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An Extension of Recurrent Iterated Function System — From the Viewpoint of Graph Theory and Product Space*

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Abstract Fractal Interpolation Function is the attractor of Iterated Function System (IFS), to overcome the weakness of IFS that IFS only reflect the similarity between global region and local region of a graph, Barnsley put forward the Recurrent Iterated Function System and the Recurrent Fractal Interpolation Function. RIFS reflect the similarity between local and local, it can make more complicated graph. In his paper, Barnsley strictly proved the existence and uniqueness of the attractor of RIFS, but he didn't give the relationship between RIFS and Recurrent Fractal Interpolation Function. Based on his research, we extended the concept of RIFS and prove the existence and uniqueness of it's attractor. Further more, we can see that under our extended definition, the Recurrent Fractal Interpolation Function can be seen as the attractor of a RIFS.

Keywords RIFS; Attractor; Recurrent Fractal Interpolation Function

1 Introduction

Iterated function system (IFS) theory is an important part of fractal theory, it is widely used in fields of image compression due to the pioneering works done by Barnsley [1, 2, 3]. When using IFS to compress an image, the image is first seen as a set named G, then an IFS will be constructed, of which the attractor (invariant sets) approximates G in a certain sense. Another field in which IFS is used is fractal interpolation and curve-fitting. The concept of Fractal interpolation function which was proposed by Barnsley in 1986 is a totally new interpolation method. It has special advantages on fitting non-smooth curves [4, 5]. In fact, the fractal interpolation function is just the attractor (invariant set) of an IFS. Whether we use IFS to compress an image or use fractal interpolation function to fit a curve, we are actually using the similarities between global regions and local regions of a graph (function). Therefore, limitations exist inevitably. Based on these considerations, Barnsley put forward another two concepts: Recurrent Iterated Function System (RIFS) and Recursive Fractal Interpolation Function.

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RIFS reflects the similarities among local regions of a graph, it can generate more complex graphics. Barnsley gave a strict proof for the existence and uniqueness of the invariant measure (invariant set) of a RIFS. However, he did not set up a direct relationship between RIFS's attractor and recursive fractal interpolation function. Based on researches of Barnsley, this paper first extends the definition of RIFS from the viewpoint of graph theory and product space, then proves the existence and uniqueness of the attractor of the extended RIFS. Further researches also find that recursive fractal interpolation function can be seen as the attractor of RIFS under the extended definition.

These finds help us to understand the concept of RIFS better, make the relationship between RIFS and recursive fractal interpolation function more legible, and also provide theoretical basis for the applications of RIFS and recursive fractal interpolation function.

2 Recursive iterative function systems

Definition 1 (Recursive Iterative Function Systems [3])**.** Assume $(X,\ d)$ is a compact metric space, $\omega_i : X \to X, i = 1,2,\ldots,N$ is a set of contractive maps, the contractive factors are $0 < s_i < 1, i = 1,2,...,N$. Denote $P = (p_{ij})_{N\times N}$ as an irreducible row-stochastic matrix, in other words, for any i, $\sum_{j=1}^{N} p_{ij} = 1$, and for $\forall i, j$, we have
(1) $p_{ij} \geq 0$;
(2) there exist a series of indicators: $i_1, i_2, \ldots, i_N$ which satisfy: $i_1 = i, i_N = j$ and $p_{i_1 i_2} \cdot p_{i_2 i_3} \cdot \ldots \cdot p_{i_{n-1} i_n} > 0$.
Then we say that $\{X, w_i, p_{ij}, i, j = 1,2,...,N\}$ is a RIFS.

To understand the concept of RIFS better, we reconsider Definition 1.

Regard the map set$\{w_1, w_2, ..., w_N\}$ as a vertex set $V = \{v_1, v_2, ..., v_N\}$, each map w_i corresponds to a vertex v_i, we can construct a graph whose vertex set is V according to the following rules:
(1) there exists a directed edge e_{ij} from v_i to v_j, iff $p_{ij} > 0$;
(2) for vertex v_i and vertex v_j, there exists a path $v_{i_1} v_{i_2} ... v_{i_n}$ satisfying $v_i = v_{i_1}, v_j = v_{i_n}$, iff there exits $i_1, i_2, ..., i_n$ satisfying $p_{i_1 i_2} \cdot p_{i_2 i_3} \cdot \ldots \cdot p_{i_{n-1} i_n} > 0$, where $i_1 = i, i_n = j$.

