The Eighth International Symposium on Operations Research and Its Applications (ISORA'09)
Zhangjiajie, China, September 20–22, 2009

A Note on The Linear Arboricity of Planar Graphs without 4-Cycles*

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Abstract The linear arboricity $la(G)$ of a graph G is the minimum number of linear forests which partition the edges of G. In this paper, it is proved that if G is a planar graph with $\Delta(G) \geq 5$ and without 4-cycles, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$. Moreover, the bound that $\Delta(G) \geq 5$ is sharp.

Keywords planar graph, linear arboricity, cycle.

1 Introduction

In this paper, all graphs are finite, simple and undirected. Given a graph $G = (V,E)$. Let $N(v) = \{u \,|\, uv \in E(G)\}$ and $N_k(v) = \{u \,|\, u \in N(v) \text{ and } d(u) = k\}$, where $d(v) = |N(v)|$ is the *degree* of the vertex v. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum (vertex) degree and the minimum (vertex) degree, respectively. A k-, k^+- or k^--vertex is a vertex of degree k, at least k, or at most k, respectively. For a real number x, $\lceil x \rceil$ is the least integer not less than x and $\lfloor x \rfloor$ is the largest integer not larger than x.

A *linear forest* is a graph such that each of its components is a path. A map φ from $E(G)$ of a graph G to $\{1,2,\ldots,t\}$ is called a *t-linear coloring* if the induced subgraph of edges having the same color i is a linear forest for any $i(1 \leq i \leq t)$. The *linear arboricity* $la(G)$ of G defined by Harary [2] is the minimum number t for which G has a t-linear coloring.

Akiyama, Exoo, and Harary [1] conjectured that $la(G) = \lceil (\Delta(G)+1)/2 \rceil$ for any regular graph G. It is obvious that $la(G) \geq \lceil \Delta(G)/2 \rceil$ for any graph G and $la(G) \geq \lceil (\Delta(G)+1)/2 \rceil$ for every regular graph G. Hence the conjecture is equivalent to the following conjecture.

Conjecture A. For any graph G, $\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.

The linear arboricity has been determined for some class of graphs (see [4]). Conjecture A has already been proved to be true for all planar graphs, see [3] and [5]. Wu [3] proved that for a planar graph G with girth g and maximum degree Δ, $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$ if $\Delta \geq 13$, or $\Delta \geq 7$ and $g \geq 4$, or $\Delta \geq 5$ and $g \geq 5$, or $\Delta \geq 3$ and $g \geq 6$. In [4], It is proved that if G

*This work was partially supported by National Natural Science Foundation of China(10471078, 10631070, 60673059).

is a planar graph with $\Delta(G) \geq 7$ and without 4-cycles, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$. In this paper, we improve the result and obtain the following result.

Theorem 1.
If G is a planar graph with $\Delta(G) \geq 5$ and without 4-cycles, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$.

The theorem is a corollary of Theorem 2. Let G be the line graph of a 3-regular planar graph of girth 5, e. g. the line graph of dodecahedron. It is easy to prove that G is the 4-regular planar graph without 4-cycles and it follows that $la(G) = 3$. So the bound that $\Delta(G) \geq 5$ in Theorem 1 is sharp.

2 Main Result and its proof

Given a t-linear coloring φ and a vertex v of a graph G, we denote $C^i_\varphi(v)$ the set of colors appears i times at v, where $i = 0, 1, 2$. Then $|C^0_\varphi(v)| + |C^1_\varphi(v)| + |C^2_\varphi(v)| = t$ and $|C^1_\varphi(v)| + 2|C^2_\varphi(v)| = d(v)$, so that

