Community Identification of Complex Network

Xiang-Sun Zhang

http://zhangroup.aporc.org

Chinese Academy of Sciences

2008.10.31, OSB2008

Outline

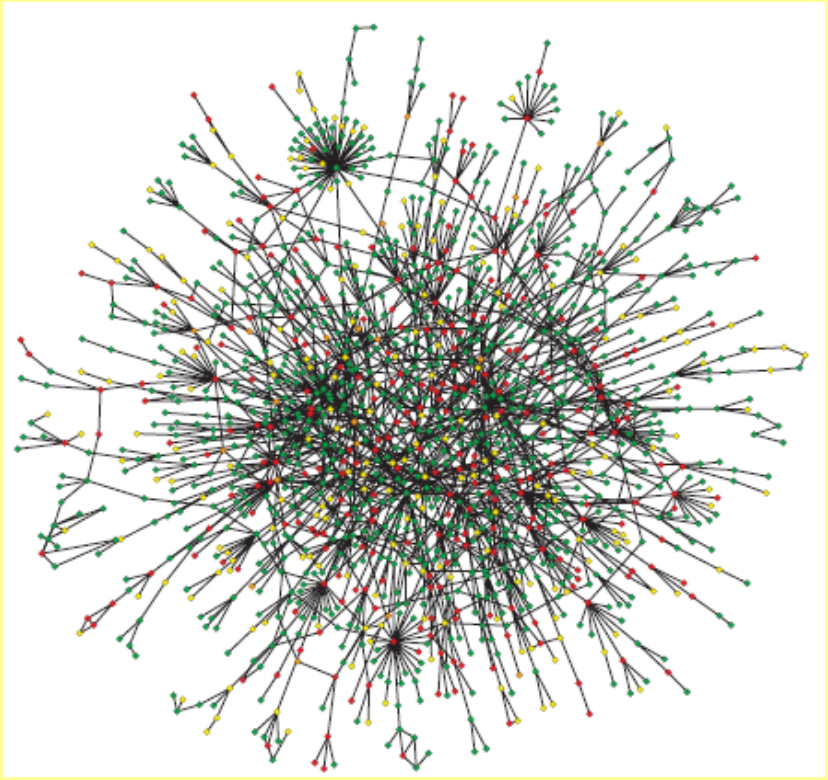
- Background
- Community identification definition
- Community identification methods
- Modularity measures for network community
- Conclusion

Complex networks

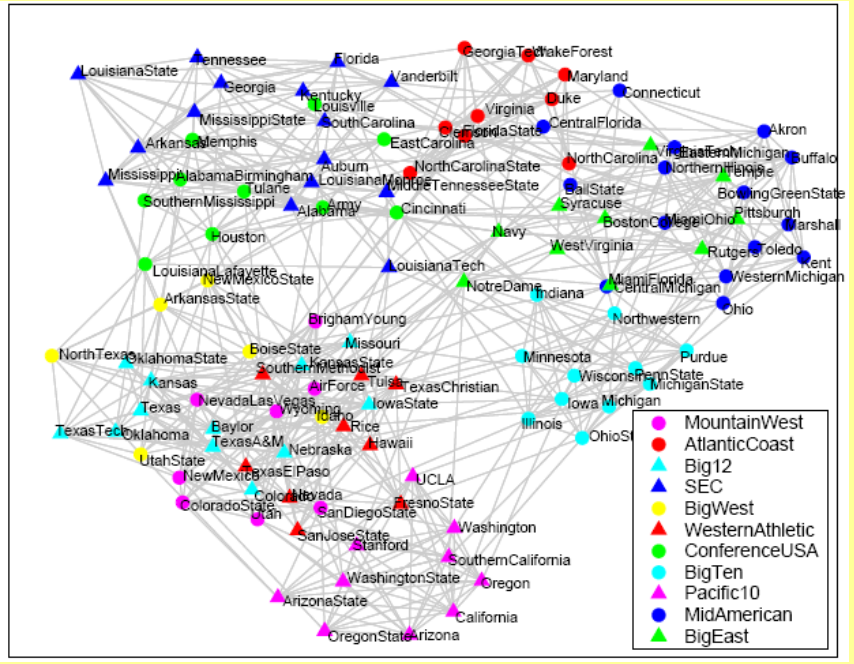
- Many systems can be expressed by a network, in which nodes represent the objects and edges denotes the relations between them.
- Social networks such as scientific collaboration network, food network, transport network, etc.
- Technological networks such as web network, software dependency network, IP address network, etc.
- Biological networks such as protein interaction networks, metabolic networks, gene regulatory networks, etc.
- …