Denote the constructed graph as $G = (G_v, G_e)$, where $G_v = V$. From its construction, we know that G is a strongly-connected directed graph. Inversely, give a strongly-connected directed graph G, for each vertex v_i, if there is an edge e_{ij} from v_i to v_j, then give this edge a weight $p_{ij} \in (0,1)$, otherwise, set $p_{ij} = 0$. For each i, these p_{ij}s satisfy the condition $\sum_{\{j|e_{ij}\in G_e\}} p_{ij} = 1$. Now a matrix $(p_{ij})_{N\times N}$ is constructed, and from the strong-connectivity of G we know that for any $i$and j, there must exit a series of indicators, say $i_1, i_2, ..., i_n$, which satisfy the condition $p_{i_1 i_2} p_{i_2 i_3} ... p_{i_{n-1} i_n} > 0$, where $i_1 = i, i_n = j$.

To summarize, a RIFS can be defined from the viewpoint of graph theory as follows.

Definition 2. $\{X, \{w_i\}_{i=1}^{N}; G\}$ is a RIFS, iff
1) $w_i : X \to X$ is a group of contractive maps, in which $s_i, i = 1,2,...,N$ are corresponding contractive factors;
2) $G = (G_v, G_e)$ is a strongly-connected weighted directed graph that has N vertexes, the weight of edge (v_i, v_j) is p_{ij}.

It's clear that a RIFS becomes to an IFS if $p_{ij} = p_j > 0$. Example 1 will help us to comprehend RIFS more intuitive:

Example 1. Set N=3, $P_1 = (p_{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $P_2 = (p_{ij}) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{pmatrix}$, then matrix P_1 satisfies the conditions of definition 1, matrix P_2 does not satisfy. Graphics corresponding to P_1 and P_2, say G_1 and G_2 , are respectively shown in Figure 1 and Figure 2.

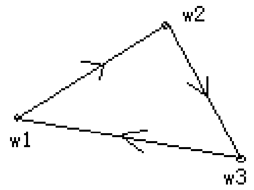

Figure 1: G_1 in Example 1

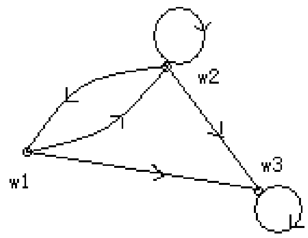

Figure 2: G_2 in Example 1

Apparently, G_1 is strongly-connected, while G_2 is weakly-connected. According to Definition 1 and Definition 2, $\{X, \{w_i\}_{i=1}^N; G_1\}$ can define a RIFS. But if we change the direction of edge (w_2, w_3), G_2 will become to a strongly-connected graph, and then, $\{X, \{w_i\}_{i=1}^N; G_2\}$ becomes to a RIFS.

3 Attractor of RIFS

For a RIFS $\{X, w_i, p_{ij}, i, j = 1, 2, ..., N\}$, consider the following random process on X: for an arbitrary point $Z_0 \in X, i_0 \in \{1, 2, ..., N\}$, choose $i_1 \in \{1, 2, ..., N\}$ with probability $p_{i_0 i_1}$, let $Z_1 = w_{i_1}(Z_0)$, then choose $i_2 \in \{1, 2, ..., N\}$ with probability $p_{i_1 i_2}$ and let $Z_2 = w_{i_2} \circ w_{i_1}(Z_0)$. Repeat this process $Z_n = w_{i_n}(Z_{n-1}) = w_{i_n} \circ w_{i_{n-1}} \circ ... \circ w_{i_1}(Z_0)$. It is clear that Z_n depends not only on Z_{n-1}, but also on the choice of i_n. This means $\{Z_n\}$ is not a Markov process on X. While if we define another process $\widetilde{Z_n} = (Z_n, i_n)$ on $\widetilde{X} = X \times \{1, 2, ..., N\}$, then $\{\widetilde{Z_n}\}$ turn to a Markov process on $\widetilde{X}$, the corresponding transition probability is

$$\widetilde{p}((x,i), \widetilde{B}) = \sum_{j=1}^{N} p_{ij} I_B(w_j x, j), \text{ where } \widetilde{B} \subset \widetilde{X} \text{ is a Borel set.}$$

As to the random process $\{\widetilde{Z_n}\}$, according to reference [3], we have the following theorem.

