The Eighth International Symposium on Operations Research and Its Applications (ISORA'09)
Zhangjiajie, China, September 20–22, 2009

Some Results on Edge Coloring Problems with Constraints in Graphs*

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Abstract In this paper some new results on the acyclic-edge coloring , f-edge coloring, g-edge cover coloring, (g, f)-coloring and equitable edge-coloring of graphs are introduced. In particular, some new results related to the above colorings obtained by us are given. Some new problems and conjectures are presented.

Keywords acyclic-edge coloring, f-edge coloring, g-edge cover coloring, (g, f)-coloring, equitable edge-coloring

1 Introduction

The edge coloring problem has interesting real life applications in optimization and network design, such as file transfers in computer networks [5] . For the file transfer problem in computer networks, each computer x has a limited number $f(x)$ of communication ports. For each pair of computers there are a number of files which are transferred between the pair of computers. In such a situation the problem is how to schedule the file transfers so as to minimize the total time for the overall transfer process. The file transfer problem in which each file has the same length can be formulated as an edge coloring [5].

Let G be a graph. We denote by $V(G)$ the set of vertices of a graph G and by $E(G)$ its set of edges. For a vertex $v \in V(G)$, $N(v)$ denotes the set of vertices adjacent to v, and $d(v) = |N(v)|$ denotes the degree of v. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum degree and the minimum degree of G. Throughout this paper, We only consider the graphs, which allow multiple edges but no loops. If G has no multiple edges, G is called a simple graph. Given two vertices $u, v \in V(G)$, let $E(uv)$ be the set of edges joining u and v in G. The multiplicity $\mu(uv)$ of edge uv is the size of $E(uv)$. Set $\mu(v) = max\{\mu(uv) : u \in V\}$, $\mu(G) = max\{\mu(v) : v \in V\}$, which are called the multiplicity of vertex v and the multiplicity of graph G, respectively. An edge-coloring of G is an assignment of colors to the edges of G. Associate positive integer $1, 2, \dots$ with colors, and we call C a k-edge-coloring of G if C: $E \rightarrow \{1, 2, \dots, k\}$. Let $i_C(v)$ denote the number of edges of G incident with vertex v that receive color i by the coloring C. For simplification, we write $i(v) = i_C(v)$ if there is no obscurity.

Let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(v) \le f(v)$ for each vertex v of $V(G)$. A (g, f)-coloring of graph G is a edge coloring of G such that

