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Entropy Methods for Equilibrium Programming with Its Application in Traffic

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Abstract In this paper, we first develop the concept of the maximum entropy function for equilibrium programming by employing prior distribution information and Kullback entropy. It plays a key role in entropy function method for solving equilibrium programming. After analyze and discuss the relation with the model transformation equivalent applied in mixed traffic assignment problems, and finally we give an optimal model and effective algorithm for mixed traffic assignment problems.

Keywords Operations research; equilibrium programming; entropy function; mixed traffic assignment.

1 Introduction

Parametric optimal problems are widely met in various decision situations. And equilibrium programming (EP) is the most important one. It plays important role in the economic mathematics, game theory, optimization theory and networks equilibrium problem [1-3]. Variational inequality (VI) problems is a generalization of an optimal problem, when the continuous mapping $F(x)$ is a gradient of differentiable convex function, the VI problems is equivalent to a mathematical programming problem [4,5]. The relation between the EP and VI problems and the integration model of mathematical programming with its applications in the economic equilibrium was presented by Carey in 1977, and an equivalent representation of differentiable optimization model is shown through the regularization method by Fukushima in 1992. Entropy function method formed in 1987 is a very effective one to solve some optimization problems and it provides a new approach for theory analysis and numerical calculation of optimization problems [6]. The method is generalized to max-min problems for continuous bounded set in reference [7] and an approach based on entropy function for VI problems is presented in reference [8]. The entropy function method for solving the max-min problems has shown its advantages in dealing with the un-differential property aspects. It is meaningful both in methodology and applications to investigate the entropy function method for EP.

EP has many applications in equilibrium situations of science, engineering and

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economics. And traffic assignment problem is often modeled as an EP, which is equivalent to VI problems [9]. Sensitivity analysis of separable traffic equilibrium equilibria with application were presented by M. Josefsson and M. Patriksson [10]. The mixed traffic assignment problems (MTAP) is naturally modeled as an EP and it is usually solved by the traditional diagonal method [11]. It can also be solved directly by entropy function method.

The paper is organized as follows. In section 2, the modelling and entropy function method for EP are presented. And the entropy function method for VI problems and relation between EP and VI problems are clarified. In section 3, the application of EP modeling with the entropy function method for the MTAP is introduced, and we give an optimal model for the MTAP and effective algorithm for the model in the last part.

2 The Entropy Method for Equilibrium Programming

EP is a cross area of mathematical programming and game theory. And it has been widely used in social economics and system engineering. For example, the traffic flow on networks composed of two different traffic classes, individual users and systematic users, noting that they obey the laws of user equilibrium (UE) and system optimal (SO) respectively, we can model the mixed traffic equilibrium situation as an EP.

Though we can transform the EP into a VI problem using the equivalent result of the EP and VI problems, and we can solve the EP with the entropy function methods for the VI problems [9,11]. The EP can also be solved by entropy function methods directly.

Let's consider a general EP as follow. There are two independent decision makers 1 and 2 in the system of EP at least.

Considering functions $f(x,y), g(x,y): X\times Y \to R$, where $X \subseteq R^n$ and $Y \subseteq R^m$ are compact.

$$\max_{x\in X_y} f(x,y), \tag{2.1}$$

$$\max_{y\in Y_x} g(x,y), \tag{2.2}$$

where $f(x,y)$ and $g(x,y)$ are the objective functions of the decision makers 1 and 2 respectively, and x,y are the decision variables. X_y, Y_x are the feasible sets of decision 1 and 2 for any reasonable fixed x,y in the whole decision area [1].

Let $F(y) = \max_{x\in X_y} f(x,y)$ be the optimal function for $f(x,y)$, and $G(x) = \max_{y\in Y_x} g(x,y)$ be the optimal function for $g(x,y)$.

In order to solve the EP, we can regard the EP as a parametric optimal problems and then use the entropy function methods for parametric optimal problems to get the information coherence distribution function [1,12].

Considering prior distribution density function $\lambda_y^0(x)$, it represents the probability density of $f(x,y)$ getting its extreme value at point x for any y in the feasible set. It satisfies

$$\lambda_y^0(x) > 0, \int_{X_y} \lambda_y^0(x)dx = 1.$$

The problem is equivalent to solving the following optimal control problem.

$$\max_{\lambda_x(y)} \int_{X_y} \lambda_y(x) f(x,y)dx, \tag{2.3}$$

where $\lambda_y(x)$ satisfies $\lambda_y(x) \geq 0, \forall x \in R^n, \int_{X_y} \lambda_y(x)dx = 1$, for any x.