$$2|C^0_\varphi(v)| + |C^1_\varphi(v)| = 2t - d(v). \tag{1}$$

If a color $i \in C^1_\varphi(v)$, then denote by (v,i) the edge colored with i and incident with v. For any two vertices u and v, let $C_\varphi(u,v) = C^2_\varphi(u) \cup C^2_\varphi(v) \cup (C^1_\varphi(u) \cap C^1_\varphi(v))$, that is, $C_\varphi(u,v)$ is the set of colors that appear at least two times at u and v. A *monochromatic path* is a path of whose edges receive the same color. For two different edges e_1 and e_2 of G, they are said to be in the *same color component*, denoted $e_1 \leftrightarrow e_2$, if there is a monochromatic path of G connecting them. Furthermore, if two ends of e_i are known, that is, $e_i = x_i y_i$ $(i = 1, 2)$, then $x_1y_1 \leftrightarrow x_2y_2$ denotes more accurately that there is a monochromatic path from x_1 to y_2 passing the edges x_1y_1 and x_2y_2 in G(that is, y_1 and x_2 are internal vertices in the path). Otherwise, we use $x_1y_1 \not\leftrightarrow x_2y_2$(or $e_1 \not\leftrightarrow e_2$) to denote that such monochromatic path passing them does not exist. Note that $x_1y_1 \leftrightarrow x_2y_2$ and $x_1y_1 \leftrightarrow y_2x_2$ are different.

Theorem 2.
Suppose that $t \geq 3$ is an integer and G is a planar graph with maximum degree $\Delta(G) \leq 2t$ and without 4-cycles. Then G has a t-linear coloring.

Proof. Let $G = (V, E)$ be a minimal counterexample to the theorem, and we assume that G has been embedded in the plane. A face of G is said to be *incident* with all edges and vertices on its boundary. The degree of a face f, denote by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k-, k^+- or k^--face is face of degree k, at least k or at most k, respectively. Two faces sharing an edge e are said to be *adjacent*. Let $L = \{1, 2, \cdots, t\}$ be the color set. First, we prove some claims for G.

Claim 1. *For any $uv \in E(G)$, $d_G(u) + d_G(v) \geq 2t + 2 \geq 8$.*

Proof of Claim 1. Suppose that G has an edge uv with $d_G(u) + d_G(v) \leq 2t + 1$. Then $G' = G - uv$ has a t-linear coloring φ by the minimality of G. Since $d_{G'}(u) + d_{G'}(v) = d(u) + d(v) - 2 \leq 2t - 1$, $|C_\varphi(u,v)| < t$. Now we color uv with a color from $L \backslash C_\varphi(u,v)$. Thus φ is extended to a t-linear coloring of G, a contradiction. □

By the claim, we have

(1) $\delta(G) \geq 2$, and
(2) any two 3^--vertices are not adjacent, and
(3) any 3-face is incident with three 4^+-vertices, or one 3^--vertex and two 5^+-vertices.

Claim 2. *Every vertex is adjacent to at most two* 2*-vertices. At the same time, if a vertex v is adjacent to two* 2*-vertices, then for any* 2*-vertex x incident with v, $N(x) = \{v, x'\}$, we have $x'v \notin E(G)$.*

Proof of Claim 2. Suppose, to be contrary, that G does contain a vertex v that it is adjacent to three 2-vertices x, y, z. let x', y', z' be the other neighbors of x, y, z, respectively. Since G is minimal, $G^* = G - vx$ has a t-linear coloring φ. Without loss of generality, assume $\varphi(xx') = 1$. If there is a color c such that $c \in C^0_\varphi(v)$, or $c \in C^1_\varphi(v) \backslash \{1\}$, or $c = 1 \in C^1_\varphi(v)$ but $xx' \not\leftrightarrow (v,1)$, then color directly vx with c. So $C^0_\varphi(v) = \emptyset$, $C^1_\varphi(v) = \{1\}$ and $xx' \leftrightarrow (v,1)$. This implies that $\varphi(vy) \neq 1$ or $\varphi(vz) \neq 1$. Assume that $\varphi(vy) \neq 1$. Thus we can recolor vy with 1 and color vx with $\varphi(vy)$ (Note that if $\varphi(yy') = 1$, then $yy' \not\leftrightarrow x'x$). So φ is extended to a t-linear coloring of G, a contradiction. Hence every vertex is adjacent to at most two 2-vertices.

