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Path and Cycle Factors of Cubic Graphs

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Abstract For a set $\mathscr{F}$ of connected graphs, a spanning subgraph F of a graph is called an $\mathscr{F}$-factor if every component of F is isomorphic to a member of $\mathscr{F}$. It was recently shown that every 2-connected cubic graph has a $\{C_n|n \geq 4\}$-factor, where C_n denote the cycle of order n. Kano et al.have conjectured that every 3-connected cubic graph of order at least six has a $\{C_n|n \geq 5\}$-factor. In this paper,we give a proof of this conjecture.

Keywords path factor; cycle factor; cubic graph

1 Introduction

We consider finite graphs without loops or multiple edges. A 3-regular graph is called a cubic graph. We denote by C_n and P_n the path and the cycle of order n, respectively. For a set $\mathscr{F}$ of connected graphs, a spanning subgraph F of a graph G is called an $\mathscr{F}$-factor of G if every component of F is isomorphic to one of members in $\mathscr{F}$. Then a $\{C_n|n \geq 3\}$-factor is nothing but a 2-factor, which is a spanning 2-regular subgraph. In this paper we consider cycle-factors and path-factors of cubic graphs, whose components are cycles and paths, respectively. We begin with some known results on these factors.

Theorem 1. *(Kaneko [1]) Every connected cubic graph has a $\{P_n|n \geq 3\}$-factor.*

Theorem 2. *(Petersen [2]) Every 2-connected cubic graph has a $\{C_n|n \geq 3\}$-factor.*

Kawarabayashi et al [3] showed the next theorem.

Theorem 3. *(i) Every 2-connected cubic graph has a $\{C_n|n \geq 4\}$-factor.*
(ii) Every 2-connected cubic graph of order at least six has a $\{P_n|n \geq 6\}$-factor.

In this paper,we shall prove the following result,which is conjectured by kano et al [4].

Theorem 4. *Every 3-connected cubic graph of order at least six has a $\{C_n|n \geq 5\}$-factor.*

The following corollary follows immediately from theorem 4.

Corollary 5. *Every 3-connected cubic graph of order at least six has a $\{p_n|n \geq 7\}$-factor.*

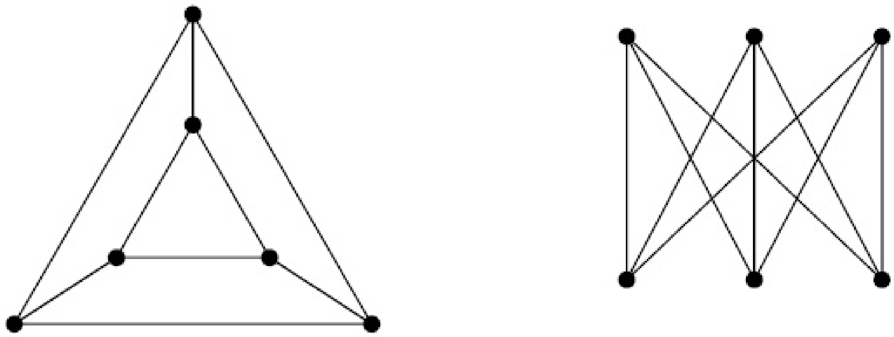

Figure 1: 3-connected cubic graph of order six.

2 Proof of Theorem 4

For a vertex v of a graph G, we denote the order of G by $|G|$, which is equal to $|V(G)|$.

Proof.Let G be a 3-connected cubic graph. We prove theorem 4 by induction on the order $|G|$. There exist two 3-connected cubic graphs of order six(see figure 1). Apparently, both of them have $\{C_n|n \geq 5\}$-factor. So we may assume $|G| \geq 8$. G is 3-connected, and so G has a $\{C_n|n \geq 4\}$-factor F by Theorem 2. We may assume that F contains a component D isomorphic to C_4 since otherwise F is the desired $\{C_n|n \geq 5\}$-factor. Let $V(D) = a,b,c,d$,since graph G is 3-connected, both ac and bd are not the edge of G. We assume that ar, bs, ct, du be the edges of $G - E(D)$ incident with $V(D)$ (see Figure 2). Since $G - E(F)$ is a 1-factor of G, ar, bs, ct, du is a set of independent edges, and so r, s, t, u are all distinct vertices of G. Let H be the graph obtained from G by removing the four vertices a, b, c, d and their incident edges, and by adding two new vertices v and w together with five new edges rv, tv, vw, uw, sw(see Figure 2). Now,we will prove that H is a 3-connected cubic graph. Since G is 3-connected graph, for any two disjoint vertex subsets A and B of $V(G)$, there are at least three distinct paths from A to B. And the number of distinct paths passing through D is at most two. To prove H is 3-connected graph,we only need to prove the case of there are just two paths passing through D from A to B since otherwise H is apparently 3-connected graph. For this case, we assume two subcase of three paths from A to B in graph G. Case1:$AB, AraduB, AsbctB$ in graph G,we can find three paths between A and B in graph H which is $AB, ArvtB, AswuB$(see figure 3). Case2:$AB, ArabsB, AudctB$ in graph G,we can find three paths between A and B in graph H which is AB, $ArvtB$, $AuwsB$ (see figure 4). So H is also 3-connected graph. Then H is a 3-connected cubic graph, and $|H| = |G| - 2$. Hence H has a $\{C_n|n \geq 5\}$-factor F_H by induction. We can obtain the desired $\{C_n|n \geq 5\}$-factor of G from F_H by the method in [4]. Consequently Theorem 4 is proved.

