The Eighth International Symposium on Operations Research and Its Applications (ISORA'09)
Zhangjiajie, China, September 20–22, 2009

A Study of First-price Sealed-bid Procurement Auctions for Divisible Items*

Yihua Wei[1] Qiying Hu[2] Wuyi Yue[3]

[1]College of Management, Shanghai University, Shanghai 200444, China
[2]School of Management, Fudan University, Shanghai 200433, China
[3]Department of Intelligence and Informatics, Konan University, Kobe 658-8501, Japan

Abstract In this paper, we study first-price sealed-bid procurement auctions for divisible items. The auctions are divided into two cases: the auctioneer either does or does not compensate for an oversupply of the quantity purchased. The bidding behaviors of bidders are modeled, and then the optimal bidding strategy for bidders is obtained. It is shown that the bidder's bidding price exists and is increasing in his cost. Moreover, we show that the bidder can gain more profit in cases where the auctioneer compensates the overage supply quantity.

Keywords Procurement auctions; First-price sealed-bid; Optimal bidding strategy.

1 Introduction

Auctions have been widely used by many companies and government departments to purchase goods or services. In this paper, we study first-price sealed-bid procurement auctions for divisible items. In practice, many items are divisible, e.g., electric power. Moreover, multiple items can be seen as divisible items when the purchase quantity is large enough.

In a sealed-bid procurement auction for divisible items, each bidder submits his bid to the auctioneer (who acts as the buyer). The bid includes a bidding price and the quantity to be supplied. Usually, the auctioneer needs a large enough number of items or quantity such that any one bidder can not satisfy the bidder's need. After bidders' bid, the auctioneer purchases a demand quantity from bidders' based on ascending order of bidding prices submitted by bidders. The bidding price of the last bidder who satisfies the demand quantity is said to be *the market clearing price.* All bidders whose bidding prices are lower than the market clearing price sell their supply quantities to the auctioneer, and so they are winners or winning bidders. In the first-price sealed-bid auctions, the trade-off price is the winning bidders' own bid prices.

In auctions for divisible items, it is usually impossible that the winning bidders' total supply quantity is exactly equal to the auctioneer's demand. There is a gap between the total supply of all winning bidders and the demand of the auctioneer, and so the auctioneer

*The project was supported in part by the National Natural Science Foundation of China through grant 70571049, Project of Shanghai Education Committee of 06AS134, China, and by GRANT-IN-AID FOR SCIENCE RESEARCH (No. 21500086) and the Hirao Taro Foundation of KUAAR, Japan.

may need only a part of some bidder's supply. That is, there is an oversupply. In order to bridge this gap, the auctioneer has alternatives. The first is to compensate the oversupply with a trade-off price. This is widely used in procurement auctions for such items as electric power, for which once the bidder produces his product, the production capacity must not be less than a certain quantity. The other is not to compensate for an oversupply.

Zhang, et al. (2000) discussed a game between power generators and the power company, in which generators submit their bids to the power company, and each bid includes a quantity to be supplied and a cost for generating that power. To minimize the total cost, the power company determines the clearing price and each generator's supply quantity. Song and Liu (2000a) assumed that the supply quantity is a liner function of the price, and other bidder's biding is represented as a discrete random variable. Song and Liu (2000b) expressed the procurement auction as a multi-period decision-making problem, and used a discrete-state and discrete-time Markov decision process to select the optimal policy from the discrete bidding strategy sets.

Ausubel and Cramton (2004) studied the theoretical and practical implementation of a forward auction clearing price auction mechanism for divisible items. There are studies on the procurement auctions for multiple items. Chen, Roundy, Zhang, et al. (2005) considered Vickrey auctions for procuring multiple items in supply chain settings by incorporating transportation costs into auctions. Engelbrecht-Wiggans and Katok (2006) studied a hybrid auction mechanism based on clearing price auctions for multiple items under the assumption that each bidder has only one unit item. Parkes and Kalagnanam (2005) studied multi-attribute auctions that extend traditional auction settings to allow negotiation over non-price attributes such as weight and color. But under their framework there is only one winner and so the auction corresponds to single item auctions.