*This research was supported by NNSF (10871119)and RSDP (200804220001) of China

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each color appears at each vertex v at least $g(v)$ and at most $f(v)$ times. The minimum number of colors used by a (g,f)-coloring of G is denoted by $\chi'_{gf}(G)$ which is called the (g,f)-chromatic index of G. The maximum number of colors used by a (g,f)-coloring of G is denoted by $\overline{\chi'}_{gf}(G)$ which is called the upper (g,f)-chromatic index of G. The (g,f)-coloring is a generalization of edge-coloring and edge cover coloring, which arises in many applications, such as the network design, the file transfer problem on computer networks and so on [24, 25, 26]. The (g,f)-chromatic index was first posed in [25]. Liu gave the concept of the upper (g,f)-chromatic index. A proper edge-coloring is a (g,f)-coloring for the case $g(v)=0$ and $f(v)=1$ for each vertex $v\in V(G)$. The edge coloring problem was posed in 1880 in relation with the well-known four-color conjecture. In 1964, Vizing [29] proved the famous Vizing's theorem: $\Delta(G)\le\chi'(G)\le\Delta(G)+1$ for any simple graph G where $\chi'(G)$ is the chromatic index of G. An proper edge coloring is called acyclic if there is no 2-colored cycle in G. In other words, if the union of any two color classes induces a subgraph of G which is a forest. The acyclic edge chromatic index of G, denoted by $a'(G)$, is the least number of colors in an acyclic edge coloring of G. The acyclic edge coloring problem was posed by Alon et al. [1] and they proved that $a'(G)\le 64\Delta(G)$ by probabilistic arguments. An f-coloring of a graph G is a (g,f)-coloring for the case $g(v)=0$ for each vertex $v\in V(G)$. The minimum number of colors needed to an f-coloring of G is called the f-chromatic index of G and is denoted by $\chi'_f(G)$. The f-coloring of a graph G was considered in [7]. Hakimi and Kariv [7] studied f-colorings of graphs and obtained many interesting results, most of which are the generalizations of the classical results on proper edge-coloring. An edge cover coloring is a (g,f)-coloring for the case $g(v)=1$ and $f(v)\ge d(v)$ for each vertex $v\in V(G)$. The edge cover chromatic index of G, denoted by $\chi'_c(G)$, is the maximum number of colors in an edge cover coloring of G. In 1974, Gupta [6] firstly studied the edge cover coloring and got the famous Gupta's theorem: $\delta(G)-1\le\chi'_c(G)\le\delta(G)$ for any simple graph G. A g-edge cover coloring is a (g,f)-coloring for the case $f(v)\ge d(v)$ for each vertex $v\in V(G)$. The g-cover chromatic index of G, denoted by $\chi'_{gc}(G)$, is the maximum number of colors in a g-edge cover coloring of G. There are many interesting problems about g-edge cover coloring to consider. In this paper, some new results on acyclic-edge coloring, f-edge coloring, the g-edge cover coloring, (g,f)-coloring and equitable edge-coloring of graphs are introduced. In particular, some new results related to the above colorings obtained by us are given. Some new problems and conjectures are presented.

2 Acyclic Edge Coloring

We have known that a proper edge coloring of a graph G is called acyclic if there is no 2-colored cycle in G. The acyclic edge chromatic number of G, denoted by $a'(G)$, is the least number of colors in an acyclic edge coloring of G. Acyclic edge coloring was introduced by Alon et al. [1] and they proved that $a'(G)\le 64\Delta(G)$ by probabilistic arguments. Molloy and Reed [21] showed that $a'(G)\le 16\Delta(G)$ using the same proof. In 2001, Alon, Sudakov and Zaks [2] gave the following conjecture.

Conjecture 2.1. $a'(G)\le\Delta(G)+2$ for all graphs G.

It is trivial if $\Delta(G)\le 2$. Burnstein showed that $a'(G)\le 5$ if $\Delta(G)=3$ in [4]. This conjecture was also proved to be true for almost all $\Delta(G)$-regular graphs, and all $\Delta(G)$-regular graphs G, whose girth is at least $c\Delta(G)\log\Delta(G)$ for some constant c [2]. Recently, Hou et al. considered the acyclic edge coloring of planar graphs and got many interesting results.

Theorem 1. [13] *If G is a planar graph, then $a'(G) \leq \max\{2\Delta(G)-2, \Delta(G)+22\}$.*

Theorem 2. [13] *If G is a planar graph with girth $g(G) \geq 5$, then $a'(G) \leq \Delta(G)+2$.*

Theorem 3. [13] *If G is a planar graph with girth $g(G) \geq 7$, then $a'(G) \leq \Delta(G)+1$.*

Theorem 4. [13] *If G is a planar graph with girth $g(G) \geq 16$ and $\Delta(G) \geq 3$, then $a'(G) = \Delta(G)$.*

Theorem 5. [12] *Let G be a planar graph with girth $g(G)$. If (1) $g(G) \geq 6$, or (2) $\Delta(G) \geq 11$ and $g(G) \geq 5$, then $a'(G) \leq \Delta(G)+1$.*

Theorem 6. [12] *Let G be a planar graph with girth $g(G)$. If (1) $\Delta(G) \geq 4$ and $g(G) \geq 16$, or (2) $\Delta(G) \geq 5$ and $g(G) \geq 10$, or (3) $\Delta(G) \geq 6$ and $g(G) \geq 8$, or (4) $\Delta(G) \geq 12$ and $g(G) \geq 7$, then $a'(G) = \Delta(G)$.*

Theorem 7. [12] *If G is a planar graph without 4-cycles and 5-cycles, then $a'(G) \leq \Delta(G)+2$.*

Theorem 8. [12] *If G is a planar graph without 4-cycles and 6-cycles, then $a'(G) \leq \Delta(G)+2$*

Theorem 9. [13] *If G is a series-parallel graph, then $a'(G) \leq \Delta(G)+1$.*

Other results on acyclic edge colorings can be found in [2], [4] and [14]. We present the following problems.