Examples

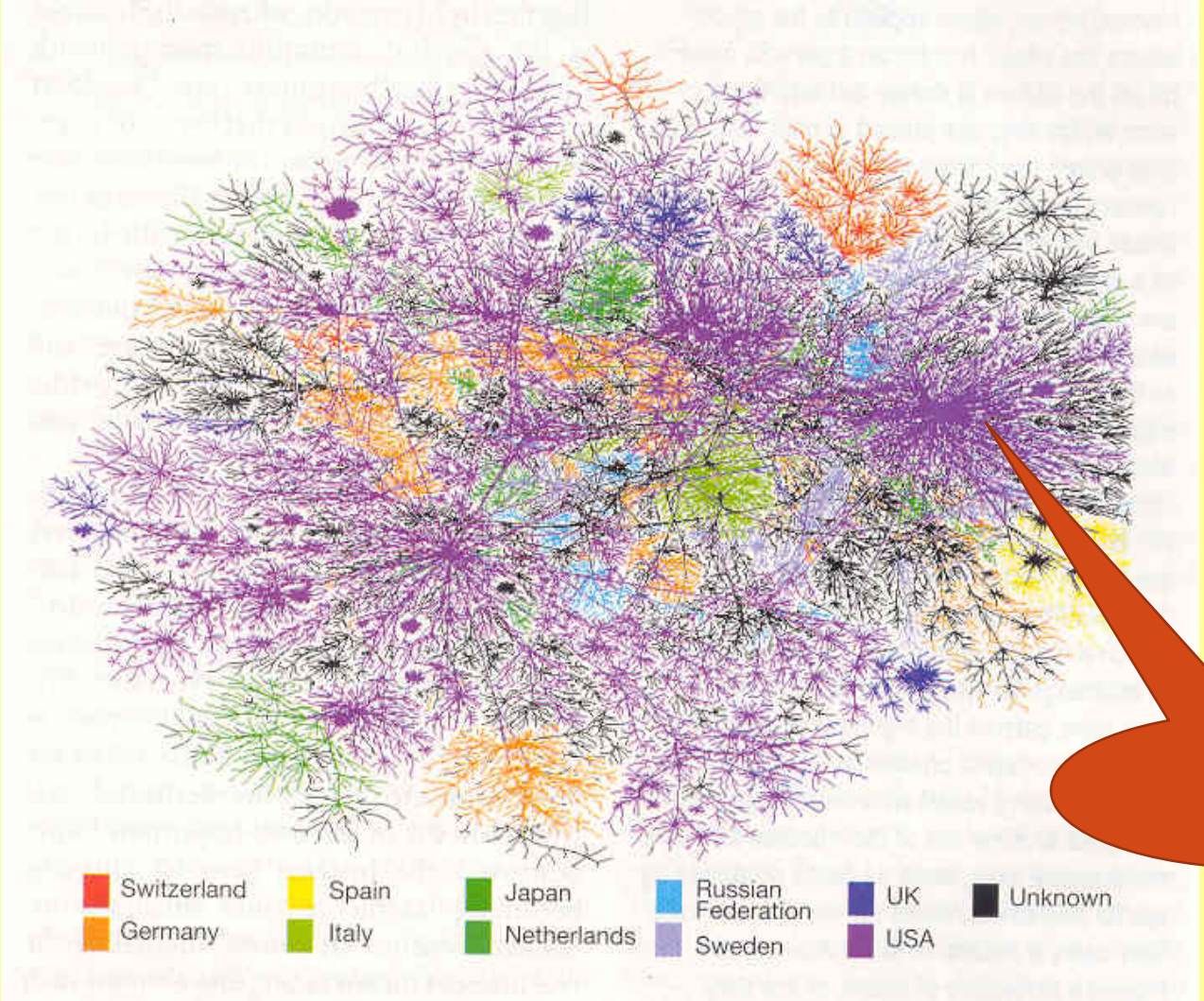
- Yeast protein interaction network (A.-L. Barabási, NATURE REVIEWS GENETICS, 2004)



- Football team network (S. White, P. Smyth, SIAM conference, 2004)



Computer IP address network

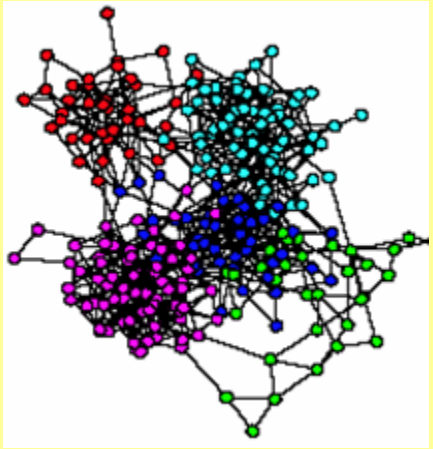


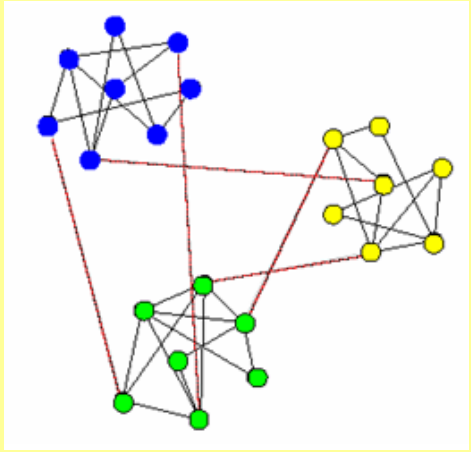
Common topological properties

- small-world property: most nodes are not neighbors of one another, but most nodes can be reached from every other by a small number of steps
- scale-free property: degree distribution follows a power law, at least asymptotically. That is, $P(k) \sim k^{-\gamma}$, where $P(k)$ is the fraction of nodes in the network having k connections to other and γ is a constant.
- …

Modularity/Community structure

- Modularity/Community structure : common to many complex networks. It means that complex networks consist of groups of nodes within which the connection is dense but between which the connection is relatively sparse.





Community structure

- Nodes in a same tight-knit community tend to have common properties or attributes

