The Third International Symposium on Optimization and Systems Biology (OSB'09)
Zhangjiajie, China, September 20–22, 2009

Consensus of A Modified Time-Delayed Vicsek Model*

Yao Chen Jinhu Lü†

The Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P. R. China

Abstract A multi-agent system (MAS) is a system composed of multiple interacting intelligent agents. Over the past few years, the consensus (or synchronization) of MAS has received an increasing attention in various disciplines. It is well known that Vicsek model is a classical model for describing the collective behaviors of MAS. This paper further investigates the consensus of a modified time-delayed Vicsek model. It should be especially pointed out that the common connectivity assumption for the consensus of MAS has been removed in this paper. Moreover, two simple consensus criteria are then attained for the consensus of a modified time-delayed Vicsek model.

Keywords Vicsek model; Consensus; Connectivity; Time-delay

1 Introduction

It is well known that a multi-agent system (MAS) is a system composed of multiple interacting intelligent agents [1-13]. Consensus is a typical kind of collective behavior and a general agreement among all group members of a given community. Over the past few years, the consensus (or synchronization) of MAS has received an increasing attention in various disciplines [2, 4, 12, 13].

To characterize the collective behavior of MAS, many mathematical models have been introduced over the last three decades, including Boid model [2], Vicsek model [1], Couzin-Levin model and its modified model [12]. One typical kind of interesting collective behavior of the above models is that they can reach consensus via the local interactions among agents. Recently, some interesting results are reported on the consensus of Vicsek model and its variants [3, 4, 6, 8, 9, 11]. It should be especially pointed out that most of the above results are obtained based on the common connectivity assumption [3, 4, 6, 8, 9]. In 2003, Jadbabie and his colleagues further investigated the consensus of a linearized Vicsek model under the condition of the connectivity assumption [4]. However, the connectivity is not a necessary condition for the consensus of the original Vicsek model. Therefore, a natural challenging question is "Can we attain some simple consensus criteria for a MAS without the connectivity assumption?"

---

*This work was supported by the National Natural Science Foundation of China under Grants 60821091 and 60772158, the National Basic Research (973) Program of China under Grant 2007CB310805, and the Important Direction Project of Knowledge Innovation Program of Chinese Academy of Sciences under Grant KJCX3-SYW-S01.

†Email: jhlu@iss.ac.cn

In 2007, Liu and Guo further studied the connectivity and synchronization of Vicsek model [3]. Their results show that the connectivity and consensus of MAS can be realized simultaneously when the common velocity of all agents is less than some given constant [3]. Following this line, the current paper obtained two simple consensus criteria based on the consensus of a modified time-delayed Vicsek model without the connectivity assumption.

The left paper is then organized as follows. Section 2 describes the consensus problem of MAS. The main results are deduced in Section 3. Section 4 gives a simple example. Finally, some concluding remarks are drawn in Section 5.

2 Formulation of the Problem

To begin with, it is necessary to briefly review Vicsek model. In 1995, Vicsek introduced a simple mathematical model to describe the collective behaviors of multiple autonomous particles [1].

Suppose that there are N autonomous particles moving on a plane, denoted by $\mathscr{V} = \{1, 2, ..., N\}$. At time t, the heading angle and position of the ith particle are described by $\theta_i(t)$ and $(x_i(t), y_i(t))$, respectively. For the ith particle, let $N_i(t) = \{j \in \mathscr{V} : d_{ij}(t) < r\}$, where $d_{ij}(t) = \sqrt{(x_i(t)-x_j(t))^2 + (y_i(t)-y_j(t))^2}$. Hereafter, $N_i(t)$ is called the neighboring region of the ith particle. Assume that each particle only gets information from the neighboring particles within its neighboring region.

Naturally, all connections among these N particles form a graph. Denote $\mathscr{E}_t = \{(i, j) : j \in N_i(t)\}$, then $\mathscr{G}_t = \{\mathscr{V}, \mathscr{E}_t\}$ is the graph generated by the neighboring edges.

According to [1], the position and heading angle of the ith particle are updated by the following simple rules

$$x_i(t+1) = x_i(t) + v\cos\theta_i(t), \tag{1}$$

$$y_i(t+1) = y_i(t) + v\sin\theta_i(t), \tag{2}$$

$$\theta_i(t+1) = \arctan\frac{\sum_{j\in N_i(t)}\sin\theta_j(t)}{\sum_{j\in N_i(t)}\cos\theta_j(t)}. \tag{3}$$

Hereafter, the consensus or synchronization is defined by $\lim_{t\to\infty}|\theta_i(t) - \theta_j(t)| = 0$ for any $i, j \in \mathscr{V}$. According to [1], all particles can reach consensus or synchronization under the condition of high density and low noise.

