

Actions of Excitement and Fast Inhibition in Neuron Net

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Abstract In this paper, we study how the parameters decide the coupling effect between two neuron cells. The neuronal network is described by the systems of equations. Then the fast inhibitory models of two neurons are explored. We find three important sections of the initial values, namely inhibitory section, synchronization section, not synchronization section. These results are helpful for studying multi-neurons models.

Keywords Coupling effect; Neuron cells; Fast inhibitory models; Synchronization

1 Introduction

The dynamics of neuron cells can be studied by using mathematical models [1]. There are three kinds of neuron cells in the neuron network. They are the excitable, the fast inhibition, and the slow inhibition ones, respectively. The dynamics of the cells are described with the following model equations

$$\left\{ \begin{array}{l} v'_k = f(v_k, w_k) + I + \rho - g_e(v_k - v_{exc})H\left(\sum_{i \in \mathcal{E}_k^e} H(v_i - \theta_{ee})\right) \\ - g_f(v_k - v_{inh})H\left(\sum_{i \in \mathcal{E}_k^f} H(x_i - \theta_{fe})\right) - g_\sigma(v_k - v_{inh})H\left(\sum_{i \in \mathcal{E}_k^\sigma} s_i\right), \\ w'_k = \varepsilon g(v_k, w_k), \\ x'_j = \phi\left(H\left(\sum_{i \in \mathcal{Z}_j^f} H(v_i - \theta_{ef})\right) - x_j\right), \\ y'_l = \phi\left(H\left(\sum_{i \in \mathcal{Z}_l^\sigma} H(v_i - \theta_{e\sigma})\right) - y_l\right), \\ s'_l = \phi(1 - s_l)H(y_l - \theta_{\sigma e}) - \varepsilon k_\sigma s_l. \end{array} \right. \quad (1)$$

where $1 \leq k \leq N_e$, $1 \leq j \leq N_f$, $1 \leq l \leq N_\sigma$; $k, j, l \in N$, the nature number set. I stands for the positive input to the k -th neuron cell, such as electric signal, ρ is Gaussian noise, H is the Heaviside step function,

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

ε is a positive small number. $\phi, g_e, g_f, g_\sigma, k_\sigma$ are positive constants, being the same order with ε . v_k, w_k are the state variables of the k -th excitable neuron cell, x_j describes the state of the j -th fast inhibition cell, and s_l describes the state of the l -th slow inhibition cell, y_l controls the slow inhibition action to take place as $y_l > \theta_{e\sigma}$. The thresholds parameters $\theta_\alpha, \alpha \in \{ee, ef, fe, e\sigma, \sigma e\}$ and v_{exc}, v_{inh} are constants. Their values are displaced as in Fig.2. The detail biological meanings of the parameters can be found in [1]. Many authors studied the solutions of equation (1) to understand the behaviors of neuron nets [2~6].

The dynamics of an isolated excitable neuron cell e_k is described by equations

$$\begin{aligned} v'_k &= f(v_k, w_k) + I \\ w'_k &= \varepsilon g(v_k, w_k) \end{aligned} \quad (2)$$

In the $v-w$ phase plane, the v -zero nullclining curve $C \equiv \{(v, w) : f(v, w) + I = 0\}$ is cubic-like and $f_w < 0$, typically, as $f(v, w) = -w - v^3 + kv + c = 0$. The curve C has one minimum point (v_{Lk}, w_{Lk}) with $f_{vv}(v_{Lk}, w_{Lk}) > 0$ (left knee) and one maximum point (right knee) which separate the curve into the left branch, the middle segment, and the right branch. The w -zero nullclining curve $D \equiv \{(v, w) : g(v, w) = 0\}$ is monotonously increasing and $g_w < 0$, typically, as $g(v, w) = -w + \tanh(v) = 0$. It is easy to see [7] that equation (2) has unique stable equilibrium point as curves C and D intersect at the left branch, a limit cycle surrounding the intersect point proved that the intersect point is located at the middle segment of curve C . In the last case, the cell is called in firing and the limit position of the limit cycle, as $\varepsilon \rightarrow 0$, is called the singular periodic orbit. (see Fig.1).

