}(K)$, and note that the denominator represents the mean time to replacement when the total damage exceeds a failure level K. Thus, there exists a positive T_1^* $(0 < T_1^* \leq \infty)$ which minimizes $C_1(T)$ in (3).

We find an optimal T_1^* which minimizes $C_1(T)$. Differentiating $C_1(T)$ with respect to T and setting it equal to zero,

$$\begin{aligned} &(c_K - c_T)\left\{Q_1(T)\sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t \right. \\ &\left. - \sum_{j=0}^{\infty} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)]\right\} \\ &+c_M \sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)][h(T) - h(t)]\mathrm{d}t = c_T, \end{aligned} \tag{4}$$

where

$$Q_1(T) \equiv \frac{\sum_{j=0}^{\infty} f^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)]}{\sum_{j=0}^{\infty} G^{(j)}(K)[F^{(j)}(T) - F^{(j+1)}(T)]}.$$

It can be clearly seen that if $Q_1(T)$ is strictly increasing and $h(t)$ is increasing, or $Q_1(T)$ is increasing and $h(t)$ is strictly increasing, then the left-hand side of (4) is strictly increasing from 0 to

$$\begin{aligned} &(c_K - c_T)\left\{\frac{Q_1(\infty)}{\lambda}[1+M(K)] - 1\right\} + c_M\left\{\frac{h(\infty)}{\lambda}[1+M(K)] \right. \\ &\left. - \sum_{j=0}^{\infty} G^{(j)}(K)\int_0^{\infty}[F^{(j)}(t) - F^{(j+1)}(t)]h(t)\mathrm{d}t\right\}. \end{aligned} \tag{5}$$

Thus, if (5) is greater than c_T, then there exists a finite and unique T_1^* $(0 < T_1^* < \infty)$ which satisfies (4). In this case, the expected cost rate is

$$C_1(T_1^*) = (c_K - c_T)Q_1(T_1^*) + c_M h(T_1^*). \tag{6}$$

Furthermore, letting T_1 be a solution of equation

$$Q_1(T)\sum_{j=0}^{\infty}G^{(j)}(K)\int_0^T[F^{(j)}(t)-F^{(j+1)}(t)]\mathrm{d}t$$
$$-\sum_{j=0}^{\infty}F^{(j+1)}(T)[G^{(j)}(K)-G^{(j+1)}(K)]=\frac{c_T}{c_K-c_T}, \tag{7}$$

then $T_1 > T_1^*$, and letting T_2 be a solution of equation

$$\sum_{j=0}^{\infty}G^{(j)}(K)\int_0^T[F^{(j)}(t)-F^{(j+1)}(t)][h(T)-h(t)]\mathrm{d}t=\frac{c_T}{c_M}, \tag{8}$$

then $T_2 > T_1^*$. Both T_1 and T_2 would be useful for computing T_1^* as its upper bounds.

On the other hand, when $H(t)=\alpha t$, *i.e.*, $h(t)=\alpha$ $(\alpha>0)$, from (5), if $Q_1(\infty)[1+M(K)]>\lambda c_K/(c_K-c_T)$, then there exists a finite T_1^* $(0<T_1^*<\infty)$ which satisfies (4). In addition, when $F(t)=1-\mathrm{e}^{-\lambda t}$ and $G(x)=1-\mathrm{e}^{-x/\mu}$, it was shown in [1, p.48] that $G^{(j)}(x)=\sum_{i=j}^{\infty}[(x/\mu)^i/i!]\mathrm{e}^{-x/\mu}$ and $M(x)=x/\mu$,

$$Q_1(T)=\frac{\lambda\sum_{j=0}^{\infty}[(\lambda T)^j/j!][G^{(j)}(K)-G^{(j+1)}(K)]}{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]G^{(j)}(K)}$$

is strictly increasing from $\lambda\mathrm{e}^{-K/\mu}$ to λ. Thus, if $K/\mu>c_T/(c_K-c_T)$, then there exists a finite and unique T_1^* $(0<T_1^*<\infty)$ which satisfies (4).

3 Periodic Model

It is assumed that each amount W_n $(n=1,2,\cdots)$ of damage due to shocks is measured only at periodic times nT_0 $(n=1,2,\cdots)$ for a given T_0 $(0<T_0<\infty)$ and has an identical distribution $G_T(x)\equiv\Pr\{W_n\le x\}$ with mean μ_T. The other assumptions are the same as those in the standard model. Suppose that the unit is replaced at time NT_0 or at failure, whichever occurs first. Then, the expected cost rate is, from [1, p.84],

$$\widetilde{C}_2(N)=\frac{c_K-(c_K-c_N)G_T^{(N)}(K)}{T_0\sum_{n=0}^{N-1}G_T^{(n)}(K)}, \tag{9}$$

where c_N = replacement cost at time NT_0.