The purpose of this paper is to give a theoretical analysis for purchasing divisible items through the first-price auction, where the auctioneer either does or does not compensate the overage supply quantity. The rest of the paper is organized as follows. In Section 2, we give the system model. Sections 3 studies the non-compensated case, and Section 4 studies the compensated case. Section 5 discusses the property of bidding prices in the case of two bidders. In Section 6, a numerical analysis is given to illustrate the models and results. Section 7 gives the conclusion.

2 System Model

An auctioneer plans to purchase some divisible items with a quantity of D via a first-price sealed-bid auction. It is assumed that there are $n+1$ supply bidders attending the auction. Each bidder's bidding price and bidding (supply) quantity are private and symmetric, i.e., each bidder knows his own bidding price and bidding quantity, yet only knows other bidders' bidding prices and bidding quantities as random variables which are draw identically and independently from distribution functions $F(\cdot)$ and $G(\cdot)$, respectively. Suppose $F(\cdot)$ has a finite support set $[\underline{b}, \overline{b}]$.

Suppose that a bidder, say bidder A, submits his bidding price b and supply quantity q, respectively, and that the biding prices of all other n bidders are $b_1 \leq b_2 \leq \ldots \leq b_n$ in an ascending order, with the corresponding supply quantities being $q_1, q_2, \ldots, q_n$. Note that $b_0 = \underline{b}$ and $b_{n+1} = \overline{b}$. Moreover, suppose that bidder A has a unit production cost c.

Obviously, for the auction we are concerned with here, the bidder's expected profit is increasing with his supply quantity. So the optimal bidding quantity is just equal to his production capacity. Hence, the only problem left for the bidder is to decide his optimal bidding price.

Before further discussion, we will introduce the following well-known lemma (see Deng and Liang (1998)).

Lemma 1.
Assume that $X_1, X_2, \ldots, X_n$ are independently identically distributed (i.i.d.) variables drawn from the distribution function $F(x)$ and the probability density function $f(x)$. Then, the probability density function (p.d.f.) of the kth lowest variable of $X_1, X_2, \ldots, X_n$ is

$$f_k(x) = n\binom{n-1}{k-1} F(x)^{k-1}(1-F(x))^{n-k} f(x).$$

3 Non-compensating the Overage Supply Quantity

Obviously, for bidder A's bid b, there must exist some $j = 0, 1, \ldots, n$ such that $b \in (b_j, b_{j+1}]$. Consider the following two cases.

(i) If the first j bidder's total supply quantity does not satisfy the demand, i.e., $\sum_{i=1}^{j} q_i < D$, then bidder A wins the auction, and his clearing quantity is q if his supply quantity q is less than the demand gap $D - \sum_{i=1}^{j} q_i$; otherwise, his clearing quantity is $D - \sum_{i=1}^{j} q_i$. Hence, the profit of bidder A is

$$(b-c)\min\left\{q, D - \sum_{i=1}^{j} q_i\right\} = \begin{cases} (b-c)q, & q \le D - \sum\limits_{i=1}^{j} q_i \\ (b-c)\left(D - \sum\limits_{i=1}^{j} q_i\right), & 0 < D - \sum\limits_{i=1}^{j} q_i \le q. \end{cases}$$

(ii) If the first j bidders' total supply quantity satisfies the demand, i.e., $\sum_{i=1}^{j} q_i \ge D$, then bidder A loses the auction and his profit is 0.

Let $x^+ = \max\{x, 0\}$ be the positive part of real x. Thus, $D_j := \left(D - \sum_{i=1}^{j} q_i\right)^+$ is the gap between the demand and the total supply quantity of the first j bidders. Due to $b \in (b_j, b_{j+1}]$, bidder A's profit is $(b-c)\min\{q, D_j\}$.