The real orbit of equation (2) is closely near the singular one for small $\varepsilon > 0$. The procedures of jump-up and jump-down are fast. The evolution of the cell state along the left branch and the right branch take a long time period.

In the neuron net an excitable neuron cell may receive more than one kind of inference from the other cells such as excitement, fast inhibition, and slow inhibition. These inferences are described by the added terms in the right side of first equation of equations (1), respectively. When input I is small the configuration of curves C and D is as the first case and the cell stays at the stable equilibrium state. When input I is large enough or cell receives excitements the curve C will be lifted up and the equilibrium point locates at middle segment, the cell firing. When cell receives inhibitions the curve C will be shifted down and the left knee will get down too, jump-up will be delayed. If the inhibition term is large enough the equilibrium point will return the left branch again and cell stays at rest. In a big network, the excitable neuron cell e_k may connect to the cell excitable e_j through the two chains $e_k = e^1 \rightarrow e^2 \rightarrow \dots \rightarrow e^M = e_j$ and $e_k = \hat{e}^1 \rightarrow \hat{e}^2 \rightarrow \dots \rightarrow \hat{e}^N \rightarrow f \rightarrow e_j$, where $e_i, \hat{e}_i, i(l) = 1, 2, \dots, M (N)$ are excitable cells, f is fast inhibition cell. The problem is whether or not e_j fires when cell e_k is firing. The goal of this paper is to study the effects of the fast inhibition and the excitement on an excitable neuron cell.

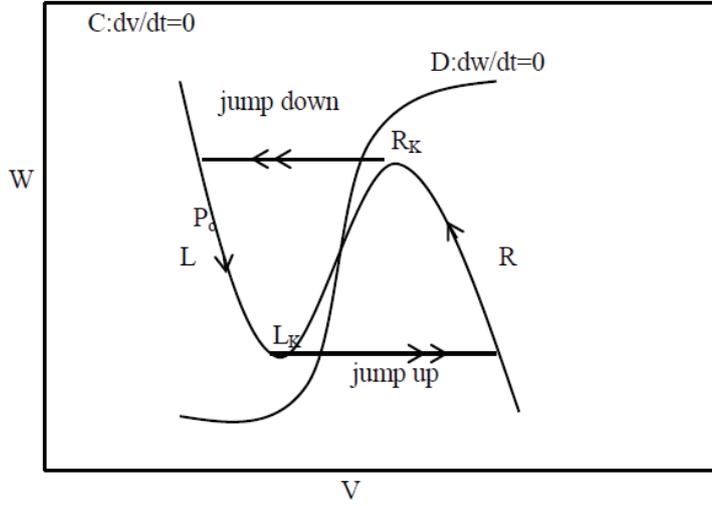


Figure 1: Nullclines for a single oscillator. A singular periodic orbit shown by arrow curves

2 The Main Results

It is clear that e_k will excite e_j if $M \leq N$ because the excitement route is shorter than the inhibition one. For the case $M = N + 1$, the problem can be reduced to the connection case $e_k \rightarrow e_j$; $e_k \rightarrow f \rightarrow e_j$ since the equations for $e^1 \dots e^{M-1}$; $\hat{e}^1 \dots \hat{e}^N$ are the same form and we can take e^{M-1} , \hat{e}^N as e_k . The equations of the reduced net is

$$\left\{ \begin{array}{l} \varepsilon \dot{V}_1 = f(v_1, w_1) + I = F_1(v_1, w_1), \\ \dot{W}_1 = g(v_1, w_1), \\ \varepsilon \dot{V}_2 = f(v_2, w_2) - g_e(v_2 - v_{exc})H(v_1 - \theta_{ee}) - g_f(v_2 - v_{inh})H(x - \theta_{fe}) = F_2(v_1, w_2), \\ \dot{W}_2 = g(v_2, w_2), \\ \varepsilon \dot{x} = \phi(H(v_1 - \theta_{ef}) - x) \end{array} \right. \quad (3)$$

where the time scale has been changed into $\tau = \varepsilon t$. If the left knee of nullclining curve $C_2 = \{(v, w); F_2(v, w) = 0\}$ is lower than that of $C_1 = \{(v, w); F_1(v, w) = 0\}$ then e_2 is inhibited.