Problem 2.1. Find the the sufficient conditions for a graph G to have $a'(G) = \chi'(G)$?

Problem 2.2. Find acyclic edge chromatic numbers of graphs embedded in a surface.

Problem 2.3. Find the relationship between $a'(G)$ and $\chi'(G)$ for a graph.

3 f-Coloring

Let G be a graph and f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$. An *f-coloring* of G is an edge coloring of G such that each vertex v has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f-color G is called the *f-chromatic index* of G, and denoted by $\chi'_f(G)$. If $f(v) = 1$ for all $v \in V(G)$, the f-coloring problem is reduced to the proper edge-coloring problem. In 1986 Hakimi and Kariv obtained the following result.

Theorem 10. [7] *Let G be a graph. Then*

$$\Delta_f(G) \leq \chi'_f(G) \leq \max_{v \in V(G)} \{\lceil \frac{d(v)+\mu(v)}{f(v)} \rceil\}.$$

Theorem 11. [7] *Let G be a simple graph. Then*

$$\Delta_f(G) \leq \chi'_f(G) \leq \max_{v \in V(G)} \{\lceil \frac{d(v)+1}{f(v)} \rceil\} \leq \Delta_f(G)+1.$$

We say that a graph G is of f*-class* 1 if $\chi'_f(G) = \Delta_f(G)$, and of f*-class* 2 otherwise. Let $V_0^*(G) = \{v \in V(G) : \frac{d(v)}{f(v)} = \Delta_f(G)\}$. The f*-core* of a graph G is the subgraph of G induced by the vertices of $V_0^*(G)$ and is denoted by G_{Δ_f}. We call a graph H *edge-orderable*, if the edges of H can be ordered $e_1, e_2, \ldots, e_{\varepsilon(H)}$ in such a way that, for every j, $1 \le j \le \varepsilon(H)$, edge e_j has an end vertex v_j such that, at every vertex $u \in N_H(v_j)$, there is an edge e_i with $i \ge j$. A graph G is $\Delta_f(G)$*-peelable*, if all the vertices of G can be iteratively peeled off using the following peeling operation: Peel off a vertex v, which has at most one remaining neighbor of f-ratio $\Delta_f(G)$, where the f-ratio of a vertex v is $d(v)/f(v)$.

X. Zhang and G. Liu [39, 41, 42] gave a series of sufficient conditions for a simple graph G to be of f-class 1 based on the f-core of G.

Theorem 12. [37] *Let G be a simple graph. If $V_0^*(G) = \emptyset$, then G is of f-class* 1.

Theorem 13. [38] *Let G be a simple graph. If G_{Δ_f} is a forest, then G is of f-class* 1.

Theorem 14. [39] *Let G be a simple graph. If G_{Δ_f} is edge-orderable, then G is of f-class* 1.

Theorem 15. [39] *Let G be a simple graph. If G is $\Delta_f(G)$-peelable, then G is of f-class* 1.

Note that Theorem 3.6 is strictly stronger than each of Theorem 3.3- 3.5. Given a partial edge coloring of graph G, we call a connected subgraph H of G an obstruction to a partial f-coloring, if $\varepsilon(H)$ is odd and $d_H(v) = 2f(v)$ for each $v \in V(H)$. A graph G is called an f -regular graph if $d(v) = \Delta_f(G)f(v)$ for all vertices $v \in V(G)$.