By summarizing the two cases above, we get the profit of bidder A as follows:

$$r^{FN}(b; c, q, b_1, \ldots, b_n; q_1, \ldots, q_n) = (b-c)\min\{q, D_j\},\ b_j < b \le b_{j+1}, j = 0, 1, \ldots, n. \quad (1)$$

Therefore, the expected profit of bidder A is

$$\begin{aligned} R^{FN}(b; c, q) &= E_{b_j, q_j, j=1,2,\ldots,n} r^{FN}(b; c, q, b_1, \ldots, b_n; q_1, \ldots, q_n) \\ &= (b-c)\sum_{j=0}^{n} E\min\{q, D_j\}[\Pr\{b \le b_{j+1}\} - \Pr\{b \le b_j\}]. \end{aligned}$$

For convenience, denote that

$$a_j(q) := E\min\{q, D_{j-1}\} - E\min\{q, D_j\}, \quad j = 1, 2, \ldots, n, \quad a_{n+1}(q) := E\min\{q, D_n\}.$$

The distribution function of b_j, denoted by $F_j(b) := Pr\{b_j < b\}$, can be computed from Lemma 1. Let $\bar{F}_j(b) = 1 - F_j(b)$ and $f_j(b)$ be the probability density function of b_j. Then the bidder's expected profit above can be rewritten as

$$R^{FN}(b;c,q) = (b-c)\sum_{j=1}^{n+1} a_j(q)\bar{F}_j(b). \tag{2}$$

Hence, bidder A's problem is to find an optimal bidding price b to maximize his expected profit, i.e.,

$$R^{FN}(c,q) \quad := \quad \max_{\underline{b}\le b\le \overline{b}} R^{FN}(b;c,q). \tag{3}$$

Following the first order condition, the maximum of $R^{FN}(b;c,q)$ must satisfy that

$$b - \frac{\sum\limits_{j=1}^{n} a_j(q)\bar{F}_j(b) + a_{n+1}(q)}{\sum\limits_{j=1}^{n} a_j(q) f_j(b)} = c. \tag{4}$$

In general, the solutions to the above equation are not unique. Let B be the set of all solutions to equation (4) that lie in $[\underline{b},\overline{b}]$. If B is not empty, let $b' = \arg\max_{b\in B} R^{FN}(b;c,q)$ be the bidding price with the maximal value among B. Then the bidder's optimal bidding strategy is given by

$$b^{FN}(c,q) = \arg\max\{R^{FN}(b';c,q), R^{FN}(\underline{b};c,q), R^{FN}(\overline{b};c,q)\} \tag{5}$$

where due to $\bar{F}_j(\underline{b}) = 1$ and $\bar{F}_j(\overline{b}) = 0$ for all $j \le n$,

$$R^{FN}(\underline{b};c,q) = (\underline{b}-c)E\min\{q,D\}, \quad R^{FN}(\overline{b};c,q) = (\overline{b}-c)E\min\left\{q, \left(D - \sum_{i=1}^{n} q_i\right)^+\right\}.$$

We now can prove the following theorem.

Theorem 1.
The bidder's optimal bidding strategy $b^{FN}(c,q)$ exists and is increasing in c. Moreover, $b^{FN}(c,q)$ is an interior point of $[\underline{b},\overline{b}]$ when $c \in (\underline{b},\overline{b})$ and $D < \sum_{i=1}^{n} q_i$.

Proof. (1) The existence of the optimal bidding strategy is obvious since $R^{FN}(b;c,q)$ is continuous in the closed interval $[\underline{b},\overline{b}]$.