Theorem 1 Let $F_1(v_{Lk}, w_{Lk}) = 0$. If $g_e(v_{Lk} - v_{exc}) + g_f(v_{Lk} - v_{inh}) > 0$ then the left knee of $C_2 = \{(v, w); F_2(v, w) = 0\}$ is lower than (v_{Lk}, w_{Lk}) .

Proof Consider the curve $\tilde{C}_2 = \{(v, w); F_2(v, w, g_e, g_f) + I = 0\}$. It is the graph of the implicit function $w = W(v, g_e, g_f)$ defined by equation $F_2 + I = 0$ since $\partial F_2 / \partial w = f_w < 0$. For $g_e = g_f = 0$, at the left knee (v_{Lk}, w_{Lk}) there hold $W_v = 0, W_{vv} > 0$. The Jacobian of functions $F_2(v, w, g_e, g_f) + I$ and $F_{2v}(v, w, g_e, g_f)$ is

$$A = \left\langle \begin{array}{cccc} f_v(v, w) - g_e - g_f, & f_w(v, w), & -(v - v_{exc}), & -(v - v_{inh}) \\ f_{vv}(v, w), & f_{vw}(v, w), & -1, & -1 \end{array} \right\rangle$$

At the point $(v_{Lk}, w_{Lk}, 0, 0)$ there hold $F_2 + I = 0, F_{2v} = 0, f_v(v_{Lk}, w_{Lk}) = 0$, and $A = \left\langle \begin{array}{cccc} f_v(v_{Lk}, w_{Lk}), & f_w, & -(v_{Lk} - v_{exc}), & -(v_{Lk} - v_{inh}) \\ f_{vv}, & f_{vw}, & -1, & -1 \end{array} \right\rangle$ being of rank 2. Therefore, the coordi-

nates of the left knee can be defined as the functions of $v_2 = v(g_e, g_f), w_2 = w(g_e, g_f)$ having the derivatives

$$\frac{\partial w_2}{\partial g_e} = \frac{(v_2 - v_{exc})}{f_w(v_2, w_2)}, \frac{\partial w_2}{\partial g_f} = \frac{(v_2 - v_{inh})}{f_w(v_2, w_2)},$$

For $|g_e|, |g_f|$ being small enough there hold

$$w_2(g_e, g_f) = w_{Lk} + \frac{\partial w_2}{\partial g_e}(0, 0)g_e + \frac{\partial w_2}{\partial g_f}(0, 0)g_f + o(g_e, g_f)$$

$$\frac{\partial w_2}{\partial g_e}(0, 0) = \frac{(v_2 - v_{exc})}{f_w(v_{Lk}, w_{Lk})}, \frac{\partial w_2}{\partial g_f}(0, 0) = \frac{(v_2 - v_{inh})}{f_w(v_{Lk}, w_{Lk})}, \quad f_w(v_{Lk}, w_{Lk}) < 0$$

Thus, if $(v_{Lk} - v_{exc})g_e + (v_{Lk} - v_{inh})g_f > 0$ then $w_2(g_e, g_f) < w_{Lk}$. This implies that the left knee of curve $\tilde{C}_2 = \{(v, w); F_2(v, w, g_e, g_f) + I = 0\}$

$$(v_2 = v(g_e, g_f), w_2 = w(g_e, g_f))$$

is lower than (v_{Lk}, w_{Lk}) . Since I is positive and $f_w < 0$ the left knee of $C_2 = \{(v, w); F_2(v, w) = 0\}$ is lower than $(v_2 = v(g_e, g_f), w_2 = w(g_e, g_f))$.