Now suppose that there is a vertex v such that v is adjacent to two 2-vertices x, y and two neighbors of y are adjacent. Let $\{x'\} = N(x) \backslash v$, $\{y'\} = N(y) \backslash v$. Then $y'v \in E(G)$. Since G is minimal, $G^* = G - vx$ has a t-linear coloring φ. Without loss of generality, assume $\varphi(xx') = 1$. It follows from the above argument that we have $C^0_\varphi(v) = \emptyset$, $C^1_\varphi(v) = \{1\}$ and $xx' \leftrightarrow (v,1)$. If $\varphi(vy) = 1$, then $\varphi(yy') = 1$ (since $xx' \leftrightarrow (v,1)$) and it follows that we can recolor vy', vy with 1, yy' with $\varphi(vy')$, and color vx with $\varphi(vy')$. Otherwise, we can recolor vy with 1 and color vx with $\varphi(vy)$. Thus we obtain a t-linear coloring of G, a contradiction. We complete the proof of Claim 2. □

Claim 3. *For every* 3*-face $uvwu$,* $\max\{d(u), d(v), d(w)\} \geq 5$.

Proof of Claim 3. Suppose, to be contrary, that there is a 3-face $uvwu$ such that $\max\{d(u), d(v), d(w)\} \leq 4$. By Claim 1, we have $d(u) = d(v) = d(w) = 4$. Since G is minimal, $G' = G - uv$ has a t-linear coloring φ. If there is a color α such that $\alpha \notin C_\varphi(u,v)$, or $\alpha \in C^1_\varphi(u) \cap C^1_\varphi(v)$ but $(u,\alpha) \not\leftrightarrow (v,\alpha)$, then we can color uv with α to obtain a t-linear coloring of G, a contradiction. So $C_\varphi(u,v) = L$, and for any $\alpha \in C^1_\varphi(u) \cap C^1_\varphi(v)$, we have $(u,\alpha) \leftrightarrow (v,\alpha)$. Since $d_{G'}(u) = d_{G'}(v) = 3$, $L = \{1,2,3\}$ and $\max\{C^2_\varphi(u), C^2_\varphi(v)\} \leq 1$.

Suppose that $\varphi(uw) = \varphi(vw)$. Without loss of generality, assume that $\varphi(uw) = 1$. If $|C^2_\varphi(u)| = 0$, then we can recolor uw with a color from $\{2,3\} \backslash C^2_\varphi(w)$, and color uv with 1. Otherwise, assume that $C^2_\varphi(u) = \{2\}$. It follows that $C^2_\varphi(v) = \{3\}$. Since $d(w) = 4$, $|C^2_\varphi(w)| \leq 2$. Without loss of generality, assume that $3 \notin C^2_\varphi(w)$. Thus, we can recolor uw with 3, color uv with 1.

Suppose that $\varphi(uw) \neq \varphi(vw)$. Without loss of generality, assume that $\varphi(uw) = 1$ and $\varphi(vw) = 2$. If $1 \in C^2_\varphi(u)$, then $2 \in C^2_\varphi(v)$, and then we can recolor vw with 1, uw with 2 and color uv with 1. Otherwise, $C^2_\varphi(w) = \{1,2\}$ and we can recolor uw with 3, color uv with 1.

By the above steps, φ is extended to a t-linear coloring of G, a contradiction. Hence Claim 3 is true. □

By Euler's formula $|V(G)|-|E(G)|+|F(G)|=2$, we have

$$\sum_{v\in V}(d(v)-4)+\sum_{f\in F}(d(f)-4)=-4(|V(G)|-|E(G)|+|F(G)|)=-8<0.$$

We define *ch* to be the *initial charge* by $ch(x)=d(x)-4$ for each $x\in V(G)\bigcup F(G)$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x\in V(G)\cup F(G)$ according to the discharging rules below. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x\in V(G)\cup F(G)} ch'(x)=\sum_{x\in V(G)\cup F(G)} ch(x)=-8. \qquad (*)$$

If we can show that $ch'(x)\geq 0$ for each $x\in V(G)\cup F(G)$, then we obtain a contradiction to $(*)$, completing the proof. The discharging rules are defined as follows.