Corrallary5 follows immediately from the next Lemma 6 and the Theorem 4.

Lemma 6. *[4] Let $k \geq 3$ be an integer. If a 2-connected cubic graph G of order at least $k+2$ has a $\{C_n|n \geq k\}$-factor, then G has a $\{P_n|n \geq k+2\}$-factor.*

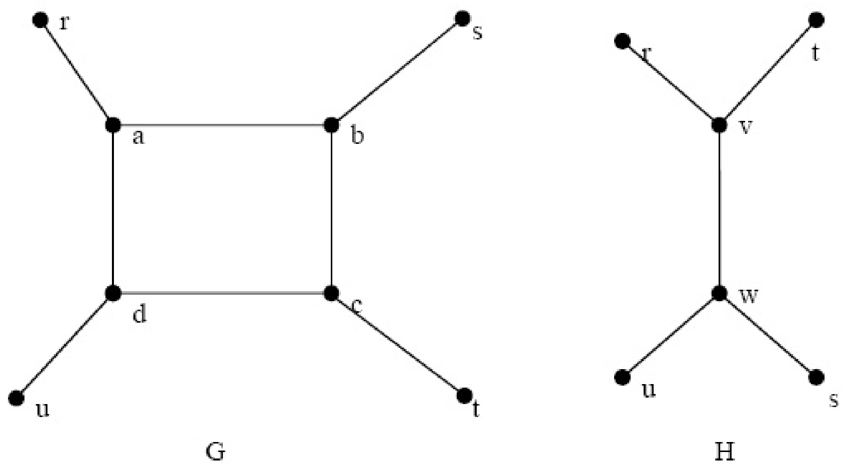

Figure 2: Cubic graph G and H.

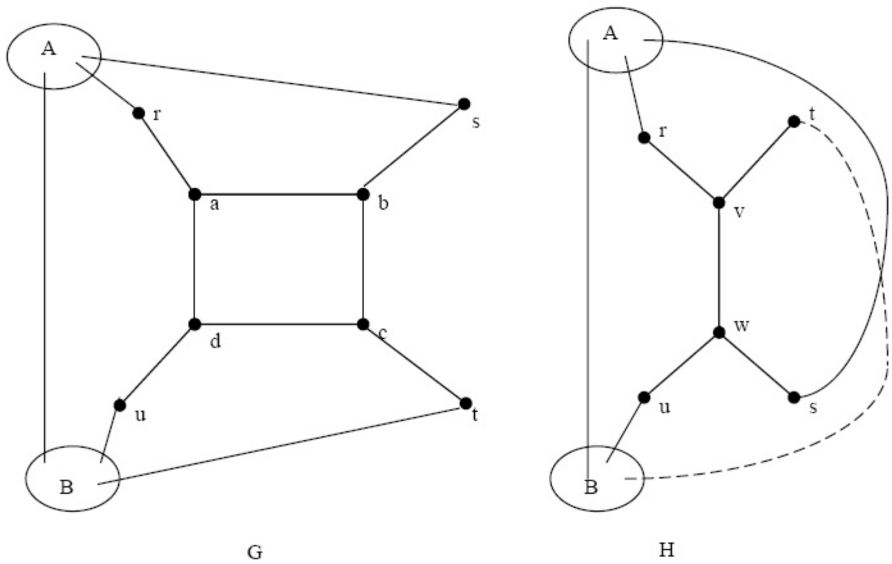

Figure 3: 3-connected cubic graph G and H.

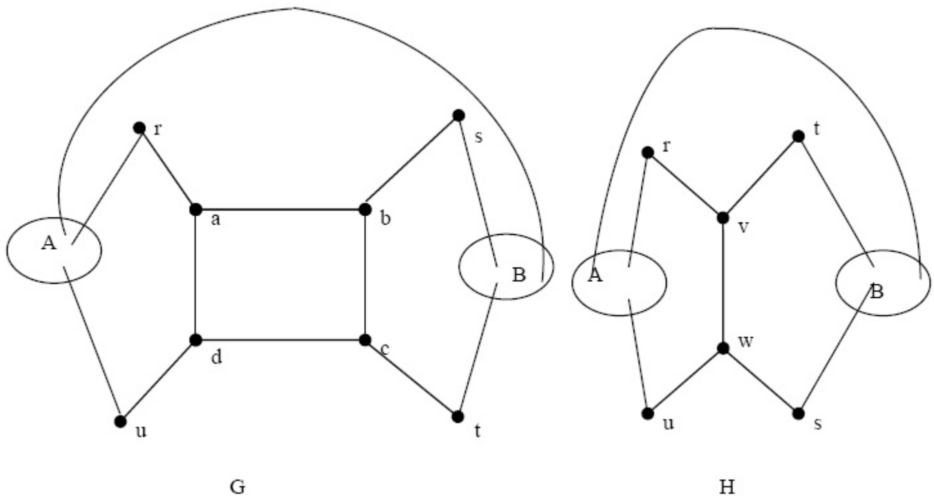

Figure 4: 3-connected cubic graph G and H.

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