Theorem 2 A sufficient condition to assume that the inhibition action $e_1 \rightarrow f \rightarrow e_2$ happens before excitement action $e_1 \rightarrow e_2$ is $\frac{1}{\phi} \ln \frac{1}{1-\theta_{fe}} < \int_{\theta_{ef}}^{\theta_{ee}} \frac{dv_1}{F_1(v_1, w_{Lk})}$.

Proof Consider equation (2) and the Fig.2. The singular orbit for e_1 is sketched. In the procedure of e_1 jumping up, at the moment of $\tau_0, v_1(\tau_0) = \theta_{ef}, x(\tau_0) = x_0$. At the time moment $\tau_1 > \tau_0, v_1(\tau_1) = \theta_{ee}$ the excitement action takes place, at time moment $\tau_2, x(\tau_2) = \theta_{fe}$ the inhibition action takes place,

The time for cell e_1 to evolve from $v_1 = v_{Lk}$ to $v_1 = \theta_{ef}$ is $\Delta\tau_0 = \int_{v_{Lk}}^{\theta_{ef}} \frac{\epsilon dv}{F_1(v, w_{Lk})}$, from θ_{ef} to θ_{ee}

$$\Delta\tau_1 = \tau_1 - \tau_0 = \int_{\theta_{ef}}^{\theta_{ee}} \frac{\epsilon dv}{F_1(v, w_{Lk})} \tag{4}$$

While exchanging from x_0 to $\theta_{fe}, v_1 \geq \theta_{ef}, H(v_1 - \theta_{ef}) = 1$ steadily, by the equation on x ,

$$\frac{1 - x_0}{1 - \theta_{fe}} = \exp\left[\frac{\phi}{\epsilon} (\tau_2 - \tau_0)\right] \tag{5}$$

From (5) we get $\Delta\tau_2 = \tau_2 - \tau_0 = \frac{\epsilon}{\phi} \ln \frac{1-x_0}{1-\theta_{fe}} \leq \frac{\epsilon}{\phi} \ln \frac{1}{1-\theta_{fe}}$. Comparing (4) and (5), a sufficient condition for that the inhibition action takes place earlier than the excitement action does is

$$\frac{1}{\phi} \ln \frac{1}{1-\theta_{fe}} \leq \int_{\theta_{ef}}^{\theta_{ee}} \frac{dv}{F_1(v, w_{Lk})}$$

Remark The condition can be satisfied by taking ϕ large enough, actually, this means that x changes rapidly.

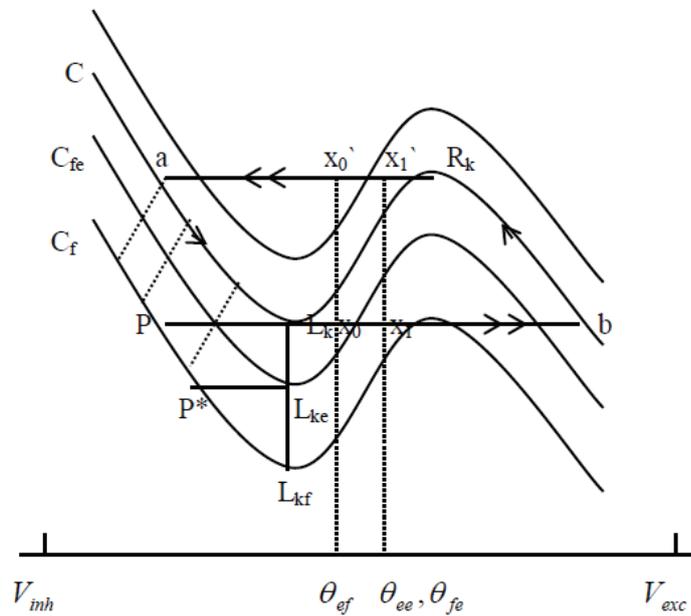


Figure 2: The orbit of (3) on slow time scale $\tau = \varepsilon t$

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