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The Valuation of Callable Financial Commodities with Two Stopping Boundaries

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1. Introduction

◆ What is a callable financial commodity?

— set in the derivative internally

— possess the right of cancellation

◆ Many commodities have been developed to

— access specific market segments

— meet specific needs of various investors

— extract and decompose risk-return profiles of derivatives

◆ Two players in the risk game

Player Ⅰ : issuer or firm (seller)

Player Ⅱ : investor (buyer)

◆ Financial commodities issued by institutions

– to meet investment objectives of clients

– { callable for the seller
  putable for the buyer }

◆ Stochastic game as a coupled optimal stopping problem

– the seller wishes to minimize the issuing cost, seek for an optimal call time (stopping time)

– the buyer tries to maximize the payoff function seek for an optimal exercise time (stopping time)

⇓

Non-cooperative Dynkin game

(Coupled stopping game)

◆ Many methodologies and techniques have been developed for valuing the financial commodities

◆ Transformation of the optimal stopping problem into the free boundary problem

◆ Deriving the optimal stopping boundaries

⇩

A saddle point provides optimal stopping rules and equals the value of the financial commodity

2. Model Formulation

Trading periods : $[0,T]$ or $[0,\infty)$

Riskless asset : $B(t)$

$$dB(t) = r(t)B(t)dt, \quad B(0) > 0, \quad r(t) \geq 0, \tag{2.1}$$

Risk asset : $X(t)$

$$dX(t) = (r(t) - \delta(t))X(t)dt + \kappa(t)X(t)d\tilde{W}_t, \tag{2.2}$$

where $\kappa(\cdot) > 0$ is the volatility and $\tilde{W}_t$ is the standard Brownian motion under the risk neutral probability $\tilde{P}$.

Stopping times $\begin{cases} \sigma \text{ (player I)} \\ \tau \text{ (player II)} \end{cases}$

The payoff:

$$R_t(\sigma,\tau) = \int_t^{\sigma\wedge\tau} e^{\int_s^{\sigma\wedge\tau} r(u)du} c(s)ds + f(\sigma, X(\sigma))1_{\{\sigma<\tau\}} + g(\tau, X(\tau))1_{\{\tau\le\sigma<T\}} + h(X(T))1_{\{\sigma\wedge\tau=T\}} \tag{2.3}$$

Assumption 2.1

1) The payoff functions $f(t,x), g(t,x)$ and $h(x)$ are monotone in x.

2) The inequalities among f, g and h hold as follows: For each t

$$f(t,x) > g(t,x) \ge h(x) \qquad \forall\ x \ge 0.$$

Remarks 2.2

1) If $\sigma = \tau$, the buyer has priority over the seller. When $\sigma \wedge \tau = \min(\sigma, \tau) = T$, the payoff is assumed to be $h(X(T))$.

2) $R_0(\sigma, \tau)$has the lower and upper bounds.

3) If it is optimal for the seller not to cancel before the maturity, callable securities may be reduced to the usual non-callable one which is an American type.

4) If it is optimal for both the seller and the buyer not to exercise before the maturity, securities reduce to the European type.

5) Even if it is optimal for the buyer not to exercise before the maturity, the seller still faces the problem of selecting an optimal stopping time σ.

Theorem 2.3

For $X(t) = x$, define

$$\overline{V}(t,x) = \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \tilde{E}\left[e^{-\int_t^{\sigma\wedge\tau} r(s)ds} R_t(\sigma,\tau) | X(t) = x\right] \tag{2.4}$$

and

$$\underline{V}(t,x) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_{t,T}} \tilde{E}\left[e^{-\int_t^{\sigma\wedge\tau} r(s)ds} R_t(\sigma,\tau) | X(t) = x\right]. \tag{2.5}$$

Then, this game possesses the value which is given by

$$V(t,x) = \overline{V}(t,x) = \underline{V}(t,x), \ 0 \le t \le T. \tag{2.6}$$

Moreover, the optimal stopping times for the seller and the buyer are

$$\begin{aligned} \hat{\sigma}_t &= \inf\left\{\sigma \ge t : \ V(\sigma, X(\sigma)) = f(\sigma, X(\sigma)) + \int_t^\sigma e^{\int_s^\sigma r(u)du} c(s)ds\right\} \wedge T, \\ \hat{\tau}_t &= \inf\left\{\tau \ge t : \ V(\tau, X(\tau)) = g(\tau, X(\tau)) + \int_t^\tau e^{\int_s^\tau r(u)du} c(s)ds\right\} \wedge T. \end{aligned} \tag{2.7}$$