Theorem 1.

(i). *For the above Markov process $\{\widetilde{Z_n}\}$, there exists an unique stationary initial distribution (distribution of $\widetilde{Z_0}$) which makes the Markov process $\{\widetilde{Z_n}\}$ a stationary random process.*

(ii). *Denote the projection from the above distribution to X as μ, then we have: for any initial value (x_0, i_0), distribution of the trajectory $x_0, w_{i_1}(x_0), w_{i_2} \circ w_{i_1}(x_0), ...$, converges to μ with probability 1. μ is called the invariant measure of the RIFS. If we denote the support of μ as A, then A is called the invariant set or the attractor of the RIFS.*

Now we show that how the trajectory and further the attractor A are determined by matrix (p_{ij}) or graph G through an example.

Example 2. Let $X = [0,1]$, $w_1(x) = \frac{1}{3}x$, $w_2(x) = \frac{1}{3} + \frac{1}{3}x$, $w_3(x) = \frac{2}{3} + \frac{1}{3}x$. Give an initial point (x_0, i_0), $i_0 = 1$ for example, then we have

$$i_1 = 2,\ x_1 = w_2(x_0),$$

$$i_2 = 3,\ x_2 = w_3 \circ w_2(x_0),$$

$$i_3 = 1,\ x_3 = w_1 \circ w_3 \circ w_2(x_0),$$

$$i_4 = 2,\ x_4 = w_2 \circ w_1 \circ w_3 \circ w_2(x_0),$$

$$\ldots,$$

$$x_{3n} = (w_1 \circ w_3 \circ w_2)^{\circ n}(x_0),$$

$$x_{3n+1} = w_2 \circ (w_1 \circ w_3 \circ w_2)^{\circ n}(x_0),$$

$$x_{3n+2} = w_3 \circ w_2 \circ (w_1 \circ w_3 \circ w_2)^{\circ n}(x_0).$$

In other words, if we start from $w_1(i_0 = 1)$, only three compounds $(w_3 \circ w_2, w_1 \circ w_3, w_2 \circ w_1)$ are allowed. In contrast, if the RIFS becomes an IFS, in other words, graph G becomes to a complete graph, then all possible compounds of w_1, w_2, w_3 are allowed, that is why the attractor of the IFS is $[0,1]$.

4 A kind of decomposition of the attractor of RIFS

In this part, we will look at the RIFS from the viewpoint of product space, and give a kind of decomposition of its attractor. The decomposition establishes theoretical foundation for the extension of the definition of RIFS given by Barnsley.

First, we define product space and give its attractor .

Assume that $(X_j, d_j), j \in \{1,2,...,N\}$ is a compact metric space, H_j is a group of nonempty compact subset of X_j, h_j is the Hausdorff distance on H_j. It is easy to prove that (H_j, h_j) is a compact metric space. We name (H_j, h_j) as the fractal space corresponding to (X_j, d_j).

Now, we define a product space

$$\widetilde{H} = H_1 \times H_2 \times ... \times H_N = \{(A_1, A_2, ..., A_N) | A_j \in H_j, j = 1,2,...,N\},$$

the distance on this product space is defined as $\widetilde{h}(A,B) = \max_j h_j(A_j, B_j)$, for any $A = (A_1, A_2, ..., A_N) \in \widetilde{H}$ and $B = (B_1, B_2, ..., B_N) \in \widetilde{H}$. With this distance, we can immediately prove that $(\widetilde{H}, \widetilde{h})$ is a compact metric space.