Theorem 16. [41] *Let G be a graph. If there is no obstruction in G, then $\chi'_f(G) = \Delta_f(G)$.*

Theorem 17. [41] *Let G be a graph with m edges. Then G is of f-class 2 if $m > \Delta_f(G)\lfloor \frac{f(V(G))}{2} \rfloor$.*

Theorem 18. [41] *Let G be a graph with $f(V(G))$ odd. If $\sum\limits_{v \in V(G)} (\Delta_f(G)f(v) - d(v)) < \Delta_f(G)$, then G is of f-class* 2.

Theorem 19. [38]*If G is an f-regular graph with $f(V(G))$ odd, then G is of f-class 2.*

Zhang and Liu [40] completely solved the classification problem on f-colorings for complete graphs and obtained the following results.

Theorem 20. [40] *Let G be a complete graph K_n. If k and n are odd integers, $f(v) = k$ and $k \mid d(v)$ for all $v \in V(G)$, then G is of f-class* 2. *Otherwise, G is of f-class* 1.

A simple graph G is called f-critical if G is of f-class 2 and $\chi'_f(G-e) < \chi'_f(G)$ for every edge $e \in E(G)$. Zhang and Liu studied the properties of f-critical graphs. Vizing's Adjacency Lemma [30] can be derived from Theorem 3.13 when $f(v) = 1$ for all $v \in V(G)$.

Theorem 21. [44] *If G is an f-critical graph and u, v are adjacent vertices of G, then $d(u) + d(v) \ge (f(u) + f(v) - 1)\Delta_f(G) + 2$.*

Theorem 22. [44] *Let G be an f-critical graph and u,w be adjacent vertices of G. Then w is adjacent to at least $f(w)(f(u)\Delta_f(G) - d(u) + 1)$ vertices of f-ratio $\Delta_f(G)$ different from u.*

In the following we define a new f-coloring with some constraints. A super f-coloring of G is a f-coloring with the property that parallel edges receive distinct colors. Let $\chi_f''(G)$ denote the minimum positive integer k for which a super f-coloring of G exists. $\chi_f''(G)$ is called the super f-chromatic index of G. Super f-coloring is a generalization of f-coloring. A super f-coloring is a proper edge-coloring when $f(v) = 1$ for all $v \in V(G)$. Now we present the following Conjecture.

Conjecture 3.1. Let G be a graph. If $1 \le f(v) \le \lfloor (d(v) - \mu(v))/\mu(G) \rfloor$ for each $v \in V(G)$, then

$$\chi_f''(G) \le \max_{v \in V(G)} \{\lceil \frac{d(v) + \mu(v)}{f(v)} \rceil\}$$

Conjecture 3.2. If $\chi_f''(G)$ exists and $d(v)(\mathrm{mod} f(v)) \ge \mu(v)$ for each $v \in V$, then

$$\chi_f''(G) = \chi_f'(G)$$

Conjecture 3.3. Let G be a graph. Then there exists a supper f-coloring if and only if $\chi_f'(G) \ge \mu(G)$. Furthermore, if there exists a supper f-coloring, then $\chi_f''(G) = \chi_f'(G)$.

4 g-Edge Cover Coloring

Let us recall that a g-edge cover coloring is a edge coloring of G such that each color appears at each vertex v at least $g(v)$ times.. The g-cover chromatic index of G, denoted by $\chi_{gc}'(G)$, is the maximum number of colors in a g-edge cover coloring of G. An edge cover coloring is a g-edge cover coloring where $g(v) = 1$ for each vertex $v \in V(G)$. The edge cover chromatic index of G, denoted by $\chi_c'(G)$, is the maximum number of colors in an edge cover coloring of G. In 1974, Gupta gave the following result in [6].

Theorem 23. (Gupta's Theorem [6]) *For any graph G,*

$$\min\{d(v) - \mu(v) : v \in V\} \le \chi_c'(G) \le \delta.$$

Song and Liu generalized the edge cover coloring to g-edge cover coloring in [27] and determined the g-edge cover chromatic index of some kinds of graphs and gave a lower bound of the g-edge cover chromatic index.