It is well known that it is very difficult to theoretically analyze Vicsek model because of the nonlinearity in (3). To simplify Vicsek model, Jadbabie and his colleagues proposed the following linearized Vicsek model [4]:

$$\theta_i(t+1) = \frac{1}{n_i(t)}\sum_{j\in N_i(t)}\theta_j(t), \tag{4}$$

where $n_i(t)$ is the number of neighboring particles in $N_i(t)$.

In [3], Liu and Guo introduced the following interesting propositions.

Proposition 1.
For the Vicsek model (1)-(3), let $\theta_i(0) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with $i \in \mathscr{V} = \{1,2,...,N\}$, and the initial

neighbor graph $\mathscr{G}_0 = \{\mathscr{V}, \mathscr{E}_0\}$ be connected. Then all headings of the system (1)-(3) will synchronize, if the absolute velocity v satisfies:

$$v \le \frac{d}{\Delta_0}(\frac{\cos\bar{\theta}}{N})^N,$$

where

$$\bar{\theta} = \max_{i\in\mathscr{V}} |\theta_i(0)|,\ d = r - \max_{(i,j)\in\mathscr{E}_0} d_{ij}(0),\ \Delta_0 = \max_{i,j}\{\tan\theta_i(0) - \tan\theta_j(0)\}.$$

Similarly, for the linearized Vicsek model (4), they attain the corresponding results as follows [3].

Proposition 2.
For the linearized Vicsek model (1), (2) and (4), let $\theta_i(0) \in [0, 2\pi)$ and the initial neighboring graph be connected. Then all headings will synchronize if the absolute velocity v satisfies:

$$v \le \frac{d(\frac{1}{N})^N}{2\pi},$$

where d is given in Proposition 1.

It is well known that the time delay is ubiquitous in various information systems, such as MAS. Hereafter, a modified time-delayed Vicsek model is described by (1), (2) and

$$\theta_i(t+1) \quad = \quad \arctan\frac{\sum_{j\in N_i(t)} \sin\theta_j(t-\tau_j^i(t))}{\sum_{j\in N_i(t)} \cos\theta_j(t-\tau_j^i(t))}. \tag{5}$$

And its corresponding linearized updating rule is given by

$$\theta_i(t+1) = \frac{1}{n_i(t)} \sum_{j\in N_i(t)} \theta_j(t-\tau_j^i(t)). \tag{6}$$

where $\tau_j^i(t)$ is the transmission time delay from particle j to i. Moreover, $\tau_j^i(t)$ is a time-varying function.

In the following, we will further investigate the consensus of a modified time-delayed Vicsek model (1), (2), (5) and its corresponding linearized model (1), (2), (6), respectively.

3 Main Results

To begin with, a key lemma is firstly introduced for the proof of the following main theorems.

Lemma 1.
Consider N coupled sequence $\{x_i(t)\}$ satisfying $x_i(t+1) = \sum_{j\in N_i(t)} a_{ij}(t)x_j(t-\tau_j^i(t))$. If

i) *$\mathscr{G}(A(t))$ is always connected;*
ii) *$0 \le \tau_j^i(t) < B$ for any $i \ne j$ with some integer $B > 0$;*

iii) $\tau_i^i(t) = 0$ *for any* $i \in \mathscr{V}$*;*
iv) $a_{ij}(t) \geq \alpha$ *for some* $\alpha \in (0,1)$ *when* $j \in N_i(t)$*;*
v) $\sum_{j \in N_i(t)} a_{ij}(t) = 1$ *for any* $i \in \mathscr{V}$ *and* $t \geq 0$*;*
vi) $i \in N_i(t)$ *for any* $i \in \mathscr{V}$*.*

Let

$$\Delta_t = \max_{i \in \mathscr{V}, t-B<\tau\leq t} x_i(\tau) - \min_{i \in \mathscr{V}, t-B<\tau\leq t} x_i(\tau).$$