The expected number of occurrences of minimal repairs due to damage 2 is

$$\sum_{n=0}^{N-1}H[(n+1)T_0][G_T^{(n)}(K)-G_T^{(n+1)}(K)]+H(NT_0)G_T^{(N)}(K)$$
$$=\sum_{n=0}^{N-1}[H((n+1)T_0)-H(nT_0)]G_T^{(n)}(K). \tag{10}$$

Therefore, adding the minimal repair cost to $\widetilde{C}_2(N)$ in (9),

$$C_2(N)=\frac{c_K-(c_K-c_N)G_T^{(N)}(K)+c_M\sum_{n=0}^{N-1}[H((n+1)T_0)-H(nT_0)]G_T^{(n)}(K)}{T_0\sum_{n=0}^{N-1}G_T^{(n)}(K)}. \tag{11}$$

We find an optimal N_2^* which minimizes $C_2(N)$. From the inequality $C_2(N+1) - C_2(N) \geq 0$,

$$(c_K - c_N)\left\{Q_2(N+1)\sum_{n=0}^{N-1} G_T^{(n)}(K) - [1 - G_T^{(N)}(K)]\right\}$$
$$+c_M\left\{[H((N+1)T_0) - H(NT_0)]\sum_{n=0}^{N-1} G_T^{(n)}(K)\right.$$
$$\left.-\sum_{n=0}^{N-1}[H((n+1)T_0) - H(nT_0)]G_T^{(n)}(K)\right\} \geq c_N, \tag{12}$$

where

$$Q_2(N) \equiv \frac{G_T^{(N-1)}(K) - G_T^{(N)}(K)}{G_T^{(N-1)}(K)}.$$

Denoting the let-hand side in (12) by $L_2(N)$,

$$L_2(N) - L_2(N-1) = \sum_{n=0}^{N-1} G_T^{(n)}(K)\Big((c_K - c_N)[Q_2(N+1) - Q_2(N)]$$
$$+c_M\big\{[H((N+1)T_0) - H(NT_0)] - [H(NT_0) - H((N-1)T_0)]\big\}\Big).$$

Therefore, if $Q_2(N)$ is strictly increasing and $h(t)$ is increasing, or $Q_2(N)$ is increasing and $h(t)$ is strictly increasing, then the left-hand side of (12) is strictly increasing to $L_2(\infty)$. Thus, if $L_2(\infty) > c_N$, then there exists a finite and unique minimum N_2^* $(1 \leq N_2^* < \infty)$ which satisfies (12). Furthermore, letting N_1 be a solution of the equation

$$Q_2(N+1)\sum_{n=0}^{N-1} G_T^{(n)}(K) - [1 - G_T^{(N)}(K)] \geq \frac{c_N}{c_K - c_N}, \tag{13}$$

then $N_1 \geq N_2^*$, and letting N_2 be a solution of the equation

$$[H((N+1)T_0) - H(NT_0)]\sum_{n=0}^{N-1} G_T^{(n)}(K) - \sum_{n=0}^{N-1}[H((n+1)T_0) - H(nT_0)]G_T^{(n)}(K) \geq \frac{c_N}{c_M}, \tag{14}$$

then $N_2 \geq N_2^*$. In particular, when $G_T^{(j)}(x) = \sum_{i=j}^{\infty}[(x/\mu_T)^i/i!]\mathrm{e}^{-x/\mu_T}$,

$$Q_2(N) = \frac{(K/\mu_T)^{N-1}/(N-1)!}{\sum_{n=N-1}^{\infty}(K/\mu_T)^n/n!}$$

is strictly increasing from e^{-K/μ_T} to 1 [1, p.24]. Thus, if $K/\mu_T > c_N/(c_K - c_N)$, then there exist a finite and unique minimum N_1 which satisfies (13). On the other hand, when $H(t) = \alpha t$, if $Q_2(\infty)[1 + M_T(K)] > c_K/(c_K - c_N)$, then there exists a finite N_2^* $(1 \leq N_2^* < \infty)$ which satisfies (12). In addition, when $G(x) = 1 - \mathrm{e}^{-x/\mu_T}$, $N_2^* = N_1$.

4 Continuous Model

We consider two continuous damage models where the total damage $Z(t)$ increases linearly with time t according two probabilistic laws [3,p.26].

4.1 Model 1

Suppose that the total amount of damage increases with t [1, p.26]: It is assumed that the total damage at time t is $Z(t) = At$, where A is a random variable whose distribution is $W_A(x) \equiv \Pr\{A \le x\}$. Then, the probability that the unit does not fail in $(0,t]$ is

$$\Pr\{Z(t) \le K\} = \Pr\{At \le K\} = \Pr\{A \le K/t\} = W_A(K/t). \tag{15}$$

Suppose that the unit is replaced at time T or when the total damage exceeds K, whichever occurs first. Then, the mean time to replacement is

$$TW_A(K/T) + \int_0^T t\mathrm{d}[1 - W_A(K/t)] = \int_0^T W_A(K/t)\mathrm{d}t. \tag{16}$$