(2) In order to show the monotone of $b^{FN}(c,q)$, due to Theorem 3.3 in Hu and Liu (2000) about properties of modular functions, it suffices to verify that $R^{FN}(b;c,q)$ is submodular in (b,c). But this is obvious from the following fact:

$$\frac{\partial^2 R^{FN}(b;c,q)}{\partial b \partial c} = \sum_{j=1}^{n} a_j(q) f_j(b) \ge 0.$$

(3) When $c \in (\underline{b},\overline{b})$ and $D < \sum_{i=1}^{n} q_i$, it is easy to see that $R^{FN}(\underline{b};c,q) = R^{FN}(\overline{b};c,q) = 0$. So, $b^{FN}(c,q) \in (\underline{b},\overline{b})$. □

The theorem above tells us that bidders always have their optimal bidding prices, which increase with their production costs. Moreover, when the unit cost lies in the interval $[\underline{b},\overline{b}]$ and the total production capacity of all bidders can satisfy the auctioneer's demand, the optimal bidding price lies also in the interior of the interval $[\underline{b},\overline{b}]$.

4 Compensating for an Oversupply

In this section, we consider the case where the auctioneer compensates for an oversupply. Then, once a bidder wins the auction, his clearing quantity equals his bidding quantity. Similar to equation (2), we can obtain the expected profit of bidder A as follows:

$$R^{FC}(b;c,q) = (b-c)q\sum_{j=0}^{n}\Pr\left\{D > \sum_{i=1}^{j} q_i\right\}\Pr\{b_j < b \leq b_{j+1}\}.$$

For convenience, denote that $A_j := \Pr\{\sum_{i=1}^{j-1} q_i < D \leq \sum_{i=1}^{j} q_i\}$ for $j = 1,2,\ldots,n$, and $A_{n+1} := \Pr\{D > \sum_{i=1}^{n} q_i\}$. Then, bidder A's expected profit can be rewritten as follows:

$$R^{FC}(b;c,q) = q(b-c)\sum_{j=1}^{n+1} A_j\bar{F}_j(b) := qR^{FC}(b;c). \tag{6}$$

So, bidder A's problem is as follows:

$$R^{FC}(c,q) \quad = \quad q\max_{\underline{b}\leq b\leq\overline{b}} R^{FC}(b;c). \tag{7}$$

Certainly, the optimal bidding price, denoted by $b^{FC}(c)$, is independent of q, the supply quantity. Following the first order condition, one can obtain the following equation for the optimal bidding price:

$$b - \frac{\sum\limits_{j=1}^{n} A_j\bar{F}_j(b) + A_{n+1}}{\sum\limits_{j=1}^{n} A_j f_j(b)} = c. \tag{8}$$

This is similar to equation (4) for the non-compensating case with the unique difference being that A_j here is independent of q.

Let B_1 be the set of all solutions of the above equation that lie in $[\underline{b},\overline{b}]$. If B_1 is not empty, let

$$b_1'(c) = \arg\max_{b\in B_1} R^{FC}(b;c)$$

be the bidding price with the maximal value among B_1. Then the bidder's optimal bidding strategy is

$$b^{FC}(c) = \arg\max\{R^{FC}(b'(c);c), R^{FC}(\underline{b};c), R^{FC}(\overline{b};c)\} \tag{9}$$

where

$$R^{FC}(\underline{b};c) = (\underline{b}-c), R^{FC}(\overline{b};c) = (\overline{b}-c)P\left\{\sum_{i=1}^{n} q_i < D\right\}.$$

Similar to Theorem 1, we can prove the following theorem.

Theorem 2.
The bidder's optimal bidding strategy $b^{FC}(c)$ exists, is independent of q, and is increasing in c.

The following result compares the bidder's profits under the two cases discussed in Theorems 1 and 2, respectively.

Theorem 3.
$R^{FC}(c,q) \geq R^{FN}(c,q)$, i.e., bidders can gain more profit in the case where the auctioneer compensates for an oversupply.