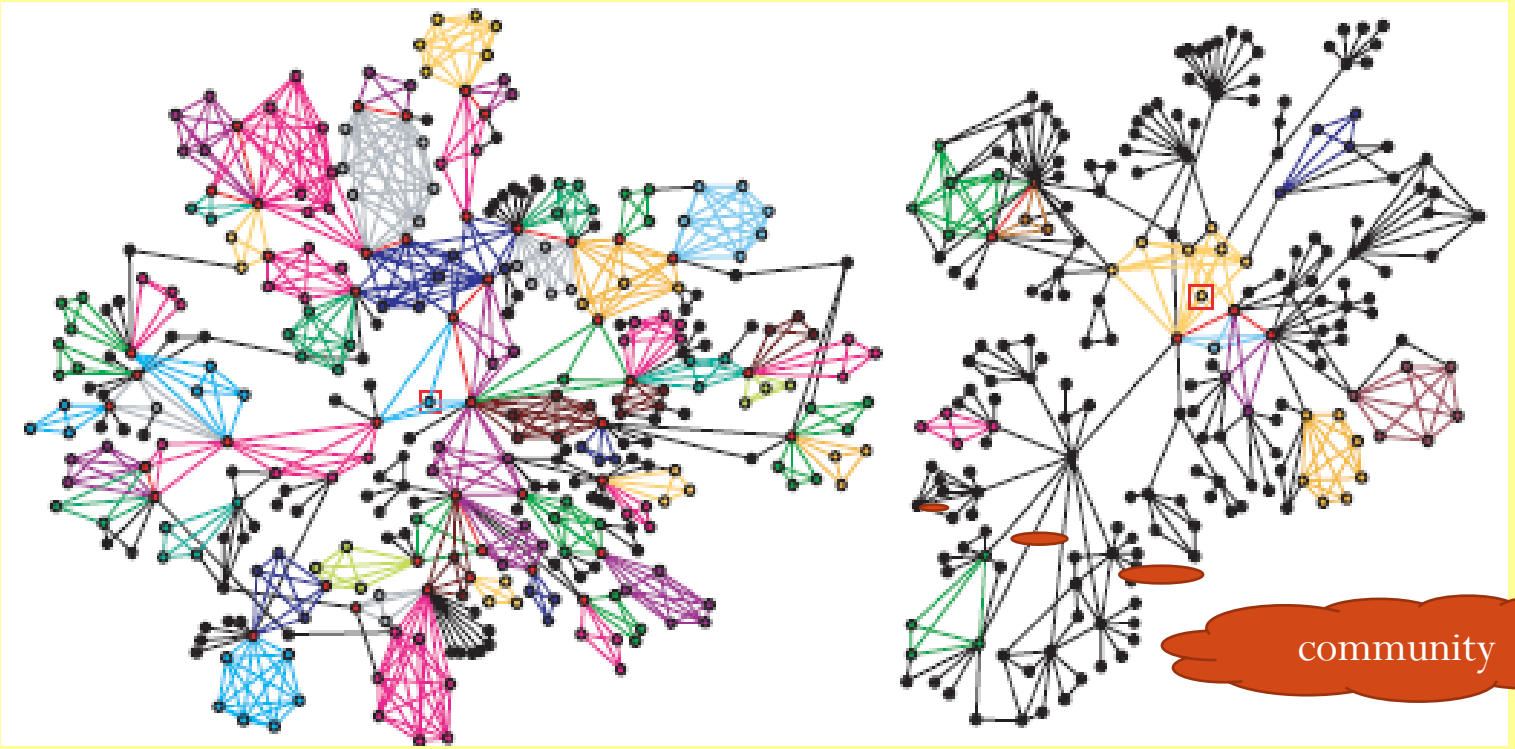
- Modules/communities in biological networks or other types of networks usually have functional meaning

Community identification

- Identifying community structure of a complex network is fundamental for uncovering the relationships between sub-structure and function of the network.

- In biological network research, it is widely believed that the modular structures are formed from the long evolutionary process and corresponds to biological functions.

Community of complex networks

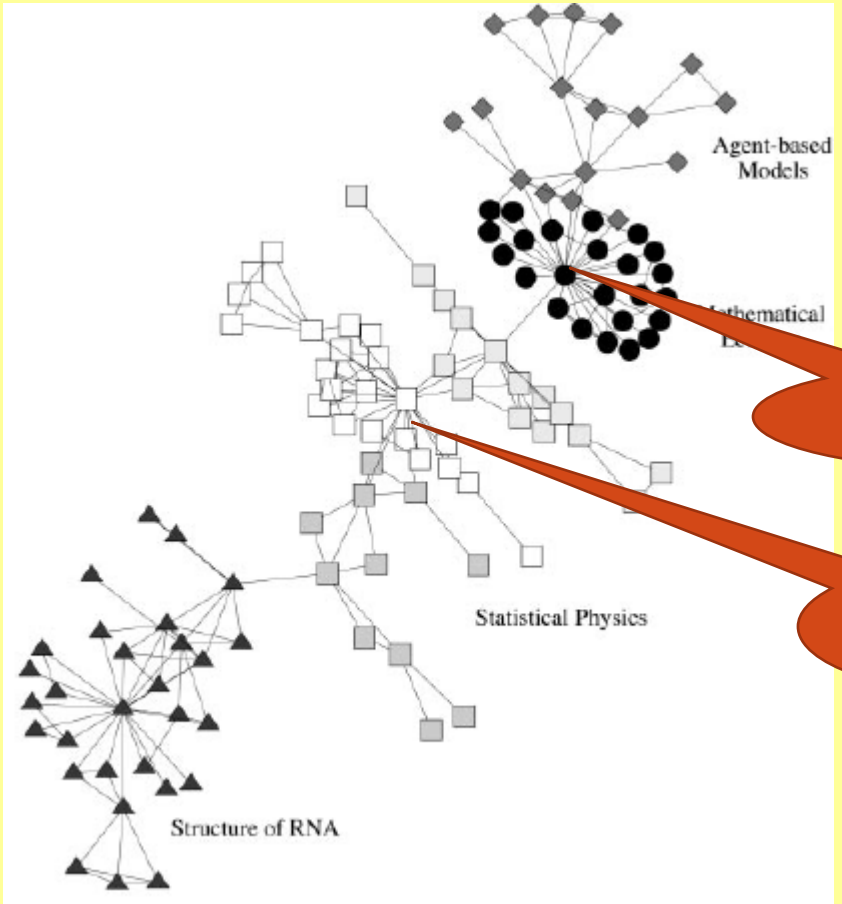


Paper-cooperation network

Phone network

Significance of community structure

- Common functions of many complex networks
- Global network structure and function decomposition