Give an indicator set $I = \{(i,j) | i,j = 1,2,...,N\}$ which satisfies the condition that for every $i \in \{1,2,...,N\}$, there exists $j \in \{1,2,...,N\}$ such that $(i,j) \in I$. In other words, for any $i \in \{1,2,...,N\}$, we have $I(i) = \{j | (i,j) \in I\} \neq \phi$. For any element (i,j) of I, assume w_{ij} is the contractive map from X_j to X_i, the contractive factor is $0 < s_{ij} < 1$. Thus we have $d_i(w_{ij}(x), w_{ij}(y)) \leq s_{ij} d_j(x,y)$ for any x, y in X_j. For an arbitrary set from H_j, say A, define a map W_{ij}: $W_{ij}(A) = w_{ij}(A) = \{w_{ij}(x)|x \in A\}$. Then W_{ij} is a contractive map, and $h_{ij}(W_{ij}(A), W_{ij}(B)) \leq s_{ij}h_j(A,B)$. Now we can define an operator W on the product space $\widetilde{H}$:

$$W(A) = (\bigcup_{j\in I(1)} W_{1j}(A_j), ..., \bigcup_{j\in I(N)} W_{Nj}(A_j)), \text{ for any } A \in \widetilde{H}.$$

It is easy to prove that W is a contractive map from $\widetilde{H}$ to $\widetilde{H}$with the distance $\widetilde{h}$, the contractive factor is $s = \max\{s_{ij}, (i, j) \in I\} < 1$. Thus, according to the Banach fixed point theorem, we come to the conclusion that, there exists a unique element $A = (A_1, A_2, ..., A_N) \in \widetilde{H}$ which satisfies

$$A_i = \bigcup_{j\in I(i)} W_{ij}(A_j) \, foralli = 1,2,...,N.$$

That is to say $W(A) = A$, and $\lim_{n\to\infty} W^{\circ n}(A_0) = A$ for any A_0 from $\widetilde{H}$.

Now, let's take a look at the relationship between the attractor of RIFS and the attractor of the above product space. Review Definition 1, set $(X_j, d_j) = (X, d)$, $I = \{(i, I(i))|i = 1,2,...,N\}$ where $I(i) = \{j : p_{ji} > 0\}$, then from Definition 1 we know certainly that $I(i) \neq \phi$. Let $w_{ij} = w_i$, (H,h) be the corresponding fractal space, $\widetilde{H} = \underbrace{H \times H \times ... \times H}_{N}$; $\widetilde{h}(A,B) = \max_i h(A_i, B_i)$, $W(A) = (\bigcup_{j\in I(1)} w_1(A_j), ..., \bigcup_{j\in I(N)} w_N(A_j))$. Thus we get a compact metric spaces $(\widetilde{H}, \widetilde{h})$ and a contractive map W on $\widetilde{H}$. Theorem 2 will give the relationship between the invariant set of W and the invariant set of RIFS.

Theorem 2 ([3]).

Let W be a RIFS under Definition 1 and Definition 2, and A be the attractor of W. Then, there exists a unique group of compact set $\{A_i, i = 1,2,...,N\}$ *where* $A_i \subset A$ *and* $A = \bigcup_{i=1}^{N} A_i$, $A_i = \bigcup_{j:p_{ji}>0}^{w_i(A_j)}$.

Theorem 2 tells us that the attractor A of a RIFS can be uniquely decomposed to N parts: $A_i, i = 1,2,...,N$, and the N-tuple $(A_1, A_2, ..., A_N)$ is just the attractor of $(\widetilde{H}, \widetilde{h})$. Inversely, as long as the attractor of the metric space $(\widetilde{H}, \widetilde{h})$ exists, the attractor of the RIFS $\{X, w_i, p_{ij}, i, j = 1,2,...,N\}$ exists definitely. If we get the attractor of $(\widetilde{H}, \widetilde{h})$, we get the unique decomposition of the attractor of the original RIFS .

5 Definition of extended RIFS and the existence and uniqueness of its attractor

Through above discussions, we can see that for an arbitrary compact metric space (X,d), as long as the indicator set I satisfies the condition $I(i) = \{j|(i, j) \in I\} \neq \phi$ for any i, we can construct a corresponding fractal space $(\widetilde{H}, \widetilde{h})$ and a contractive map W.