Proof. Since $\min\{q,(D-\sum_{i=1}^{j} q_i)^+\} \leq q$ if $D > \sum_{i=1}^{j} q_i$, and $= 0$ otherwise, we thus have $E\min\{q, D_j\} \leq q\Pr\{D > \sum_{i=1}^{j} q_i\}$. Substituting the above formulas into equations (2) and (6), we can obtain that for any given b, $R^{FN}(b;c,q) \leq R^{FC}(b;c,q)$. Thus,

$$R^{FN}(c,q) = \max_b R^{FN}(b;c,q) \leq \max_b R^{FC}(b;c,q) = R^{FC}(c,q).$$

This completes the proof. □

The result given in the theorem above is clear.

5 Property of Bidding Prices in the Case of Two Bidders

In this section, we will discuss property of bidding prices in the case where there are two bidders, i.e. $n = 1$. Following from equations (2) and (6), we can obtain that

$$R^{FN}(b;c,q) = \begin{cases} (b-c)[\bar{F}(b)(q-\phi(q))+\phi(q)] & q \leq D \\ (b-c)[\bar{F}(b)(D-\phi(D))+\phi(D)] & q \geq D \end{cases} \tag{10}$$

where

$$\begin{aligned} \phi(q) &= Emin\{q,(D-q_1)^+\} = \int_0^q G(D-x)dx, \\ R^{FC}(b;c) &= (b-c)[\bar{F}(b)+F(b)G(D)] \\ &= (1-G(D))(b-c)\left(\bar{F}(b)+\frac{G(D)}{1-G(D)}\right). \end{aligned} \tag{11}$$

Lemma 2.
(1) The bidder's bidding strategy $b^{FN}(c,q)$ is decreasing in q and;
(2) $b^{FN}(c,q) \leq b^{FC}(c)$, i.e. the bidder will bid higher in the case where they will be compensated.

Proof. (1) It will be proved in the following two cases.

(i) If $q \geq D$, then $R^{FN}(b;c,q) = (b-c)[\bar{F}(b)(D-\phi(D))+\phi(D)]$ is independent in q, so that $b^{FN}(c,q)$ is independent in q;

(ii) If $q \leq D$, then

$$R^{FN}(b;c,q) = (b-c)[\bar{F}(b)(q-\phi(q))+\phi(q)] = (q-\phi(q))(b-c)\left(\bar{F}(b)+\frac{1}{q/\phi(q)-1}\right).$$

Since $\phi(q) = Emin\{q,(D-q_1)^+\} < q$, then that the bidder's problem can be written as

$$R^{FN}(c,q) = \max_{\underline{b} \leq b \leq \bar{b}} R^{FN}(b;c,q) = (q-\phi(q)) \max_{\underline{b} \leq b \leq \bar{b}} \psi(b:c,q)$$

where $\psi(b:c,q)=(b-c)\left(\bar{F}(b)+\frac{1}{q/\phi(q)-1}\right)$.

Similar to the proof of Theorem 1, $\psi(b:c,q)$ is submodular in (b,c), which implies that $b^{FN}(c,q)$ is decreasing in q. So we only need to verify the submodularity of $\psi(b:c,q)$.

Because

$$\frac{dq/\phi(q)}{q}=\frac{\int_0^q G(D-x)dx-qG(D-q)}{\phi^2(q)}\geq\frac{\int_0^q G(D-q)dx-qG(D-q)}{\phi^2(q)}=0$$

means that $q/\phi(q)$ is increasing in q, then

$$\frac{\partial\psi(b:c,q)}{\partial b}=\frac{d}{db}[(b-c)\bar{F}(b)]+\frac{1}{q/\phi(q)-1}$$

is decreasing in q. This means that the submodularity of $\psi(b:c,q)$.

By summarizing the two cases above, we can obtain that $b^{FN}(c,q)$ is decreasing in q.

(2) Because $q/\phi(q)$ is increasing in q, then $\frac{1}{q/\phi(q)-1}$ is decreasing in q. And also because

$$\lim_{q\to 0}\frac{1}{q/\phi(q)-1}=\lim_{q\to 0}\frac{\phi(q)}{q-\phi(q)}=(\lim_{q\to 0}\frac{\phi'(q)}{1-\phi'(q)}=\frac{G(D)}{1-G(D)},$$

then $\frac{1}{q/\phi(q)-1}\leq\frac{G(D)}{1-G(D)}$.