Thus, by the similar method of obtaining (3), the expected cost rate is

$$C_3(T) = \frac{c_K - (c_K - c_T)W_A(K/T) + c_M \int_0^T W_A(K/t)\mathrm{d}H(t)}{\int_0^T W_A(K/t)\mathrm{d}t}. \tag{17}$$

Let $r_A(t)$ be the failure rate of $W_A(t)$, *i.e.*, $r_A(t) \equiv -W_A'(t)/W_A(t)$. Differentiating $C_3(T)$ with respect to T and setting it equal to zero,

$$\begin{aligned}&(c_K - c_T)\left\{r_A(K/T)\int_0^T W_A(K/t)\mathrm{d}t - [1 - W_A(K/T)]\right\} \\ &+ c_M \int_0^T W_A(K/t)[h(T) - h(t)]\mathrm{d}t = c_T.\end{aligned} \tag{18}$$

Thus, if $r_A(t)$ is strictly increasing and $h(t)$ is increasing or $r_A(t)$ is increasing and $h(t)$ is strictly increasing, if a solution T_3^* to (18) exists, it is unique.

Next, suppose that the unit is replaced at time NT_0 for $T_0 > 0$ and when the total damage exceeds K, whichever occurs first. Then, by the similar method of obtaining (11), the expected cost rate is

$$C_4(N) = \frac{c_K - (c_K - c_N)W_A(K/NT_0) + c_M \sum_{n=0}^{N-1}[H((n+1)T_0) - H(nT_0)]W_A(K/nT_0)}{T_0 \sum_{n=0}^{N-1} W_A(K/nT_0)}, \tag{19}$$

where define that when $n = 0$, $W_A(K/nT_0) \equiv 1$. From the inequality $C_4(N+1) - C_4(N) \ge$

0,

$$
\begin{aligned}
&(c_K - c_N)\left\{Q_4(N)\sum_{n=0}^{N-1} W_A(K/nT_0) - [1 - W_A(K/NT_0)]\right\}\\
&+c_M\left\{[H((N+1)T_0) - H(NT_0)]\sum_{n=0}^{N-1} W_A(K/nT_0)\right.\\
&\left.-\sum_{n=0}^{N-1}[H((n+1)T_0) - H(nT_0)]W_A(K/nT_0)\right\} \geq c_N,
\end{aligned}
\tag{20}
$$

where

$$
Q_4(N) \equiv \frac{W_A(K/NT_0) - W_A(K/(N+1)T_0)}{W_A(K/NT_0)}.
$$

Thus, if $Q_4(N)$ is strictly increasing and $h(t)$ is increasing, or $Q_4(N)$ is increasing and $h(t)$ is strictly increasing, if a solution N_4^* to (20) exists, its minimum is unique.

4.2 Model 2

It is assumed that $Z(t) = \mu_A t + B_t$, where B_t has a distribution $\Pr\{B_t \leq x\} \equiv W_B(x)$. Then, the probability that the unit does not fail in $(0,t]$ is

$$
\Pr\{Z(t) \leq K\} = \Pr\{B_t \leq K - \mu_A t\} = W_B(K - \mu_A t).
$$

Thus, by replacing formally $W_A(K/t)$ in (17) with $W_B(K - \mu_A t)$, the expected cost rate is

$$
C_5(T) = \frac{c_K - (c_K - c_T)W_B(K - \mu_A T) + c_M \int_0^T W_B(K - \mu_A t)\mathrm{d}H(t)}{\int_0^T W_B(K - \mu_A t)\mathrm{d}t}. \tag{21}
$$

Let $r_B(t)$ be the failure rate of $W_B(t)$, *i.e.*, $r_B(t) \equiv -W_B'(t)/W_B(t)$. Differentiating $C_5(T)$ with respect to T and setting it equal to zero,

$$
\begin{aligned}
&(c_K - c_T)\left\{r_B(K - \mu_A T)\int_0^T W_B(K - \mu_A t)\mathrm{d}t - [1 - W_B(K - \mu_A T)]\right\}\\
&+c_M \int_0^T W_B(K - \mu_A t)[h(T) - h(t)]\mathrm{d}t = c_T.
\end{aligned}
\tag{22}
$$

Thus, if $r_B(t)$ is strictly increasing and $h(t)$ is increasing or $r_B(t)$ is increasing and $h(t)$ is strictly increasing, if a solution T_5^* to (22) exists, it is unique.

5 Conclusions

We have discussed three kinds of replacement policies which are combined with additive and independent damages. Expected cost rate models have been obtained by using the techniques of cumulative processes and reliability theory. Optimal policies have been derived analytically and could be compared numerically for specified parameters. However, the minimal repair cost c_M may be a variable and some damage caused by damage 1 may be reduced by some maintenance using the opportunity time of minimal repair.

Acknowledgements

This project is supported by National Natural Science Foundation of China (70471017, 70801036); Humanities and Social Science Research Foundation of China (05JA630027); Grant-in-Aid for Scientific Research (C) of Japan Society for the Promotion of Science under Grant No. 22500897.

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