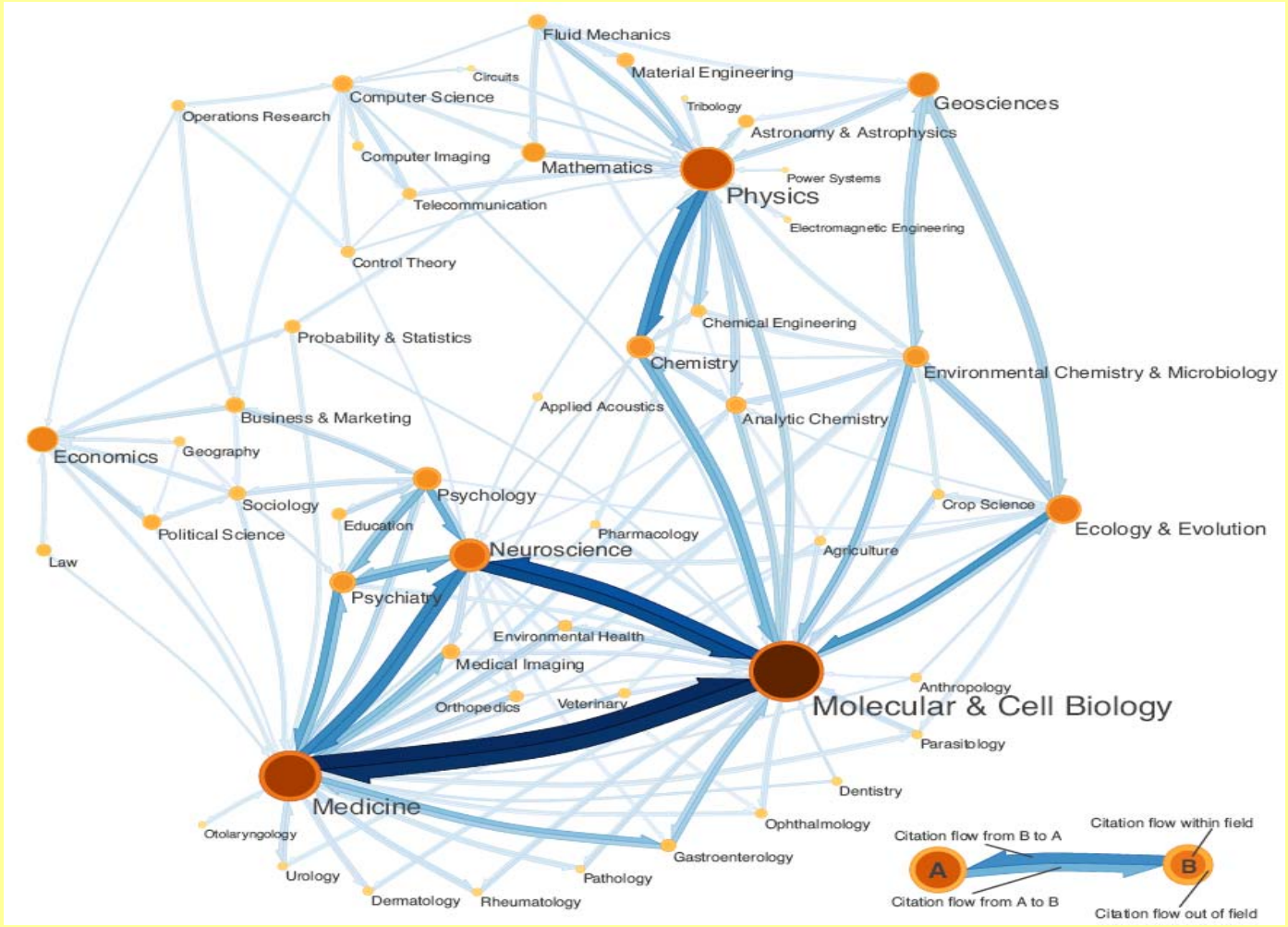
The scientific collaboration network in The Santa Fe Institute: the module denotes the groups of scientists in similar research field.

Mathematical ecology

Statistical physics

Martin Rosvall, Carl T. Bergstrom, PNAS, vol. 105, no.4. 1118-1123, 2007

A network of science based on citation patterns: **6,128 journals connected by 6,434,916 citations.**



The network is partitioned into 88 modules and 3,024 directed and weighted links, which represent a trace of the scientific activity.

Community identification definition

- Given a network/graph $N = (V, E)$, partition N into several subnetworks which satisfy community conditions

- In complex network research, a popular qualitative community definition is

  The nodes in a community are densely linked but nodes in different communities are sparsely linked

Filippo Radicchi et. al. *Proc. Natl. Acad. Sci. USA (PNAS)*, Vol.101, No.9, *2658-2663,* 2004

Community detection methods

- Some methods are based on topological properties of nodes or edges such as betweenness-based methods (Girvan, Newman, PNAS, 2002)

- Some of them are clustering-based, e.g. various spectral clustering algorithms (S. White, P. Smyth, SIAM conference, 2004)

Community detection methods

- In Newman and Girvan, ***PRE***, 2004, a modularity function Q was proposed as following