R1. Let f be a 3-face $uvwu$ with $d(u)\leq d(v)\leq d(w)$. If $d(u)=d(v)=4$, then f receives $\frac{1}{4}$ from each of u and v, receives $\frac{1}{2}$ from w. Otherwise, f receives $\frac{1}{2}$ from each of v and w.

R2. Let z be a 2-vertex. First, it receives $\frac{1}{2}$ from each of its neighbors. Then, if z is incident with a 3-face f, then it receives 1 from its incident 6^+-face. Otherwise, it receives $\frac{1}{2}$ from each of its incident faces.

R3. Let z be a 3-vertex. z receives $\frac{1}{2}$ from each of its incident 5^+-faces.

R4. Let z be a 4-vertex. Let f be a 5^+-face and z_1,z_2 be two neighbors of z incident with f. If $d(z_1)=d(z_2)=4$, then z receives $\frac{1}{5}$ from f. Otherwise, if $\min\{d(z_1),d(z_2)\}\geq 5$, then z receives $\frac{1}{3}$ from f. Otherwise, z receives $\frac{1}{4}$ from f.

Let f be a face of G. If $d(f)=3$. then $ch'(f)\geq ch(f)+\max\{2\times\frac{1}{4}+\frac{1}{2},2\times\frac{1}{2},3\times\frac{1}{3}\}=0$. If $d(f)=4$, then $ch'(f)=ch(f)=0$. Suppose that $d(f)=5$. If f is incident with at most two 4^--vertices, then $ch'(f)\geq ch(f)-2\times\frac{1}{2}=0$ by R1-R4. Otherwise, there are two adjacent 4^--vertices u,v incident with f, and it follows from Claim 1 that $d(u)=d(v)=4$. Thus, if f is incident with a 3^--vertex w, then two neighbors of w incident with f must be 5^+-vertices, and it follows that $ch'(f)\geq ch(f)-\frac{1}{2}-2\times\frac{1}{4}=0$ by R2-R4. Otherwise, all 4^--vertices incident with f are 4-vertices, and it follows from R4 that $ch'(f)\geq ch(f)-\max\{\frac{1}{3}+2\times\frac{1}{4},5\times\frac{1}{5}\}=0$. Suppose that $d(f)\geq 6$. Let a be the number of 2-vertices incident with f and a 3-face. Then $a\leq\lfloor(d(f)-2)/3\rfloor$ by Claim 2. Let b be the number of 3^--vertices which receive $\frac{1}{2}$ from f. Then $b\leq\lfloor(d(f)-2a)/2\rfloor$ by Claim 1. The number of 4-vertices incident with f is at most $d(f)-2(a+b)+1$. So $ch'(f)\geq ch(f)-a-b\times\frac{1}{2}-(d(f)-2(a+b)+1)\times\frac{1}{3}\geq 0$.

Let v be a vertex of G. If $d(v)=2$, then $ch'(v)=ch(v)+2\times\frac{1}{2}+\min\{1,2\times\frac{1}{2}\}=0$ by R2. If $d(v)=3$, then $ch'(v)=ch(v)+\min\{2\times\frac{1}{2},3\times\frac{1}{3}\}=0$ by R3. If $d(v)=4$, then $ch'(v)\geq ch(v)+\min\{\frac{1}{5}+\frac{1}{3},2\times\frac{1}{4}\}-2\times\frac{1}{4}=0$ by R3. If $d(v)=5$, then it is incident with at most two 3-faces and it follows that $ch'(v)\geq ch(v)-2\times\frac{1}{2}=0$ by R1. Suppose that $d(v)\geq 6$. If v is adjacent to at most one 2-vertex, then v is incident with at most $\lfloor\frac{d(v)}{2}\rfloor$ 3-faces. Otherwise, v is adjacent to two 2-vertices and incident with at most $\lfloor\frac{d(v)-2}{2}\rfloor$ 3-faces. So $ch'(v)\geq ch(v)-(1+\lfloor\frac{d(v)}{2}\rfloor)\times\frac{1}{2}\geq 0$. Hence we complete the proof of the theorem. □

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