Corollary 2.4

The value function $V(t,x)$ satisfies

$$\begin{aligned} & g(t,x)+\int_0^t e^{\int_s^t r(u)du} c(s)ds \\ \leq & \quad V(t,x) \\ \leq & \quad f(t,x)+\int_0^t e^{\int_s^t r(u)du} c(s)ds, \ \forall (t,x) \in [0,T]\times \mathbf{R}^+. \end{aligned}$$

Theorem 2.5 (Perpetual financial commodity)

In addition to Assumption 2.1, if the inequality

$$\lim_{t\to\infty}\left(f(t,x)+\int_0^t e^{\int_s^t r(u)du} c(s)ds \right) \leq M_x \ \forall x \geq 0 \quad (2.8)$$

holds, then there exists the limit $V_\infty(x) \equiv \lim_{t\to\infty} V(t,x)$ which satisfies equation (2.6) in Theorem 2.3.

Corollary 2.6

If $\delta(t) = 0$ and $g(t,x)$ is convex and decreasing in x for each t, then

$$V(t,x) = \operatorname*{ess\,inf}_{\sigma \in \mathcal{T}_{t,T}} \tilde{E}\left[e^{-\int_t^{\sigma} r(s)ds} R_t(\sigma, T) | X(t) = x \right],$$

that is, the coupled optimal stopping problem above in Theorem 2.3 can be reduced to the one only for the issuer (player I).

3. Some Examples

A European call option

K : exercise price

$$\begin{aligned} c(t) &= 0,\ f(t,x)=\infty,\ g(t,x)=0,\ h(x)=\max(x-K,0)=(x-K)^+ \\ V(t,x) &= xe^{-\delta(T-t)}\Phi(d_1^+(T-t,x,K)) - Ke^{-r(T-t)}\Phi(d_1^-(T-t,x,K)) \\ d_1^\pm(t,x,y) &= \frac{\log(x/y)+(r-\delta\pm\frac{1}{2}\kappa^2)t}{\kappa\sqrt{t}} \end{aligned}$$

- Lower bound: $V(t,x) \geq 0$

B American put option

$$\begin{aligned} c(t) &= 0,\ f(t,x)=\infty,\ g(t,x)=h(x)=(K-x)^+, \\ V(t,x) &= \sup_\tau \tilde{E}\left[e^{-r(\tau-t)}R_0(\infty,\tau)|X(t)=x\right] \end{aligned}$$

- Lower and upper bounds:

$$(K-x)^+ \leq V(t,x) \leq K$$

C Game put option

(Kifer(2000), Suzuki and Sawaki(2007))
$\delta = 0,\ p \geq 0$: penalty

$$\begin{aligned} c(t) &= 0,\ f(t,x) = (K-x)^+ + p,\ g(t,x) = h(x) = (K-x)^+, \\ V(t,x) &= \sup_\tau \inf_\sigma \tilde{E}\left[e^{-r(\sigma\wedge\tau - t)} R_0(\sigma,\tau) | X(t) = x\right] \\ &= \inf_\sigma \sup_\tau \tilde{E}\left[e^{-r(\sigma\wedge\tau - t)} R_0(\sigma,\tau) | X(t) = x\right] \end{aligned}$$

- Lower and upper bounds:

$$(K-x)^+ \leq V(t,x) \leq (K-x)^+ + p$$

- The optimal stopping region for the issuer:
  $V^{ap}(t,x)$:the price of American put
  $t^* \equiv \sup\{t \geq 0 |\ p \leq V^{ap}(t,K)\}$

$$\mathcal{S}_t = \begin{cases} \{K\}, & t \in [0, t^*] \\ \phi, & t \in (t^*, T] \end{cases}$$

  If $p > V^{ap}(t,K)$, $t^* = 0$, that is, $\mathcal{S}_t = \phi$.

- The optimal region for the investor: $\mathcal{S}_t = [0, x_t]$
  x_t^{ap} :optimal boundary of American put

$$x_t^{ap} \leq x_t \leq K \text{ and } \lim_{t\to T} x_t = K.$$

Theorem 3.1

$V^{ep}(t,x)$: the price of European put

$V(t,x)$ can be decomposed as follows;

$$V(t,x) = V^{ep}(t,x) + e(t,x) - d(t,x),$$

where

$$\begin{aligned} e(t,x) &= rK \int_t^T e^{-r(s-t)} \Phi(d_2(s-t,x,x_s))ds \geq 0, \\ d(t,x) &= \tilde{E}\Big[\int_t^{t^*} e^{-r(s-t)} \left(\frac{\partial V}{\partial x}(s,K+) - \frac{\partial V}{\partial x}(s,K-) \right) dL_s^x(K) \\ &\qquad\qquad \mid X(t) = x \Big] \geq 0. \end{aligned}$$

