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A New Constructive Proof to the Existence of an Integer Zero Point of a Mapping with the Direction Preserving Property*

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Abstract Let $f : Z^n \to R^n$ be a mapping satisfying the direction preserving property that $f_i(x) > 0$ implies $f_i(y) \geq 0$ for any integer points x and y with $\|x-y\|_\infty \leq 1$. We assume that there is an integer point x^0 with $c \leq x^0 \leq d$ satisfying that

$$\max_{1\leq i\leq n}(x_i - x_i^0)f_i(x) > 0$$

for any integer point x with $f(x) \neq 0$ on the boundary of $H = \{x \in R^n \mid c-e \leq x \leq d+e\}$, where c and d are two finite integer points with $c \leq d$ and $e = (1,1,\cdots,1)^\top \in R^n$. This assumption is implied by one of two different conditions for the existence of an integer zero point of the mapping in van der Laan et al. (2004). Under the assumption, there is an integer point $x^* \in H$ such that $f(x^*) = 0$. A constructive proof of the existence is derived from an application of the well-known $(n+1)$-ray algorithm for computing a fixed point. The existence result has applications in general equilibrium models with indivisible commodities.

Keywords Integer Zero Point, Direction Preserving, Simplicial Algorithm, Triangulation, Existence

1 Introduction

The problem we consider in this paper is the existence of an integer zero point of a mapping $f : Z^n \to R^n$. The interests in integer zero points or fixed points of a mapping have been inspired by the work in Iimura (2003) though the statement of the existence of a discrete fixed point in Iimura (2003) is incorrect and a corrected statement was given in Iimura et al. (2004) after an application of the integrally convex set defined in Favati and Tardella (1990). A brief introduction to the applications of discrete fixed points of a mapping in economics can be found in Iimura (2003) and references therein.

Let $f : Z^n \to R^n$ be a mapping. Following the definition in Iimura (2003), we say that $f(x)$ satisfies the direction preserving property if $f_i(x) > 0$ implies $f_i(y) \geq 0$ for any integer points x and y with $\|x-y\|_\infty \leq 1$. We assume throughout this paper that $f(x)$

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satisfies the direction preserving property. Recently, under two different conditions, based on the $2n$-ray algorithm in van der Laan and Talman (1981), a constructive proof of the existence of an integer zero point of a mapping with the direction preserving property has been obtained in van der Laan et al. (2004). Those two conditions can be stated as follows.

Condition 1.
There exist integer vectors m, x^0, and M with $m+e < x^0 < M-e$ such that, for any integer point x on the boundary of $C = \{y \in R^n \mid m \leq y \leq M\}$,

$$(x-x^0)^\top f(x) > 0.$$

Condition 2.
There exists an integer vector u with $u > e$ such that, for any two cell connected integer points x and y on the boundary of $U = \{z \in R^n \mid -u \leq z \leq u\}$,

$$f_k(x) f_k(-y) \leq 0,\ k = 1, 2, \cdots, n.$$

Given these two conditions, the following two theorems can be found in van der Laan et al. (2004).

Theorem 1.
If Condition 1 holds, there exists an integer point $x^* \in C$ such that $f(x^*) = 0$.

Theorem 2.
If Condition 2 holds, there exists an integer point $x^* \in U$ such that $f(x^*) = 0$.

We introduce in this paper a new condition for the existence of an integer zero point of the mapping, which is as follows.

Condition 3.
There is an integer point x^0 with $c \leq x^0 \leq d$ satisfying that

$$\max_{1 \leq i \leq n} (x_i - x_i^0) f_i(x) > 0$$

for any integer point x with $f(x) \neq 0$ on the boundary of $H = \{x \in R^n \mid c-e \leq x \leq d+e\}$, where c and d are two finite integer points with $c \leq d$.

Lemma 1.
Condition 1 implies Condition 3. However, Condition 3 implies neither Condition 1 nor Condition 2.

Proof. From Condition 1, we obtain that, for any integer point x on the boundary of C,

$$(x-x^0)^\top f(x) > 0,$$

which implies that $(x_i - x_i^0) f_i(x) > 0$ for at least one of $i = 1, 2, \cdots, n$. Thus, $\max_{1 \leq i \leq n} (x_i - x_i^0) f_i(x) > 0$ for any integer point x on the boundary of C. Let $H = C$, $c = m+e$, and $d = M-e$. Condition 3 follows. We remark that H in Condition 3 can be smaller than C in Condition 1. $\square$

Given Condition 3, we obtain the following theorem in this paper.