Taking the procedure as decreasing uncertainty and getting more acute information step by step from prior density distribution, so we let the corresponding Kullback entropy get its maximum.

$$K(\lambda_y^1(x), \lambda_y^0(x)) = -\int_{X_y} \lambda_y^1(x) \ln \frac{\lambda_y^1(x)}{\lambda_y^0(x)} dx.$$

Making the weights of two goals equally, we have

$$\max_{\lambda_y^1(x)} \left[\int_{X_y} \lambda_y^1(x) f(x,y)dx - \int_{X_y} \lambda_y^1(x) \ln \frac{\lambda_y^1(x)}{\lambda_y^0(x)} dx \right] \tag{2.4}$$

$$s.t. \quad \lambda_y^1(x) \geq 0, \int_{X_y} \lambda_y^1(x)dx = 1$$

and the Lagrange function for (2.4) given by the formula:

$$L(y, \lambda_y^1(x), \mu) = \int_{X_y} \lambda_y^1(x)dx - \int_{X_y} \lambda_y^1(x) \ln \frac{\lambda_y^1(x)}{\lambda_y^0(x)} dx + \mu[\int_{X_y} \lambda_y^1(x)dx - 1]. \tag{2.5}$$

From the variational principle and noting that $\int_{X_y} \lambda_y^1(x)dx = 1$, we obtain that

$$\lambda_y^1(x) = \frac{\lambda_y^0(x) \exp\ f(x,y)}{\int_{X_y} \lambda_y^0(x) \exp\ f(x,y)dx}. \tag{2.6}$$

Repeating the above steps, we have

$$\lambda_y^k(x) = \frac{\lambda_y^0(x) \exp[(k-1) f(x,y)]}{\int_{X_y} \lambda_y^0(x) \exp[(k-1) f(x,y)]dx}. \tag{2.7}$$

The Lagrange function is

$$L(y, \lambda_y^k(x), \mu) = \int_{X_y} \lambda_y^k(x) f(x,y)dx - \int_{X_y} \lambda_y^k \int_{X_y} \lambda_y^k(x) \ln \frac{\lambda_y^k(x)}{\lambda_y^{k-1}(x)} dx + \mu[\int_{X_y} \lambda_y^k(x)dx - 1]$$

$$= \frac{1}{k} \ln \int_{X_y} \lambda_y^0(x) \exp[(k-1) f(x,y)]dx. \tag{2.8}$$

The related entropy function is

$$F_k(y) = \frac{1}{k} \ln \int_{X_y} \lambda_y^0(x) \exp[kf(x,y)dx. \tag{2.9}$$

The expression is called maximum entropy function for $F(y)$.

The same as above, the information coherence distribution function at step k for the other decision variable as follows:

$$\lambda_x^k(y) = \frac{\lambda_x^0(y)\exp[kg(x,y)]}{\int_{Y_x}\lambda_x^0(y)\exp[kg(x,y)]dy}. \tag{2.10}$$

And its Lagrange functional is

$$L(x,\lambda_x^k(y),\mu) = \int_{Y_x}\lambda_x^k(y)g(x,y)dy - \int_{Y_x}\lambda_y^k\int_{Y_x}\lambda_x^k(y)\ln\frac{\lambda_y^k(y)}{\lambda_y^{k-1}(y)}dy + \mu[\int_{Y_x}\lambda_x^k(y)dy - 1]$$

$$= \frac{1}{k}\ln\int_{Y_x}\lambda_x^0(y)\exp[(k-1)g(x,y)]dy. \tag{2.11}$$

For convenience, we say that

$$\lambda_x^{k+1}(y) = \frac{\lambda_x^k(y)\exp g(x,y)}{\int_{Y_X}\lambda_x^k(y)\exp g(x,y)dy} \tag{2.12}$$

is the k 's distribution function for the solution of the problem (2.1), and yet sign it as $\lambda_y^k(x)$. And we call

$$G_k(x) = \frac{1}{k}\ln\int_{Y_x}\lambda_x^0(y)\exp[kg(x,y)]dy$$

the entropy function for $G(x)$.

From the above analysis we have the results of the entropy function method for EP.

Theorem *If* $f(x,y)$, $g(x,y)$ *are continue and the solution for the EP is unique.* $x^*(y) = (x_1^*(y),\cdots,x_m^*(y))$, $y^*(x) = (y_1^*(x),\cdots,y_m^*(x))$, *then*

$$x_i^*(y) = \lim_{k\to\infty}\int_{X_y}\lambda_y^k(x)dx, \qquad 1 \le i \le m.$$

$$y_i^*(x) = \lim_{k\to\infty}\int_{Y_x}\lambda_x^k(y)dy, \qquad 1 \le i \le m.$$

The solution for the EP can be reached from the simultaneous equations above.

3 The Application of EP in MTAP

The mixed traffic equilibrium problem is a typical one of EP. In the real traffic network, traffic flows usually consist of two or more kinds of vehicles, such as motor vehicle and non-motor vehicle running in a mixed way. Generally, the interaction between different kind vehicles is asymmetric, similar to the conclusion drawn by Smith. In this condition, we cannot establish equivalent optimization model in the traditional sense for this mixed traffic system equilibrium problem. Note that Bechmann equivalent transformation of the single kind system equilibrium model and the characteristics of mixed traffic flow, by the EP theory, we can easily set EP model for mixed traffic equilibrium problem.

3.1 Model

Here, we only discuss deterministic user equilibrium assignment problem for two kind vehicles (*e.g. motor vehicle and non-motor vehicle*).