$L_t^x(K)$: the local time of X_t at the level K

in the time interval $[0,t]$

Corollary 3.2

$e^a(t,x)$:the early exercise premium of the American put

We obtain

$$\begin{aligned} e(t,x) &> e^{ap}(t,x), \;\; t \in [0,t^*], \\ e(t,x) &= e^{ap}(t,x), \;\; t \in (t^*,T]. \end{aligned}$$

Corollary 3.3

$V_\infty^{ap}(x)$: the price of perpetual American put

x_{ap}^* : its optimal boundary

$p^* \equiv V_\infty^{ap}(K)$

For $p \geq p^*$,

$$V_\infty(x) = V_\infty^{ap}(x) = rK\int_0^\infty e^{-rt}\Phi(d_2(t,x,x_{ap}^*))dt.$$

For $p < p^*$,

$$\begin{aligned} V_\infty(x) &= V_\infty^{ap}(x) \\ &- \left(\frac{dV_\infty}{dx}(K+) - \frac{dV_\infty}{dx}(K-)\right)\tilde{E}\left[\int_0^\infty e^{-rt}dL_t^x(K)\middle|\; X(0)=x\right]. \end{aligned}$$

D Callable convertible bond (Yagi and Sawaki(2005)(2007))

F :face value; C :call price; z :dilution factor

$$\begin{aligned} c(t) &= 0,\ f(t,x) = \max(zx, C),\ g(t,x) = zx, \\ h(x) &= \min(x, \max(zx, F)) \end{aligned}$$

- Lower and upper bounds:

$$zx \leq V(t,x) \leq \max(zx, C)$$

- Optimal stopping boundaries for the issuer:

$$\begin{aligned} \mathcal{S}^I &= \{(t,x) | V(t,x) = \max(zx, C)\} \\ x_t^I &= \inf\{x | x \in \mathcal{S}_t^I\} \\ \mathcal{S}_t^I &= [x_t^I, \infty) \end{aligned}$$

- Optimal stopping boundaries for the investor:

$$\begin{aligned} \mathcal{S}^{II} &= \{(t,x) | V(t,x) = zx\} \\ x_t^{II} &= \inf\{x | x \in \mathcal{S}_t^{II}\} \\ \mathcal{S}_t^{II} &= [x_t^{II}, \infty) \end{aligned}$$

- Letting $x_t^* \equiv \min(x_t^I, x_t^{II})$, the continuing region is given by

$$\mathcal{C}_t = [0, x_t^*)$$

Theorem 3.4

$V(t,x)$ can be written as

$$V(t,x) = B(t,x) + V^{ec}(t,x) + p(t,x) - d(t,x),$$

where $B(t,x)$ is the discount bond value, $V^{ec}(t,x)$ the price of European call, $p(t,x)$ the early conversion premium and $d(t,x)$ the callable discount.

E Installment American call option (Ben(2002))

q :installment rate

$$
\begin{aligned}
c(t) &= -q, \ \ f(t,x) = \infty, \ \ g(t,x) = h(x) = (x-K)^+, \\
R_t(\tau_e, \tau_s) &= (X(\tau_e) - K)^+ 1_{\{\tau_e < \tau_s < T\}} + (X(T) - K)^+ 1_{\{\tau_e \wedge \tau_s \geq T\}} \\
&\quad - \int_t^{\tau_e \wedge \tau_s} e^{r(\tau_e \wedge \tau_s - s)} q ds, \\
V(t,x;q) &= \operatorname*{ess\,sup}_{\tau_e, \tau_s} \tilde{E}\left[e^{-r(\tau_e \wedge \tau_s - t)} R_t(\tau_e, \tau_s) | X(t) = x \right]
\end{aligned}
$$

- Optimal stopping boundary:

$$
\begin{aligned}
\mathcal{S} &= \{(t,x) | V(t,x;q) = 0\} \\
\mathcal{E} &= \{(t,x) | V(t,x;q) = (x-K)^+\}
\end{aligned}
$$

Especially, at the maturity the optimal stopping and exercise boundaries $\underline{x}_t$, $\overline{x}_t$ are as follows,

$$
\begin{aligned}
\underline{x}_T &= K \\
\overline{x}_T &= \max\left(\frac{rK - q}{\delta}, K \right)
\end{aligned}
$$