Theorem 3.
If Condition 3 holds, there exists an integer point $x^* \in H$ such that $f(x^*) = 0$.

In this paper, we present a new constructive proof of Theorem 3, which is derived from the well-known $(n+1)$-ray algorithm in van der Laan and Talman (1979). The $(n+1)$-ray algorithm is one of simplicial methods for computing a fixed point of a mapping. The simplical methods were originated by Scarf in Scarf (1967), and have been substantially developed after Scarf's work (e.g., Allgower and Georg, 2000; Dang, 1991, 1995; Dang and Maaren, 1998; Eaves, 1972; Eaves and Saigal, 1972; Forster, 1995; Kojima and Yamamoto, 1982; Kuhn, 1968; van der laan and Talman, 1979, 1981; Merrill, 1972; Scarf, 1973, 1981; Todd, 1976; Yamamoto, 1983; etc.). The basic idea of the constructive proof is as follows. It assigns to each integer point of H an integer label and subdivides H into integer simplices. Starting at x^0, the constructive proof follows a finite simplicial path that leads to an integer zero point of the mapping.

The rest of this paper is organized as follows. An integer labeling rule is introduced in Section 2. The constructive proof is given in Section 3.

2 Integer Labeling

Let $N = \{1,2,\cdots,n\}$ and $N_0 = \{1,2,\cdots,n+1\}$. Let u^i be the ith unit vector of R^n and $h^i = -u^i$ for $i = 1,2,\cdots,n$. Let $e = (1,1,\cdots,1)^\top \in R^n$ and $h^{n+1} = e$. For any subset K of N_0 with $K \neq N_0$, let

$$G(x^0,K) = \{x^0 + \sum_{k\in K} \lambda_k h^k \mid 0 \le \lambda_k,\ k \in K\}.$$

To obtain a constructive proof of Theorem 3, we need a triangulation of H that subdivides every integer unit cube contained in H into integer simplices, and $G(x^0,K)$ into integer simplices for any subset $K \subset N_0$. Here, an integer unit cube is a unit cube having only integer vertices and an integer simplex is a simplex having only integer vertices. There are several triangulations suitable for this purpose, which include the K_1-triangulation in Freudenthal (1942), the J_1-triangulation in Todd (1976), a modification of the D_1-triangulation in Dang (1991), etc. A specific choice of the triangulation plays however no dominant role at all in this paper though efficiency of simplicial methods depends critically on the underlying triangulation. For simplicity, we choose the K_1-triangulation as an underlying triangulation of the $(n+1)$-ray algorithm for our constrictive proof. For completeness of the following discussions, we introduce the K_1-triangulation here.

A simplex of the K_1-triangulation of R^n is the convex hull of $n+1$ integer vectors, $y^0, y^1, \ldots, y^n$, given by $y^0 = y$ and $y^k = y^{k-1} + u^{\pi(k)}$, $k = 1,2,\ldots,n$, where y is an integer point of R^n and $\pi = (\pi(1),\pi(2),\ldots,\pi(n))$ a permutation of elements of $N = \{1,2,\ldots,n\}$. Let K_1 be the set of all such simplices. Since a simplex of the K_1-triangulation is uniquely determined by y and π, we use $K_1(y,\pi)$ to denote it.

We say that two simplices of K_1 are adjacent if they have a common facet. We show how to generate all the adjacent simplices of a simplex of the K_1-triangulation of R^n in the following. For a given simplex $\sigma = K_1(y,\pi)$ with vertices $y^0, y^1, \ldots, y^n$, its adjacent simplex opposite to a vertex, say y^i, is given by $K_1(\bar{y},\bar{\pi})$, where $\bar{y}$ and $\bar{\pi}$ are generated in the following table.