Set $\delta_g(G) = \min\{\lfloor d_G(v)/g(v) \rfloor : v \in V(G)\}$. Let $E(i)$ be the set of edges receive color i in an edge-coloring C of G.

Theorem 24. [27] *Let G be a bipartite graph. Then $\chi_{gc}'(G) = \delta_g(G)$. Furthermore if $\chi_{gc}'(G) = k \ge 2$, there exists an g-edge cover-coloring C of G for which $||E(i)| - |E(j)|| \le 1$ for i and $j \in \{1,2,\ldots,k\}$ and $|i_C(v) - j_C(v)| \le 1$ for all $v \in V$, i and $j \in \{1,2,\ldots,k\}$.*

Theorem 25. [27] *Let G be a graph. If $g(v)$ is positive and even for all $v \in V$, then*

$$\chi_{gc}'(G) = \delta_g(G).$$

Theorem 26. [27] *Let G be a graph. Let* $1 \le g(v) \le d(v)$ *for all* $v \in V$. *Then*

$$\chi'_{gc}(G) \ge \min_{v \in V}\{\lfloor (d(v) - \mu(v))/g(v) \rfloor\}.$$

Set $\delta'_g = \min_{v\in V}\{\lfloor (d(v)-1)/g(v)\rfloor\}$. Xu obtained the following result in [35].

Theorem 27. [35] *Let G be a graph. Let* $1 \le g(v) \le d(v)$ *for all* $v \in V$ *and* $1 \le \delta'_g \le 4$. *Then* $\chi'_{gc}(G) \ge \delta'_g$.

Other results on g-edge cover colorings can be found in [19]. We say that a graph G is of *gc-class* 1 if $\chi'_{gc}(G) = \delta'_g$, and of *gc-class* 2 otherwise. A simple graph G is called gc-critical if G is of gc-class 2 and $\chi'_{gc}(G+e) > \chi'_{gc}(G)$ for every edge e which is not in $E(G)$. Now we present the following Problems.

Problem 4.1. Find the sufficient conditions for a graph G to be of *gc-class* 1 (of *gc-class* 2).

Problem 4.2. Study the properties of graphs which are gc-critical.

5 Other Edge Coloring with Constraints

In this section we consider problems on the (g, f)-coloring and equitable edge-coloring of graphs. A (g, f)-coloring of graph G is a edge coloring of G such that each color appears at each vertex v at least $g(v)$ and at most $f(v)$ times. The minimum number of colors used by a (g, f)-coloring of G is denoted by $\chi'_{gf}(G)$ which is called the (g, f)-chromatic index of G. The maximum number of colors used by a (g, f)-coloring of G is denoted by $\overline{\chi'}_{gf}(G)$ which is called the upper (g, f)-chromatic index of G. A (g, f)-factor of G is a spanning subgraph H of G satisfying $g(v) \le d_H(v) \le f(v)$ for each $v \in V(G)$. If a graph G itself is a (g, f)-factor, then G is called a (g, f)-graph. Set $\Delta_f(G) = \max\{\lceil d_G(v)/f(v)\rceil : v \in V(G)\}$. A (g, f)-factorization of a graph G is a partition $\{F_1, F_2, ..., F_m\}$ of $E(G)$ such that F_i is a (g, f)-factor for $1 \le i \le m$. Clearly, a graph G has a (g, f)-coloring if and only if it has a (g, f)-factorization. We have the following basic results for (g, f)-coloring.