Theorem 3.5

$$
\begin{aligned}
V(t,x;q) &= V^{ec}(t,x) - q\int_t^T e^{-r(u-t)}\Phi(d_1^-(u-t,x,\underline{x}_u))du \\
&\quad + \int_t^T \{\delta x e^{-\delta(u-t)}\Phi(d_1^+(u-t,x,\overline{x}_u)) \\
&\quad - (rK-q)e^{-r(u-t)}\Phi(d_1^-(u-t,x,\overline{x}_u))\}du
\end{aligned}
$$

F European double barrier equity linked bond

U :upper barrier; L :lower barrier

$$
\begin{aligned}
c(t) &= 0,\ \ f(t,x) = \infty,\ \ g(t,x) = 0, \\
\hat{X}(T) &= \max_{0\le t\le T} X(t),\ \ \check{X}(T) = \min_{0\le t\le T} X(t), \\
h(\hat{X}(T), \check{X}(T)) &= F1_{\{\hat{X}(T)\ge U\}} + F1_{\{\hat{X}(T)<U, \check{X}(T)\ge L\}} \\
&\quad + \min\left(F, \frac{X(T)}{X(0)}F\right)1_{\{\hat{X}(T)<U, \check{X}(T)<L\}} \\
V(t,x) &= \tilde{E}\Big[e^{-r(T-t)}\Big(F - \frac{F}{X(0)}(X(0)-X(T))^{+}1_{\{\hat{X}(T)<U\}} \\
&\quad + \frac{F}{X(0)}(X(0)-X(T))^{+}1_{\{\hat{X}(T)<U, \check{X}(T)\ge L\}}\Big| X(t)=x\Big] \\
&= e^{-r(T-t)}F - \frac{F}{X(0)}\hat{V}(t,x) + \frac{F}{X(0)}\check{V}(t,x),\ \ L \le x \le U,
\end{aligned}
$$

where $\hat{V}$ is the price of European up-and-out put option with the strike price $X(0)$ and $\check{V}$ one of European double barrier knock-out put option with strike price $X(0)$.

G PRDC

Let $X(t)$ be the exchange rate which follows the stochastic differential equation

$$dX(t) = (r - r_f)X(t)dt + \kappa X(t)d\tilde{W}_t,$$

where r is the domestic riskless interest rate and r_f the riskless rate of the counter part.

$$\begin{aligned} c(t) &= c, \;\; f(t,x) = \infty, \;\; g(t,x) = 0, \\ h(\hat{X}(T)) &= \left(\alpha\frac{X(T)}{X(0)} - \beta\right)^{+} F1_{\{\hat{X}(T)\le U\}}, \end{aligned}$$

where α, $\beta \geq 0$.

4. Analytical Properties

Assumption 4.1

$f(t,x)$, $g(t,x)$ and $h(x)$ are non-increasing in t and monotone convex in x and $c(t) = 0$. Define the stopping regions for player I and II, respectively, by

$$\begin{aligned} \mathcal{S}^I &= \{(t,x) | V(t,x) = f(t,x)\} \\ \mathcal{S}^{II} &= \{(t,x) | V(t,x) = g(t,x)\} \end{aligned}$$

and the continuing region is

$$\mathcal{C} = \{(t,x) | g(t,x) < V(t,x) < f(t,x)\}.$$

$\mathcal{S}^I_t$ and $\mathcal{S}^{II}_t$ are the truncations of $\mathcal{S}^I$ and $\mathcal{S}^{II}$ at time t.

Lemma 4.2

1)$V(t,x)$ is non-increasing in t for each x,

2)$V(t,x)$ is monotone convex in x for each t.

3)If $(t,x) \in \mathcal{C}$, then $\mathcal{L}V = 0$, where

$$\mathcal{L} = \frac{1}{2}\kappa^2 x^2 \frac{\partial^2}{\partial x^2} + (r-\delta)x\frac{\partial}{\partial x} + \frac{\partial}{\partial t} - r.$$

Lemma 4.3

If $f(t,x)$ and $g(t,x)$ are monotone and convex in x, then $\mathcal{S}_t^I$ and $\mathcal{S}_t^{II}$ are connected sets, that is, the stopping region never possesses the detached region.

Lemma 4.4

If the issuer does not hold the call right, it is reduced to be American type. Let $V^a(t,x)$ be its price and, $V^a_\infty(x)$ the value of the perpetual financial commodity. We have

1)$V(t,x) \leq V^a(t,x) \leq V^a_\infty(x)$,

2)$\mathcal{S}^{II}_t \supseteq \mathcal{S}^a_t \supseteq \mathcal{S}^a_\infty$.