Table 1: Pivot Rules of the K_1-Triangulation

i	$\bar{y}$	$\bar{\pi}$
0	$y+u^{\pi(1)}$	$(\pi(2),\ldots,\pi(n),\pi(1))$
$1 \leq i < n$	y	$(\pi(1),\ldots,\pi(i+1),\pi(i),\ldots,\pi(n))$
n	$y-u^{\pi(n)}$	$(\pi(n),\pi(1),\ldots,\pi(n-1))$

Let $\mathscr{K}_1$ be the set of faces of simplices of K_1. A q-dimensional simplex of $\mathscr{K}_1$ with vertices y^0, y^1, ..., y^q is denoted by $< y^0, y^1, \ldots, y^q >$. The restriction of $\mathscr{K}_1$ on $G(x^0,K)$ for any subset $K \subset N_0$ is given by

$$\mathscr{K}_1|G(x^0,K) = \{\sigma \in \mathscr{K}_1 \mid \sigma \subset G(x^0,K) \text{ and } \dim(\sigma) = |K|\},$$

where $|\cdot|$ denotes the cardinality of a set and $\dim(\cdot)$ the dimension of a set. Obviously, $\mathscr{K}_1|G(x^0,K)$ is a triangulation of $G(x^0,K)$.

For $\sigma \in \mathscr{K}_1$, let $\mathrm{grid}(\sigma) = \max\{\|x-y\|_\infty \mid x \in \sigma \text{ and } y \in \sigma\}$. We define $\mathrm{mesh}(K_1) = \max_{\sigma\in\mathscr{K}_1} \mathrm{grid}(\sigma)$. Clearly, $\mathrm{grid}(\sigma) = 1$ for any $\sigma \in \mathscr{K}_1$ and $\mathrm{mesh}(K_1) = 1$.

In our constructive proof, we need an integer labeling rule that assigns an integer label to each integer point of H. Such an integer labeling rule is given in the following definition.

Definition 1.
For $x \in Z^n$, we assign to x an integer label $l(x)$ given by $l(x) = 0$ if $f(x) = 0$, and

$$l(x) = \begin{cases} \min\{k \mid f_k(x) = \max_{j\in N} f_j(x)\} & \text{if } f_j(x) > 0 \text{ for some } j \in N, \\ n+1 & \text{if } f(x) \leq 0 \text{ and } f(x) \neq 0. \end{cases}$$

Definition 2.

- *A q-dimensional simplex $\sigma =< y^0, y^1, \ldots, y^q >$ of $\mathscr{K}_1$ is complete if $l(y^i) \neq l(y^j)$ for $0 \leq i < j \leq q$, and $l(y^k) \neq 0$, $k = 0,1,\ldots,q$.*
- *A q-dimensional simplex $\sigma =< y^0, y^1, \ldots, y^q >$ of $\mathscr{K}_1$ is 0-complete if $l(y^i) \neq l(y^j)$ for $0 \leq i < j \leq q$, and there is some k satisfying that $l(y^k) = 0$.*
- *A q-dimensional simplex $\sigma =< y^0, y^1, \ldots, y^q >$ of $\mathscr{K}_1$ is almost complete if labels of $q+1$ vertices of σ consist of q different nonzero integers.*

As a direct result of Definition 2, we have

Lemma 2.
Every almost complete simplex has exactly two complete facets.

Let ∂H denote the boundary of H. Then, according to the assumption, for any integer point $x \in \partial H$ with $f(x) \neq 0$,

$$\max_{1\leq i\leq n} (x_i - x_i^0) f_i(x) > 0.$$

Lemma 3.
For any nonempty subset $K \subset N_0$, there is no complete simplex in $G(x^0,K) \cap \partial H$ carrying only integer labels in K.

Proof. Suppose that there is a complete simplex in $G(x^0,K) \cap \partial H$ carrying only integer labels in K. Let $\sigma = < y^1, y^2, \cdots, y^{|K|} >$ be such a complete simplex.

Consider $n+1 \notin K$. Without loss of generality, we assume that $K = \{1,2,\cdots,|K|\}$ and $l(y^i) = i$, $i = 1,2,\cdots,|K|$. Since $f(x)$ satisfies the direction preserving property and $\mathrm{grid}(\sigma) = 1$, hence, for any $i \in K$, $f_j(y^i) \geq 0$, $j = 1,2,\cdots,|K|$. From the definition of $G(x^0,K)$ and $n+1 \notin K$, we obtain that, for any $x \in G(x^0,K)$, $x_i - x_i^0 \leq 0$, $i \in K$, and $x_i - x_i^0 = 0$, $i \notin K$. Thus, for any $i \in K$,

$$(y_j^i - x_j^0) f_j(y^i) \leq 0,\ j = 1,2,\cdots,n.$$

Therefore, for any $i \in K$,

$$\max_{1 \leq j \leq n} (y_j^i - x_j^0) f_j(y^i) \leq 0.$$

For any $i \in K$, since $y^i \in \partial H$ and $f(y^i) \neq 0$, hence,

$$\max_{1 \leq j \leq n} (y_j^i - x_j^0) f_j(y^i) > 0.$$

A contradiction occurs. The lemma follows.