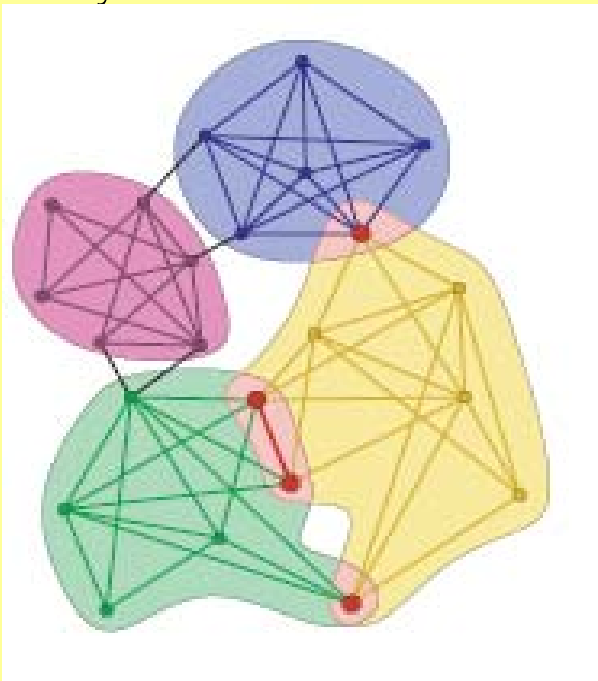
$$Q(P_k) = \sum_{c=1}^{k} \left[\frac{L(V_c, V_c)}{L(V, V)} - \left(\frac{L(V_c, V)}{L(V, V)} \right)^2 \right]$$

to measure the community structure of a network.

- A class of methods maximizing modularity Q appear. Heuristic algorithms such as Simulated Annealing, Genetic Algorithms, Local Search, etc. [Newman, PNAS, 2006; Guimera, Nature, 2005].

Overlapping/fuzzy communities

- In Palla et al., ***Nature***, 2005, a clique- percolation method was proposed for community detection



- In Reichardt, Bornholdt, ***PRL***, 2004, a Potts model was used for detecting fuzzy structure

Our work (I will not focus on)

- **Shihua Zhang, Rui-Sheng Wang, and Xiang-Sun Zhang. Identification of Overlapping Community Structure in Complex Networks Using Fuzzy c-means Clustering. *Physica A*, 2007, 374, 483–490.**

- **Shihua Zhang, Rui-Sheng Wang and Xiang-Sun Zhang. Uncovering fuzzy community structure in complex networks. *Physical Review E*, 76, 046103, 2007**

- **Rui-Sheng Wang, Shihua Zhang, Yong Wang, Xiang-Sun Zhang, Luonan Chen. Clustering complex networks and biological networks by Nonnegative Matrix Factorization with various similarity measures. *Neurocomputing*, DOI: 10.1016/j.neucom.2007.12.043**

- . . .

Mathematical community definition

- Mathematically, let

$$d_i = d_i^{in} + d_i^{out}$$

then the condition for a subnetwork $N_k = (V_k, E_k)$ being a community is

$$\sum_{i \in V_k} d_i^{in} - \sum_{i \in V_k} d_i^{out} > 0$$

i.e.

$$2|E_k| - |\overline{E_k}| > 0$$

where $\overline{E_k}$ is all edges linking V_k and $V \backslash V_k$

Filippo Radicchi et. al. *Proc. Natl. Acad. Sci. USA (PNAS)*, Vol.101, No.9, *2658-2663*, 2004

- **A popular method to partition a network into community structure is to optimize a quantity called modularity, or some alternatives, which is a measure for a given partition.**

- **Modularity definition and modularity optimization are still in the state-in-art process.**

Modularity function Q

- **Newman and Girvan (*Physical Review E*, 2004) gives a quantitative measure Q**

$$Q(N_1, \cdots, N_k) = \sum_{i=1}^{k} \left[\frac{|E_i|}{|E|} - \left(\frac{d_i}{2|E|} \right)^2 \right]$$

where $N_1, \ldots, N_k$ is a partition of N. We can prove

$$2|E_i| - |\overline{E_i}| > 0 \Rightarrow Q(N_1, \ldots, N_k) > 0$$

- **But it is not necessary that**

$$Q(N_1,\ldots,N_k) > 0 \Rightarrow 2|E_i| - |\overline{E_i}| > 0$$

- **It suggests that partition N into N_1, …, N_k such that $Q(N_1$, …, $N_k)$ is as large as possible to make sure that**

$$2|E_i| - |\overline{E_i}| > 0$$

which leads to an optimization process below

- **Step 1:** Fix k $(k = 1, \ldots, n)$, $N_1 \cup \ldots \cup N_k = N$, compute

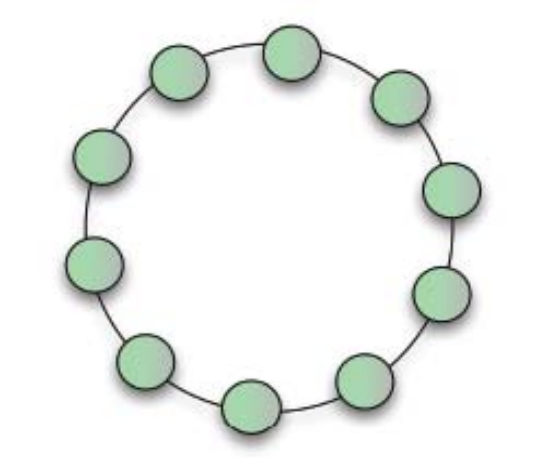
$$\max_{N_1,\ldots N_k} Q(N_1,\ldots,N_k)$$

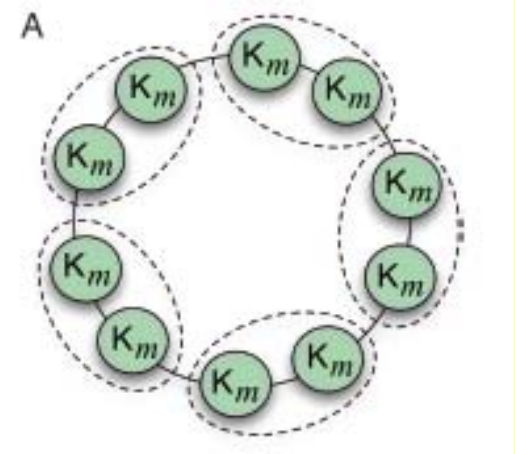
- **Step 2:** Compute

$$\max_{k\in\{1,\ldots n\}} \max_{N_1,\ldots N_k} Q(N_1,\ldots,N_k)$$

This is an enumeration algorithm, then heuristic algorithms including simulation annealing, genetic algorithm are generally used (Newman, *PNAS*, 2006; Guimera, Nature, 2005).