Denote the flows of the two kind vehicles on link path a as x_a and $\hat{x}_a$ respectively, and the travel time (*or cost*) function as $t_a(x_a, \hat{x}_a)$ and $\hat{t}_a(\hat{x}_a, x_a)$ respectively. When the one mode vehicle's flow pattern can be given, the other mode vehicle's user equilibrium flow can be obtained by solving mathematical programming problem. Therefore, the MTAP can be formulated in terms of the EP as follows:

$$(\text{MTAP}) \begin{cases} \min\limits_{x_a} \sum\limits_a \int_0^{x_a} t_a(w, \hat{x}_a)dw \\ \\ \min\limits_{\hat{x}_a} \sum\limits_a \int_0^{\hat{x}_a} \hat{t}_a(\hat{x}_a, v)dv \end{cases} \tag{3.1}$$

$$s.t. \quad q_{rs} = \sum_k f_k^{rs}, \quad \forall (r,s) \in C, \tag{3.2}$$

$$\hat{q}_{rs} = \sum_k \hat{f}_k^{rs}, \quad \forall (r,s) \in C, \tag{3.3}$$

$$x_a = \sum_{r,s} \sum_k f_k^{rs} \delta_{a,k}^{rs}, \quad \forall a \in A, \tag{3.4}$$

$$\hat{x}_a = \sum_{r,s} \sum_k \hat{f}_k^{rs} \delta_{a,k}^{rs}, \quad \forall a \in A, \tag{3.5}$$

$$f_k^{rs}, \hat{f}_k^{rs} \geq 0, \quad \forall k \in K_{rs}, \tag{3.6}$$

where A is the set of directed links, C is the set of centroid points, K_{rs} is the set of paths from r to s, q^{rs} and $\hat{q}^{rs}$ are the travel demand of the two mode vehicles from r to s respectively, f_k^{rs} and $\hat{f}_k^{rs}$ are the flows of the two mode vehicles on path k from r to s respectively, and $\delta_{a,k}^{rs}$ is the incidence function, if $a \in K_{rs}$, $\delta_{a,k}^{rs}$ =1, otherwise $\delta_{a,k}^{rs}$ =0.

According to EP theory, because the constraints (3.2)~(3.6) are linear, therefore, if t_a and $\hat{t}_a$ are continuous, and x_a and $\hat{x}_a$ are strictly monotone function, then MTAP exists unique equilibrium solution $(x_a^*, \hat{x}_a^*), \forall a \in A$ [13].

If $(x_a^*, \hat{x}_a^*)$ is a solution to MTAP, by the solution of EP, x_a^* and $\hat{x}_a^*$ are UE solutions respectively. Therefore, $(x_a^*, \hat{x}_a^*)$ satisfies Wardropian UE principle. Which can be derived from one order condition of EP solution (see Ref. [14]).

Let Jacobi matrix be a positive definite matrix, to insure the traffic flow assignment is unique. The convergence of the solution has been proved in reference [14]. The procedure is as follows.

3.2 Algorithm

Step 1: Initialize the feasible solutions $X^{(1)} = (\cdots, x_a^{(1)}, \cdots)$ and

$\hat{X}^{(1)} = (\cdots, \hat{x}_a^{(1)}, \cdots)$ to MTAP, let $n = 1$.

Step 2: Given $X^{(n)}$ and $\hat{X}^{(n)}$, to solve the following subproblem

$$min\, z^{(n)}(X^{(n)}, \hat{X}^{(n)}) = \sum_a [\int_0^{x_a} t_a(w, x_a^{(n)})dw + \int_0^{\hat{x}_a} \hat{t}_a(\hat{x}_a^{(n)}, v)dv] \tag{3.7}$$

$$s.t. \quad q_{rs} = \sum_k f_k^{rs}, \quad \forall (r,s) \in C, \tag{3.8}$$

$$\hat{q}_{rs} = \sum_k \hat{f}_k^{rs}, \quad \forall (r,s) \in C, \tag{3.9}$$

$$x_a = \sum_{r,s}\sum_k f_k^{rs}\delta_{a,k}^{rs}, \quad \forall a \in A, \tag{3.10}$$

$$\hat{x}_a = \sum_{r,s}\sum_k \hat{f}_k^{rs}\delta_{a,k}^{rs}, \quad \forall a \in A, \tag{3.11}$$

$$f_k^{rs}, \hat{f}_k^{rs} \ge 0, \quad \forall k \in K_{rs}, \tag{3.12}$$

and denote the new iteration points as $X^{(n+1)}$ and $\hat{X}^{(n+1)}$.

Step3: Terminate check.

When $\left\| X^{(n)} - X^{(n+1)} \right\| \le \varepsilon$ and $\left\| \hat{X}^{(n)} - \hat{X}^{(n+1)} \right\| \le \varepsilon$, to terminate the procedure, otherwise, let $n = n + 1$, go to Step 2.

It can be proved that the above output results are equivalent with mixed traffic user equilibrium flows.

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