Consider $n+1 \in K$. Without loss of generality, we assume that $K = \{1,2,\cdots,|K|-1,n+1\}$ and $l(y^i) = i$, $i = 1,2,\cdots,|K|-1$, and $l(y^{|K|}) = n+1$. Since $f(x)$ satisfies the direction preserving property and $\mathrm{grid}(\sigma) = 1$, hence, for any $i \in K$, $f_j(y^i) \geq 0$, $j = 1,2,\cdots,|K|-1$. From $l(y^{|K|}) = n+1$, we obtain that $f(y^{|K|}) \leq 0$. Thus, $f_j(y^{|K|}) = 0$, $j = 1,2,\cdots,|K|-1$. Since $y^{|K|} \in G(x^0,K)$, hence, $y_j^{|K|} - x_j^0 \geq 0$, $j = |K|, |K|+1, \cdots, n$. Therefore,

$$\max_{1 \leq j \leq n} (y_j^{|K|} - x_j^0) f_j(y^{|K|}) \leq 0.$$

Since $y^{|K|} \in \partial H$ and $f(y^{|K|}) \neq 0$, hence,

$$\max_{1 \leq j \leq n} (y^{|K|} - x_j^0) f_j(y^{|K|}) > 0.$$

A contradiction occurs. The lemma follows. □

As a result of the direction preserving property and $\mathrm{mesh}(K_1) = 1$, we have

Lemma 4.
There is no complete n-dimensional simplex.

Proof. Suppose that there is a complete n-dimensional simplex. Let $\sigma = < y^0, y^1, \cdots, y^n >$ be such a complete simplex. Without loss of generality, we assume $l(y^i) = i$, $i = 1,2,\cdots,n$, and $l(y^0) = n+1$. Since $f(x)$ satisfies the direction preserving property and $\mathrm{grid}(\sigma) = 1$, hence, $f(y^0) \geq 0$. From $l(y^0) = n+1$, we obtain that $f(y^0) \leq 0$ and $f(y^0) \neq 0$. A contradiction occurs. The lemma follows. □

3 A Constructive Proof of the Existence

In this section, based on the well-known (n+1)-ray algorithm in van der Laan and Talman (1979), the integer labeling rule given in Definition 1, and the results in Lemma 3 and Lemma 4, a constructive proof of Theorem 3 is obtained, which is as follows:

Initialization: Let $K = \emptyset$, $y^0 = x^0$, $\sigma_0 =< y^0 >$, $y^+ = y^0$, and $k = 0$. Go to **Step 1**.

Step 1: Compute $l(y^+)$. If $l(y^+) = 0$, the algorithm terminates and an integer zero point of $f(x)$ in H has been found. If $l(y^+) \in K$, let y^- be the vertex of σ_k other than y^+ and carrying integer label $l(y^+)$, and τ_{k+1} the facet of σ_k opposite to y^-, and go to **Step 2**. If $l(y^+) \notin K$, go to **Step 3**.

Step 2: If $\tau_{k+1} \subset G(\eta, K\backslash\{j\})$ for some $j \in K$, let $K = K\backslash\{j\}$ and go to **Step 4**. Otherwise, do as follows: Let σ_{k+1} be the unique simplex that is adjacent to σ_k and has τ_{k+1} as a facet, y^+ the vertex of σ_{k+1} opposite to τ_{k+1}, and $k = k+1$. Go to **Step 1**.

Step 3: Let $K = K \cup \{l(y^+)\}$ and $\tau_{k+1} = \sigma_k$. Let σ_{k+1} be the unique $|K|$-dimensional simplex in $G(\eta, K)$ having τ_{k+1} as a facet, and y^+ the vertex of σ_{k+1} opposite to τ_{k+1}. Let $k = k+1$ and go to **Step 1**.