Modularity Q fails to identify correct community structure in some cases



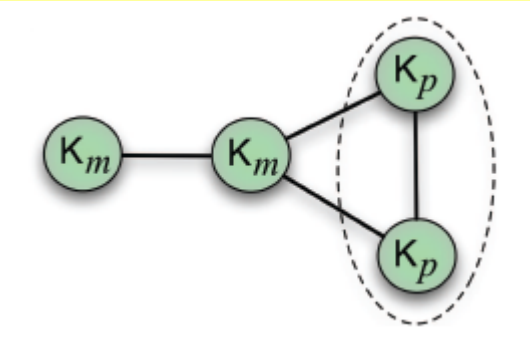


Left: a graph consists of a ring of cliques connected by single links, each clique is a qualified community.

Right: when the number of cliques is larger than about , the modularity optimization gives a partition where two cliques are combined into one community! This phenomena is called resolution limit.

Fortunato & Barthelemy, *Proc. Natl. Acad. Sci.* (2007)

Modularity Q fails to identify correct community structure in some cases



a graph consists of four cliques with different size, each clique is a qualified community.

when the clique size are quite heterogeneous, i.e. p<< m, the modularity optimization gives a partition where two small cliques are combined into one community!

We suggested a new quantitative measure

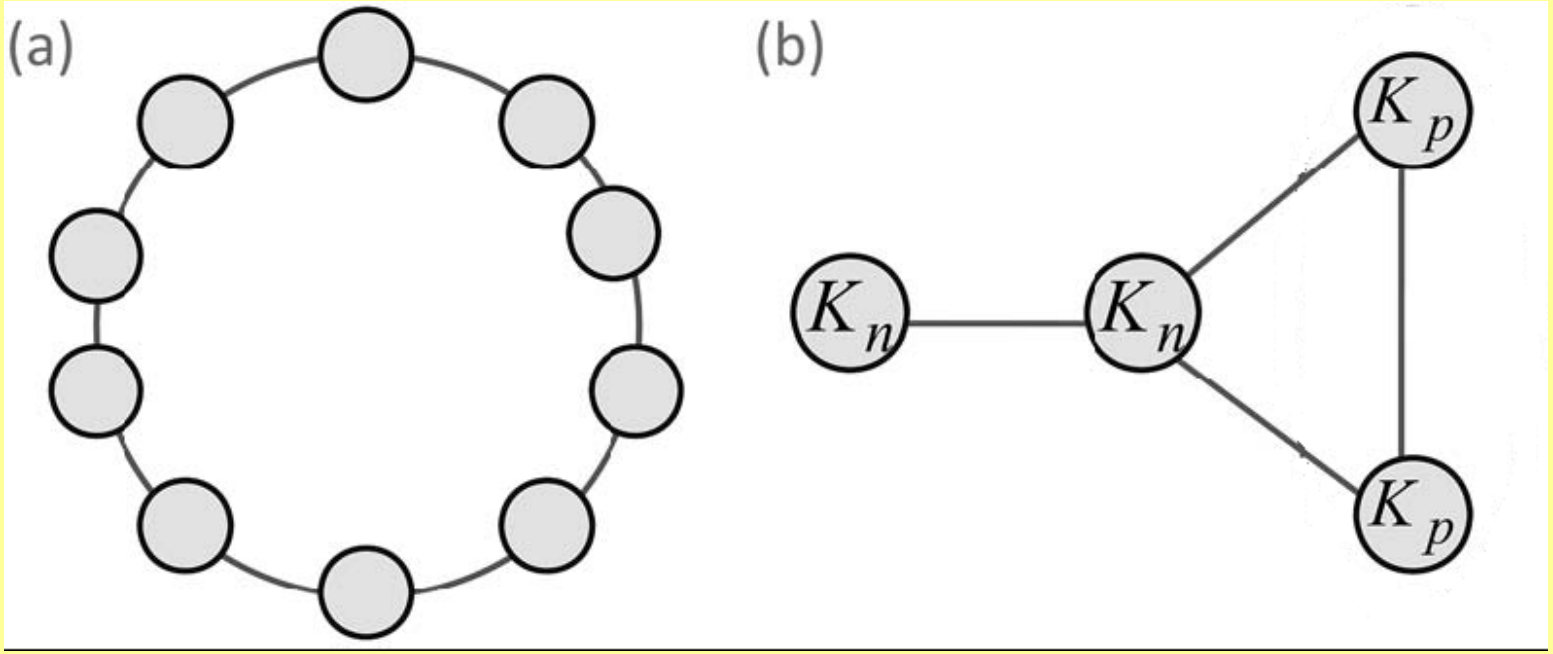
- **Modularity Density D:**

$$D(N_1,\cdots,N_k)=\sum_{i=1}^{k}\left[\frac{2|E_i|}{|V_i|}-\frac{|\bar{E}_i|}{|V_i|}\right]$$

which obviously has property:

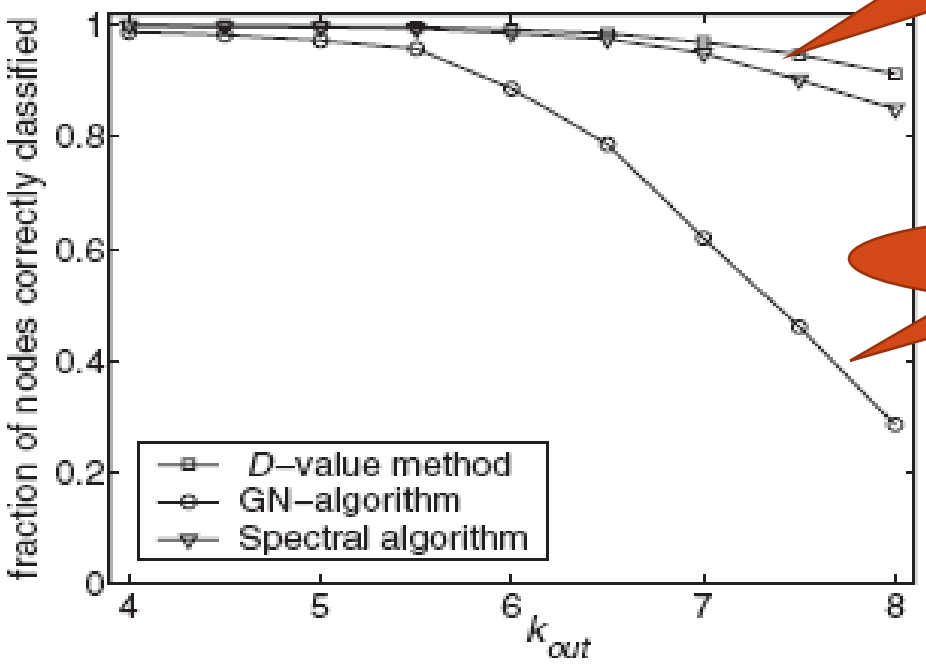
$$2|E_i|-|\overline{E_i}|>0 \Rightarrow D(N_1,\ldots,N_k)>0$$

Zhenping Li, Shihua Zhang, Rui-Sheng Wang, Xiang-Sun Zhang, Luonan Chen, Quantitative function for community detection. *Physical Review E*, 77, 036109, 2008



Modularity density *D* overcomes "resolution limit" problem in the cases of the ring of *L* cliques and the network with heterogeneous clique size

Experiment Result



D

Q

FIG. 2. Test of various methods on computer-generated networks with known community structures. It is a plot of the fraction of nodes correctly classified with respect to k_{out}. Each point is an average over 100 realizations of the networks.

Problem remained

- **Fortunato & Barthelemy, *PNAS* (2007), analyzed the "resolution limit" numerically based on some special network structures.**

- **Zhenping Li etc, *Physical Review E* (2008), suggested a new measure *D* and compare the modularity density *D* and modularity *Q* based on special network structures and numerical examples.**

- **A theoretical/mathematical framework to evaluate the different measures and display community structure properties is needed.**

A closed optimization model based on the modularity Q

- **Given a network $N = (V, E)$, $V = (v_1, \ldots, v_n)$, let (e_{ij}) be the adjacency matrix. Suppose that N is partitioned into k parts $N_1, \ldots, N_k$. Use binary integer variable x_{ij}:**

$$x_{ij} = \begin{cases} 1 & \text{if node i is in community j} \\ 0 & \text{otherwise} \end{cases}$$

The community definition then can be expressed as

$$\sum_{s,t \in V} e_{st} x_{sj} x_{tj} \geq \sum_{s,t \in V} e_{st} x_{sj} (1 - x_{tj})$$

For j=1,2,…, k

Optimization model based on Q

- A nonlinear integer programming based on Q

$$\begin{aligned} \max \quad & \sum_{j=1}^{k} \left[\frac{\sum_{s,t \in V} e_{st} x_{sj} x_{tj}}{\sum_{(s,t) \in E} e_{st}} - \left(\frac{\sum_{s,t \in V} e_{st} x_{sj}}{\sum_{(s,t) \in E} e_{st}} \right)^2 \right] \\ s.t. \quad & \sum_{j=1}^{k} x_{ij} = 1, \quad i = 1, \cdots, n \\ & x_{ij} \in \{0, \ 1\}, \ i = 1, \cdots, n, \ j = 1, \cdots, k \end{aligned}$$

Xiang-Sun Zhang and Rui-Sheng Wang, Optimization analysis of modularity measures for network community detection, *OSB 2008*.

Optimization model based on *D*

- **A nonlinear integer programming based on *D***

$$\max \quad \sum_{j=1}^{k} \left[\frac{\sum_{s,t \in V} e_{st} x_{sj} x_{tj}}{\sum_{t \in V} x_{tj}} - \frac{\sum_{s,t \in V} e_{st} x_{sj} (1 - x_{tj})}{\sum_{t \in V} x_{tj}} \right]$$