Then, one gets

$$\Delta_{kNB} \leq (1-\alpha^{(N+1)B})^k \Delta_0.$$

Theorem 1.
Consider the modified time-delayed Vicsek model (1), (2), (5). If

i) *There exists an integer* $B > 0$ *satisfying* $0 \leq \tau_j^i(t) < B$ *for any* $i \neq j$ *and* $\tau_i^i(t) = 0$ *for any* $i \in \mathscr{V}$*;*
ii) *The initial graph* $\mathscr{G}_0 = \{\mathscr{V}, \mathscr{E}_0\}$ *is connected and the absolute velocity* v *satisfies the following condition*

$$v < \frac{d}{\Delta_0^\theta (1 + NB\alpha^{-(N+1)B})}, \tag{7}$$

where

$$\begin{aligned} d &= r - \max_{(i,j)\in\mathscr{E}_0} d_{ij}(0), \\ \Delta_0^\theta &= \max_{i,j\in\mathscr{V}, -B<\tau_1,\tau_2\leq 0} \{\tan\theta_i(\tau_1) - \tan\theta_j(\tau_2)\}, \\ \alpha &= \frac{\cos(\max_{i\in\mathscr{V}, -B<\tau\leq 0} |\theta_i(\tau)|)}{N}. \end{aligned}$$

Then, all headings of the modified time-delayed Vicsek model can realize synchronization.

Proof: Let

$$\theta^M(t) = \max_{i\in\mathscr{V}, t-B<\tau\leq t} \theta_i(\tau),\ \theta^m(t) = \min_{i\in\mathscr{V}, t-B<\tau\leq t} \theta_i(\tau),\ \Delta_t^\theta = \tan\theta^M(t) - \tan\theta^m(t).$$

One has

$$\sqrt{(\cos\theta_i(t) - \cos\theta_j(t))^2 + (\sin\theta_i(t) - \sin\theta_j(t))^2} = 2|\sin\frac{\theta_i(t)-\theta_j(t)}{2}|. \tag{8}$$

According to (8), for any $i, j \in \mathscr{V}$, one gets

$$\begin{aligned} d_{ij}(t+1) &= \{(x_i(t) - x_j(t) + v(\cos\theta_i(t) - \cos\theta_j(t)))^2 + \\ &\quad (y_i(t) - y_j(t) + v(\sin\theta_i(t) - \sin\theta_j(t)))^2\}^{\frac{1}{2}} \\ &\leq d_{ij}(t) + 2v \cdot |\sin\frac{\theta_i(t)-\theta_j(t)}{2}| \\ &\leq d_{ij}(t) + v \cdot |\theta_i(t) - \theta_j(t)| \\ &\leq d_{ij}(t) + v \cdot (\theta^M(t) - \theta^m(t)). \end{aligned}$$

On the other hand, since $f(x) = \tan x - x$ is monotonous increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$, then one has

$$\tan\theta^m(t) - \theta^m(t) \le \tan\theta^M(t) - \theta^M(t).$$

That is,

$$\theta^M(t) - \theta^m(t) \le \Delta_t^\theta.$$

Thus one attains

$$d_{ij}(t+1) \le d_{ij}(t) + v \cdot \Delta_t^\theta. \tag{9}$$

The updating rule (5) can be simplified into the following form:

$$\tan\theta_i(t+1) = \sum_{j\in N_i(t)} \frac{\cos\theta_j(t-\tau_j^i(t))}{\sum_{k\in N_i(t)}\cos\theta_k(t-\tau_k^i(t))}\tan\theta_j(t-\tau_j^i(t)).$$

Denote

$$x_i(t) = \tan\theta_i(t), \quad a_{ij}(t) = \frac{\cos\theta_j(t-\tau_j^i(t))}{\sum_{k\in N_i(t)}\cos\theta_k(t-\tau_k^i(t))}.$$

Then (5) has the following form:

$$x_i(t+1) = \sum_{j\in N_i(t)} a_{ij}(t)x_j(t-\tau_j^i(t)).$$

Therefore, $\theta^M(t)$ is non-increasing and $\theta^m(t)$ is non-decreasing.

Moreover, one gets

$$a_{ij}(t) \ge \frac{\cos\bar{\theta}}{N} = \alpha$$

for all $t \ge 0$ and $j \in N_i(t)$, where $\bar{\theta} = \max\{|\theta^M(0)|, |\theta^m(0)|\}$.

According to (9), combined with the monotony of Δ_t^θ, one has

$$d_{ij}(t+1) \le d_{ij}(0) + (t+1)v\Delta_0^\theta.$$

Furthermore, from the assumptions of this theorem, one gets

$$d_{ij}(t+1) < r$$

for $0 \le t < NB$.