Following the submodularity property, it is obvious that the optimal solution of $\max_b\{(b-c)[\bar{F}(b)+y]\}$ is increasing in y. Hence, $b^{FN}(c,q)\leq b^{FC}(c)$. □

6 A Numerical Example

The buyer purchases 2,000 tons of a product, and the bidder's bidding price will not exceed 500 dollar/ton. Each bidder is uncertain about the other bidders' bidding prices and supply quantities, but knows that the bidding prices follow a uniform distribution over the interval $[10,50]$, and the supply quantity has a probability of 1/4 for 700, 800, 900, and 1,000 tons, respectively. The supply quantity (capacity) of bidder A is 900 tons.

Substituting the parameters into the non-compensating model and the compensating model respectively, the optimal bidding strategy and the expected profit of bidder A in both case can be obtained, as shown in Table 1, for varying production costs.

Table 1. Bidder's optimal bidding strategy and expected profit for the first-price auction.

Cost	Non-compensating		Compensating	
	Bidding price	Expected profit	Bidding price	Expected profit
100	257	85050	292	119841
150	282	60092	310	90112
200	309	39635	331	63779
250	339	23851	354	41506
300	369	12607	379	23891
350	401	5451	407	11318
400	434	1643	436	3759
450	467	208	467	525

The results in Table 1 show that (1) the bidder's bidding price increases with the cost, while the expected profit decreases with the cost; and (2) bidders will bid lower prices and then obtain lower profits in the non-compensating case than those in the compensating case. These illustrate the results given in Theorems 1, 2, and 3.

7 Conclusions

This paper studied first-price sealed-bid procurement auctions, in which the auctioneer either does or does not compensate for an over supply in the quantity of the item being traded. We obtained the optimal bidding strategy for the bidder (supplier), and showed that the bidding price increases along with the unit cost. Moreover, we showed that the bidder can gain more profit when the auctioneer compensates for an oversupply.

References

[1] Ausubel, L. and Cramton, P., 2004. Auctioning many divisible goods. Journal of the European Economic Association, 2 (2-3): 480 - 493.

[2] Chen, R. R., Roundy, R. O., Zhang, R. Q. and Janakiraman, G., 2005. Efficient auction mechanisms for supply chain procurement. Management Science, 51 (3): 467-482.

[3] David, A. and Wen, F., 2000. Strategic bidding in competitive electricity markets: A literature survey. Proceedings of IEEE PES 2000 Summer Meeting: 2168-2173.

[4] Deng, Y. and Liang, Z., 1998. Stochastic Point Processes and Their Applications (in Chinese). Beijing: Science Press.

[5] Engelbrecht-Wiggans, R. and Katok, E., 2006. E-sourcing in procurement: Theory and behavior in reverse auctions with noncompetitive contracts. Management Science, 52 (4): 581-596.

[6] Hu, Q. and Liu, J., 2000. An Introduction to Markov Decision Processes (in Chinese). Xian: Xidian University Press.

[7] Parkes, D. C. and Kalagnanam, J., 2005. Models for iterative multiattribute procurement auctions. Management Science, 51 (3): 435-451.

[8] Shrestha, G., Kai, S. and Goel, L., 2001. Strategy bidding for minimum power output in the competitive power market. IEEE Transaction on Power System, 16 (4): 813-818.

[9] Song, H. and Liu, C., 2000a. Future aspects of modern heuristics applications to power systems. Proceedings of IEEE PES 2000 Winter Meeting: 1307-1311.

[10] Song, H. and Liu, C., 2000b. Optimal electricity supply bidding by Markov decision process. IEEE Transaction on Power System, 15 (2): 618-624.

[11] Zhang, D., Wang, Y. and Luh, P., 2000. Optimization based bidding strategies in the deregulated market. IEEE Transaction on Power System, 15 (3): 981-986.