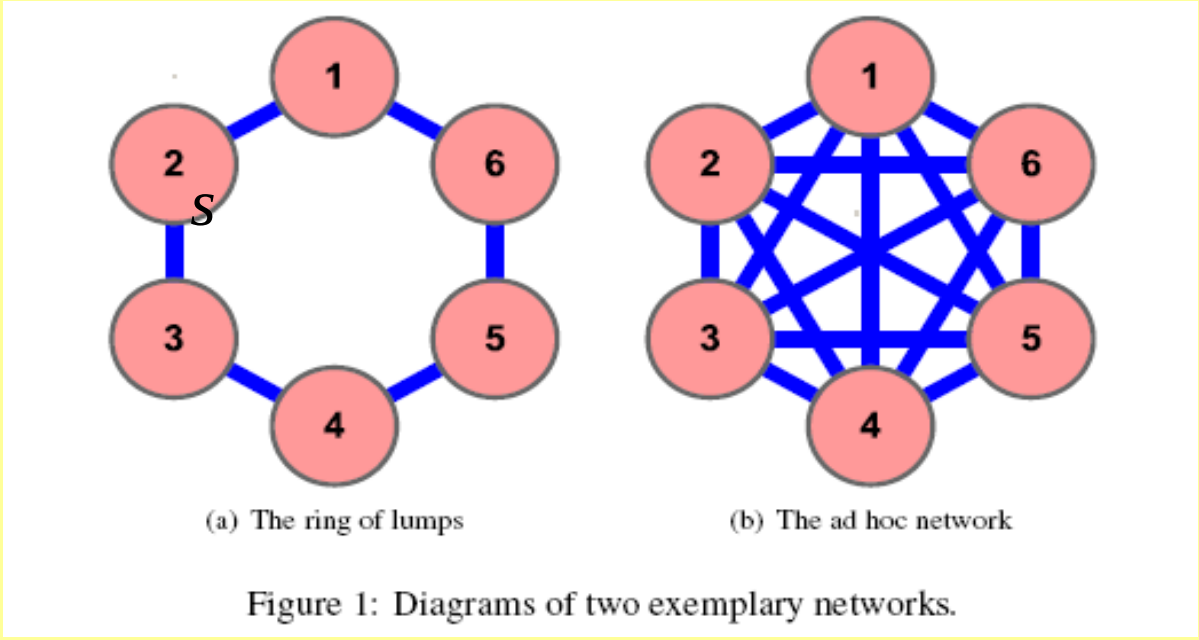
$$s.t. \quad \sum_{j=1}^{k} x_{ij} = 1, \ i = 1, \cdots, n,$$

$$x_{ij} \in \{0, \ 1\}, \ i = 1, \cdots, n, \ j = 1, \cdots, k$$

Xiang-Sun Zhang and Rui-Sheng Wang, Optimization analysis of modularity measures for network community detection, *OSB 2008*.

Convex analysis based some special structures

The following two exemplar networks are used in almost all *PNAS* papers that discuss the community identification



(a) The ring of lumps (b) The ad hoc network

Figure 1: Diagrams of two exemplary networks.

A ring of dense lumps whose adjacency matrix is:

$$A^L = \begin{pmatrix} A & M & 0 & . & . & 0 & 0 & M \\ M & A & M & . & . & 0 & 0 & 0 \\ 0 & M & A & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & A & M & 0 \\ 0 & 0 & 0 & . & . & M & A & M \\ M & 0 & 0 & . & . & 0 & M & A \end{pmatrix}$$

where $L > 4$, A is an $m \times m$ adjacency matrix to represent a connected subnetwork called as lump, then AL is an $Lm \times Lm$ matrix, M stands for a random matrix with s non-zero elements. Note that these random matrices don't have to be identical, provided that they have the same number of non-zero elements .

The second exemplary network is a special version of the *ad hoc* network (a computer-generated network). Its adjacency matrix takes the form:

$$A^L = \begin{pmatrix} A & M & M & . & . & M & M & M \\ M & A & M & . & . & M & M & M \\ M & M & A & . & . & M & M & M \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ M & M & M & . & . & A & M & M \\ M & M & M & . & . & M & A & M \\ M & M & M & . & . & M & M & A \end{pmatrix}.$$

Denote a partition as $P = \{V_1, \ldots V_K\}$, the optimization process can be written as a two-stage optimization problem:

$$Q_p : \max_k \bar{Q}(k) = \max_k \max_{\sum_{i=1}^{k} |V_i| = n} Q(V_1, V_2, \cdots, V_k);$$

$$D_p : \max_k \bar{D}(k) = \max_k \max_{\sum_{i=1}^{k} |V_i| = n} D(V_1, V_2, \cdots, V_k);$$

We denote $\bar{Q}(k)$ and $\bar{D}(k)$ as the solutions from the first-step optimization problems: with a fixed k, partition the whole network into k subnetworks $N_1 = (V_1, E_1)$, …, $N_k = (V_k, E_k)$ to maximize the quantitative functions Q and D. And $\max_k \bar{Q}(k)$ and $\max_k \bar{Q}(k)$ are the second-step optimization problems.

Convex Analysis

- **A function (or a programming) whose variables take discrete values (or, say, the sample values) is called as discrete convex (concave) function (or programming) if they can be embedded into a continuous convex (concave) function (or programming).**

- **Result 1 : For the ring of A,**

$$\max_{\sum_{i=1}^{k}|V_i|=n} Q(V_1, V_2, \ldots, V_k)$$ **is a discrete concave programming**

$$\max_{\sum_{i=1}^{k}|V_i|=n} D(V_1, V_2, \ldots, V_k)$$ **is a discrete concave programming**

$\overline{Q}(k)$ **is a discrete convex function**

$\overline{D}(k)$ **is a discrete convex function**

Convex Analysis (continued)

- Result 2 : For the *ad hoc* network,

$\max_{\sum_{i=1}^{k}|V_i|=n} Q(V_1, V_2, \ldots, V_k)$ is a discrete concave programming

$\max_{\sum_{i=1}^{k}|V_i|=n} D(V_1, V_2, \ldots, V_k)$ is a linear programming

$\overline{Q}(k)$ is a discrete convex function

$\overline{D}(k)$ is a linear function

Above analysis makes it possible that we solve the two exemplar networks analytically, then compare Q and D analytically.

Convex Analysis (continued)

Result 3 :

- for the ring of A where each A is the smallest community (known community), the modularity density model D can identify the known communities. But the modularity model Q fails if

$$s > \frac{|E|}{L-1}$$

which extends the result in Fortunato & Barthelemy, *Proc. Natl. Acad. Sci.* (2007) where s takes 1.

Further research in community identification

- The closed formulation of the Q and D optimization allows to design more efficient algorithm to solve the community identification problem

- Based on the comparison of Q and D, present new measures that exactly reflect the community definition

- Consider modularity measures in directed networks

Thanks

Welcome to visit us at

http://zhangroup.aporc.org