Hence, the connectivity of all particles in the time-delayed Vicsek model is guaranteed for $1 \le t \le NB$.

Next, one only needs to prove $d_{ij}(t) < r$ for any $(k-1)NB+1 \le t \le kNB$ and $k > 0$.

When $k = 1$, the above condition naturally holds from the above discussion.

When $k > 1$, suppose that all cases hold for $i = 1, \cdots, k$. Thus, one has

$$\begin{aligned} d_{ij}(kNB+1) &\leq d_{ij}(0) + v\sum_{t=0}^{kNB} \Delta_t^{\theta} \\ &\leq d_{ij}(0) + v\Delta_0^{\theta} + vNB\sum_{s=0}^{k-1} \Delta_{sNB}^{\theta}. \end{aligned}$$

For the case of $(k+1)$, according to Lemma 1, let $\beta = 1 - \alpha^{(N+1)B}$, then there is $\Delta_{sNB}^{\theta} \leq \beta^s \Delta_0^{\theta}$ for $1 \leq s \leq k$. When $kNB+1 \leq t \leq (k+1)NB$, one gets

$$\begin{aligned} d_{ij}(t) &\leq d_{ij}(kNB+1) + (t-kNB-1)v\Delta_{kNB+1}^{\theta} \\ &\leq d_{ij}(0) + v\Delta_0^{\theta} + vNB\Delta_0^{\theta}(1+\beta+\ldots+\beta^{k-1}) + (t-kNB-1)v\Delta_{kNB+1}^{\theta} \\ &\leq d_{ij}(0) + v\Delta_0^{\theta} + vNB\Delta_0^{\theta}(1+\beta+\ldots+\beta^{k}) \\ &< d_{ij}(0) + v\Delta_0^{\theta}(1+vNB\frac{1}{1-\beta}) \\ &\leq r. \end{aligned}$$

Therefore, the connectivity can be guaranteed for $kNB+1 \leq t \leq (k+1)NB$. That is, the case of $(k+1)$ holds. And the connectivity can be guaranteed for any $t \geq 0$.

According to Lemma 1, one has

$$\lim_{t\to\infty} |\tan\theta_i(t) - \tan\theta_j(t)| = 0$$

for any $i, j \in \mathscr{V}$. Since $\theta_i(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then one gets

$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = 0.$$

Thus all headings of particles can reach consensus. ■

Theorem 2.
Consider the corresponding linearized time-delayed Vicsek model (1), (2), and (6). Denote $\theta_i(0) \in [0, 2\pi)$. Suppose that

i) *There exists an integer $B > 0$ such that $0 \leq \tau_j^i(t) < B$ for any $i \neq j$ and $\tau_i^i(t) = 0$ for any $i \in \mathscr{V}$;*
ii) *The initial graph $\mathscr{G}_0 = \{\mathscr{V}, \mathscr{E}_0\}$ is connected and the absolute velocity v satisfies the condition*

$$v < \frac{d}{2\pi(1+N^{(N+1)B+1}B)}.$$

Then the headings of all particles can realize synchronization.

Here, the main idea and techniques for the proof of Theorem 2 are totally similar to those of Theorem 1. Therefore, the proof of Theorem 2 is omitted.

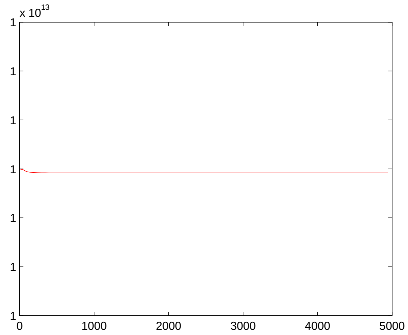

Figure 1: The trajectory of agent 1.

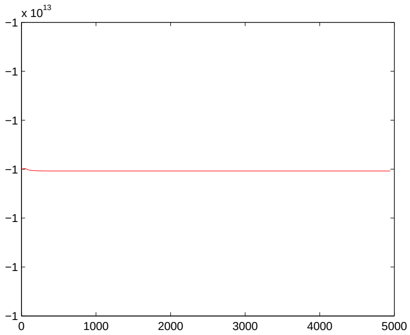

Figure 2: The trajectory of agent 2.