Theorem 28. [17] *Let G be a bipartite graph, then G has a* (g, f)*-coloring if and only if G is an* (mg, mf)*-graph for some positive integer m.*

Theorem 29. [32] *Let G be a bipartite graph, then G has a* (g, f)*-coloring if and only if* $\Delta_f(G) \le \delta_g(G)$. *And if G has a* (g, f)*-coloring, then*

$$\chi'_{gf}(G) = \Delta_f(G), \ \overline{\chi'}_{gf}(G) = \delta_g(G).$$

Theorem 30. [32] *Let G be a graph and let g and f be nonnegative integer-valued functions defined on* $V(G)$ *such that* $g(v) \le f(v)$ *for all* $v \in V(G)$. *If G has a* (g, f)*-coloring, then*

$$\chi'_f(G) \le \chi'_{gf}(G) \le \overline{\chi'}_{gf}(G) \le \chi'_{gc}(G).$$

Theorem 31. [32] *Let G be a simple graph. If* $\chi'_f(G) = \max_{v\in V}\{\lceil (d(v)+1)/f(v)\rceil\}$, $\chi'_{gc}(G) = \min_{v\in V}\{\lfloor (d(v)-1)/g(v)\rfloor\}$ *and* $\chi'_{gc}(G) \ge \chi'_f(G)$. *Then G has a* (g, f)*-coloring and* $\chi'_{gf}(G) = \chi'_f(G)$, $\overline{\chi'}_{gf}(G) = \chi'_{gc}(G)$.

Other results on the (g,f)-coloring can be found in [19]. Now we consider the equitable coloring of graphs. An edge-coloring C of G with k colors is equitable if $|i(v)-j(v)| \le 1$, where $1 \le i < j \le k$ for any $v \in V(G)$. Define $V_t(G) = \{v \in V(G) : t \mid d(v)\}$, where t is a positive integer. Call the subgraph of G induced by $V_t(G)$ the *t-core* of G. Hilton and de Werra studied the equitable edge-coloring problem of simple graphs in [11]. The following theorem is one of the main results in [11].

Theorem 32. [11] *Let G be a simple graph and let $k \ge 2$. If $k \not| d(v)$ for all $v \in V(G)$, then G has an equitable edge-coloring with k colors.*

If $k = \Delta(G)+1$, then Theorem 32 reduces to Vizing's theorem of simple graphs [29]. If $k = \delta(G)-1$, then Theorem 32 implies Gupta's theorem of simple graphs [6]. Xu and Liu discussed equitable edge-coloring of multigraphs and present the following theorem.

Theorem 33. [34] *Let G be a graph and $k \ge 2$. If $\mu(v) \le d(v)(\mathrm{mod} k) \le k-\mu(v)$ for all $v \in V(G)$, then G has an equitable edge-coloring with k colors.*

Let $M(v)$ denote the set of the colors each of which appears at most $f(v)-1$ times at vertex v. It is very likely that the above theorem can be strengthened as follows, which was made by Hilton [10] in 2005.

Conjecture 5.1. Let G be a simple graph and let $k \ge 2$. If the k-core of G is a forest, then G has an equitable edge-coloring with k colors.

Recently, this conjecture has been proved by X. Zhang and G. Liu by the following theorem.

Theorem 34. [43] *Let G be a simple graph and let $k \ge 2$. Let $f(v) = \lceil \frac{d(v)}{k} \rceil$ for each $v \in V(G)$. Let $C = \{c_1, c_2, \ldots, c_k\}$. If the edges of G can be f-colored with k colors of C in the order $e_1, e_2, \ldots, e_{\varepsilon(G)}$ in such a way that, for every j $(1 \le j \le \varepsilon(G))$, when f-coloring the jth edge $e_j = w_j v_j$, there are $M(v) \ne \emptyset$ for all $v \in N_G(w_j)$ or for all $v \in N_G(v_j)$, then G has an equitable edge-coloring with k colors.*

Finally, we present some problems for further research as follows.

Problem 6.1. Is $\chi'_{gc}(G) \ge \chi'_f(G)$ a necessary and sufficient condition for a graph G to have a (g,f)-coloring?

Problem 6.2. If G has a (g,f)-coloring, do $\chi'_{gf}(G) = \chi'_f(G)$ and $\overline{\chi'}_{gf}(G) = \chi'_{gc}(G)$ hold?

Problem 6.3. Find the relationship between the equitable edge-coloring and the (g,f)-coloring of a graph.

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