Furthermore, according to Theorem 2, the existence and uniqueness of the attractor of the RIFS $\{X, w_i, p_{ij}, i, j = 1,2,...,N\}$ can also be proved. The irreducibility of the matrix P is not required this time, the value of p_{ij} is not important either, the only emphasis is whether the row vector of P is non-zero. So, we can give the definition of a RIFS from the viewpoint of graph theory, and also give the theorem about the existence and uniqueness its attractor:

Definition 3. $\{X, (w_i\}_{i=1}^{N}; G\}$ is a RIFS, iff

(1) $w_i : X \to X, i = 1,2,...,N$ is a group of contractive map, contractive factors are s_i separately;

(2) $G = (G_v, G_e)$ is a directed graph with N vertexes, the in-degree of each vertex is larger than 1.

Theorem 3. *Under Definition 3, the attractor of a RIFS exists and is unique.*

6 Recursive fractal interpolation function

Recursive fractal interpolation function is an extension of fractal interpolation function. The latter is the attractor of IFS, we will see in this part that recursive fractal interpolation function is the attractor of RIFS.

Let $I = [0,1]$, $K = I \times R$, $w_i (i = 1,2,...,N)$ is an affine transformation and has the form of

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}, \text{ where } |d_i| < 1.$$

It's easy to prove that under the metric $d^*((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + \frac{1-a}{2c}|y_1 - y_2|$, w_i is a contractive map with $s = \max\{\frac{1+a}{2}, d\}$ as its contractive factor, where $a = \max\{|a_i|\}$, $c = \max\{|c_i|\}$, $d = \max\{|d_i|\}$. Beyond this, w_i also meets the following conditions:

Give a data set $\{(x_i, y_i)\}_{i=0}^{N}$, where $0 = x_0 < x_1 < ... < x_N = 1$ is a division of I, denote $I_i = [x_{i-1}, x_i], i = 1,2,...,N$, for any I_i,

(1) there exists an interval $J_i = [x_{l(i)}, x_{r(i)}]$ which satisfies that $x_{r(i)} - x_{l(i)} > x_i - x_{i-1}$, $l(i)$, $r(i) \in \{0,1,2,...,N\}$, and

(2) $w_i((x_{l(i)}, y_{l(i)}) = (x_{i-1}, y_{i-1})$, $w_i((x_{r(i)}, y_{r(i)}) = (x_i, y_i)$.

Reference [6] proved the existence and uniqueness of the attractor of RIFS determined by $\{w_i, i = 1,2,...,N\}$, while we will see that if we use the extended definition of RIFS and Theorem 3 in part 5, the existence and uniqueness of the recursive fractal interpolation function is apparent.

Proof. Define a directed graph G as follows:

there is a directed edge from j to i, iff $I_j \subset J_i$.

It is obvious that, the in-degree of every vertex of G is larger than 2, then $\{K, w_i, G_1, i = 1,2,...,N\}$ is a RIFS according to Definition 3. From Theorem 3, the attractor (invariant set) of this RIFS exists and is unique. From reference[1], we also know that this attractor is just the image of a continuous function F on I, and that$F$interpolates on $\{(x_i, y_i)\}_{i=0}^{N}$. So, the attractor (invariant set) of $\{K, w_i, G_1, i = 1,2,...,N\}$ is the recursive fractal interpolation function determined by $\{w_i, i = 1,2,...,N\}$.

□

The following example shows us differences between Definition 1 and Definition 2.

Example 3. Set $N = 4$, let $J_1 = J_2 = I_1 \cup I_2, J_3 = J_4 = I_3 \cup I_4$. The corresponding graph G_3 is shown in Figure 3.

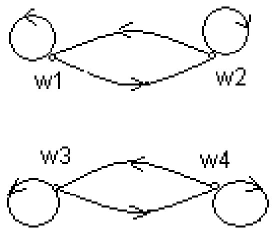

Figure 3: G_3 in Example 2

Obviously, G_3 is non-connective. However, G_3 can determine a RIFS with an unique attractor according to Definition 3.

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