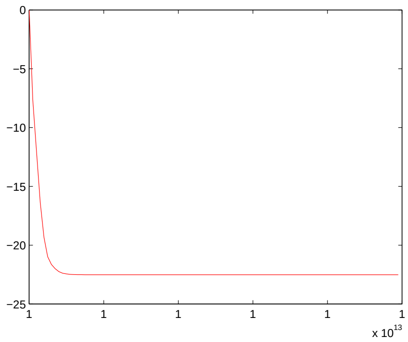

Figure 3: The trajectory of agent 3.

4 An Example

Consider 3 autonomous agents moving on the plane, labelled from 1 to 3. The initial positions of these agents are described by $(x_1(0), y_1(0)) = (0, 1)$, $(x_2(0), y_2(0)) = (0, -1)$, $(x_3(0), y_3(0)) = (1, 0)$ .And their initial headings are given by $\theta_1(0) = \frac{\pi}{6}$, $\theta_2(0) = -\frac{\pi}{6}$, $\theta_3(0) = \frac{\pi}{3}$, $\theta_1(-1) = \frac{\pi}{7}$, $\theta_2(-1) = -\frac{\pi}{7}$, $\theta_3(-1) = -\frac{4\pi}{9}$.

For simplification, let $B = 2$ and $r = 1.7$, then $\tau_j^i \in \{0, 1\}$.

According to Theorem 1, one has

$$v < \frac{d}{\Delta_0^\theta (1 + NB\alpha^{-(N+1)B})} = 5.6734 \times 10^{-12}.$$

Choose $v = 5.0 \times 10^{-12}$. Then the trajectories of 3 agents are drawn in Figs. 1-3.

5 Concluding Remarks

This paper has further investigated the consensus of a modified time-delayed Vicsek model and attained two simple consensus criteria. It should be especially pointed out that the common connectivity assumption has been removed in the above proposed consensus criteria. Moreover, it is necessary to develop an effective distributed algorithm to maintain the connectivity of the dynamic graph of MAS in the near future.

References

[1] T. Vicsek, A. Czirok, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-propelled particles," *Phys. Rev. Lett.*, vol. 75, pp. 1226-1229, 1995.

[2] C. W. Reynolds, "Flocks, herds, and schools: A distributed behavior model," *Comput. Graph*, vol. 21, no. 4, pp. 25-34, 1987.

[3] Z. Liu and L. Guo, "Connectivity and synchronization of Vicsek model," *Science in China Ser. F*, vol. 37, pp. 979-988, 2007.

[4] A. Jadbabie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Automat. Contr.*, vol. 48, no. 6, pp. 988-1001, Jun. 2003.

[5] M. Cao, A. S. Morse, and B. D. O. Anderson, "Agreeing asynchronously," *IEEE Trans. Automat. Contr.*, vol. 53, no. 8, pp. 1826-1838, Aug. 2008.

[6] A. V. Savkin, "Coordination collective motion of groups of autonomous mobile robots: Analysis of Vicsek's model," *IEEE Trans. Automat. Contr.*, vol. 49, no. 6, pp. 981-983, Jun. 2004.

[7] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis, "Convergence in multiagent coordination, consensus, and flocking," *Proc. the 44th IEEE Conf. Decision Contr. and Europ. Contr. Conf. 2005*, Seville, Spain, pp. 2996-3000, 2005.

[8] Y. Chen, J. Lü, and Z. Lin, "Consensus of discrete-time multi-agent systems with nonlinear local rules and time-varying delays," *Proc. the 48th IEEE Conf. Decision Contr. and Chin. Contr. Conf. 2009*, Shanghai, China, 2009, in press.

[9] X. Li and Y. Xi, "Flocking of multi-agent dynamic systems with guaranteed group connectivity," *J. Syst. Sci. Complexity*, vol. 21, pp. 337-346, 2008.

[10] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Automat. Contr.*, vol. 50, no. 5, pp. 655-661, Jun. 2005.

[11] S. Li and H. Wang, "Multi-agent coordination using nearest-neighbor rules: Revisiting the Vicsek model," *arXiv:cs/0407021*.

[12] J. Lü, J. Liu, I. D. Couzin, and S. A. Levin, "Emerging collective behaviors of animal groups," *Proc. the 7th World Congr. Contr. Automation*, Chongqing, China, pp. 1060-1065, 2008.

[13] J. Lü and G. Chen, "A time-varying complex dynamical network model and its controlled synchronization criteria," *IEEE Trans. Automat. Contr.*, vol. 50, no. 6, pp. 841-846, Jun. 2005.