Step 4: Let $\sigma_{k+1} = \tau_{k+1}$, y^- be the vertex of σ_{k+1} carrying integer label j, and τ_{k+2} the facet of σ_{k+1} opposite to y^-. Let $k = k+1$ and go to **Step 2**.

Theorem 4.
If Condition 3 holds, the algorithm will terminate within a finite number of iterations at an integer point $x^* \in H$ such that $f(x^*) = 0$.

Proof. Lemma 3 implies that all the simplices generated by the algorithm are contained in H. Applying Lemma 2 and following an standard argument in Todd (1976), one can derive that the algorithm will never cycle. Since H is bounded, hence, there is a finite number of simplices in H and the algorithm will terminate within a finite number of iterations. From Lemma 4, we know that there is no complete n-dimensional simplex in H. This result implies that the algorithm will terminate at an integer point $x^* \in H$ such that $f(x^*) = 0$. The theorem follows. □

References

[1] E.L. Allgower and K. Georg (1990). Numerical Continuation Methods, Springer-Verlag, 1990.

[2] C. Dang (1991). The D_1-triangulation of R^n for simplicial algorithms for computing solutions of nonlinear equations. Mathematics of Operations Research 16, 148-161.

[3] C. Dang (1995). Triangulations and Simplicial Methods, Lecture Notes in Mathematical Systems and Economics 421.

[4] C. Dang and H. van Maaren (1998). A simplicial approach to the determination of an integral point of a simplex, Mathematics of Operations Research 23, 403-415.

[5] B.C. Eaves (1972). Homotopies for the computation of fixed points. Mathematical Programming 3, 1-22.

[6] B.C. Eaves and R. Saigal (1972). Homotopies for the computation of fixed points on unbounded regions. Mathematical Programming 3, 225-237.

[7] P. Favati and F. Tardella (1990). Convexity in nonlinear integer programming, Ricerca Operativa 53, 3-44.

[8] W. Forster (1995). Homotopy methods, Handbook of Global Optimization, Eds. R. Horst and P.M. Pardalos, Kluwer Academic Publishers.

[9] H. Freudenthal (1942). Simplizialzerlegungen von Beschrankter Flachheit. Annals of Mathematics 43, 580-582.

[10] T. Iimura (2003). A discrete fixed point theorem and its applications, Journal of Mathematic Economics 39, 725-742.

[11] T. Iimura, K. Murota, and A. Tamura (2004). Discrete fixed point theorem reconsidered, METR 2004-09, University of Tokyo.

[12] M. Kojima and Y. Yamamoto (1982). Variable dimension algorithms: basic theory, interpretation, and extensions of some existing methods. Mathematical Programming 24, 177-215.

[13] H.W. Kuhn (1968). Simplicial approximation of fixed points. Proceedings of National Academy of Science 61, 1238-1242.

[14] G. van der Laan and A.J.J. Talman (1979). A restart algorithm for computing fixed points without an extra dimension. Mathematical Programming 17, 74-84.

[15] G. van der Laan and A.J.J. Talman (1981). A class of simplicial restart fixed point algorithms without an extra dimension. Mathematical Programming 20, 33-48.

[16] G. van der Laan, A.J.J. Talman, and Z. Yang (2004). Solving discrete zero point problems, TI2004-112/1, Tinbergen Institute.

[17] O.H. Merrill (1972). Applications and Extensions of an Algorithm that Computes Fixed Points of Certain Upper Semi-Continuous Point to Set Mappings. PhD Thesis, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, MI.

[18] H.E. Scarf (1967). The approximation of fixed points of a continuous mapping. SIAM Journal on Applied Mathematics 15, 1328-1343.

[19] H.E. Scarf (Collaboration with T. Hansen) (1973). The Computation of Economic Equilibria. Yale University Press, New Haven.

[20] H.E. Scarf (1981). Production sets with indivisibilities—part I: generalities. Econometrica 49 (1981) 1-32.

[21] M.J. Todd (1976). The Computation of Fixed Points and Applications. Lecture Notes in Economics and Mathematical Systems 124, Springer-Verlag, Berlin, 1976.

[22] Y. Yamamoto (1983). A new variable dimension algorithm for the fixed point problem. Mathematical Programming 25, 329-342.