Theorem 4.5

Let $p_t \equiv f(t,x) - g(t,x)$, the difference depending only on t and assume that p_t is non-increasing in t and put $K_t \equiv \arg\min_x f(t,x)$. Define

$$t^* = \sup\{t \geq 0 |\ p_t \leq V^a(t, K_t)\}$$

If $t \in [0, t^*]$,the optimal call region for the issuer is $\mathcal{S}_t^I = \{K_t\}$.
If $t \in (t^*, T]$, $\mathcal{S}_t^I = \phi$.

5. Numerical Examples

(A) Penalty costs are discounted

Game put options which pay off functions are

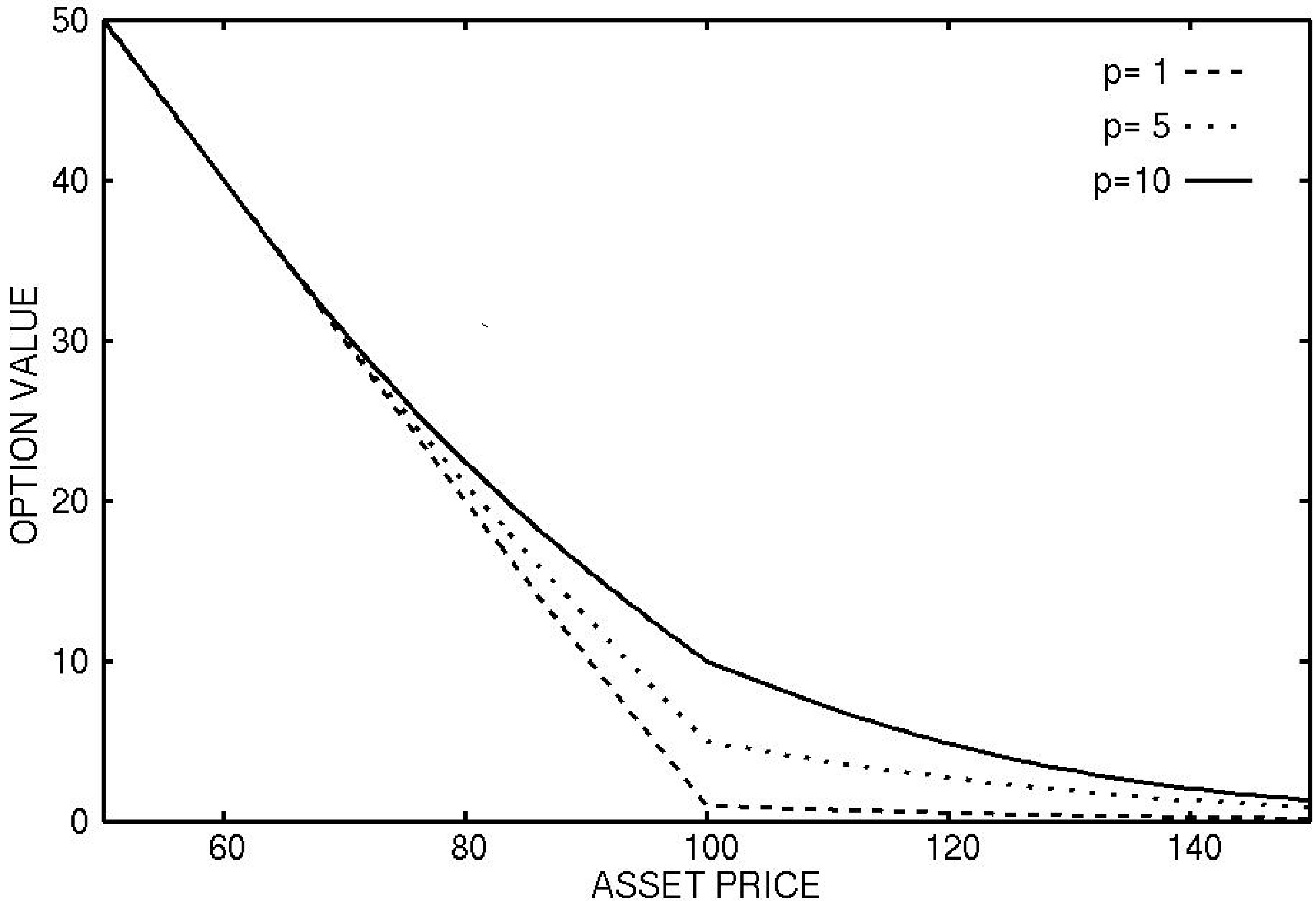
$$f(t,x) = \max\{K - x, 0\} + e^{-rt}p$$

and

$$g(t,x) = \max\{K - x, 0\}.$$

Parameters;

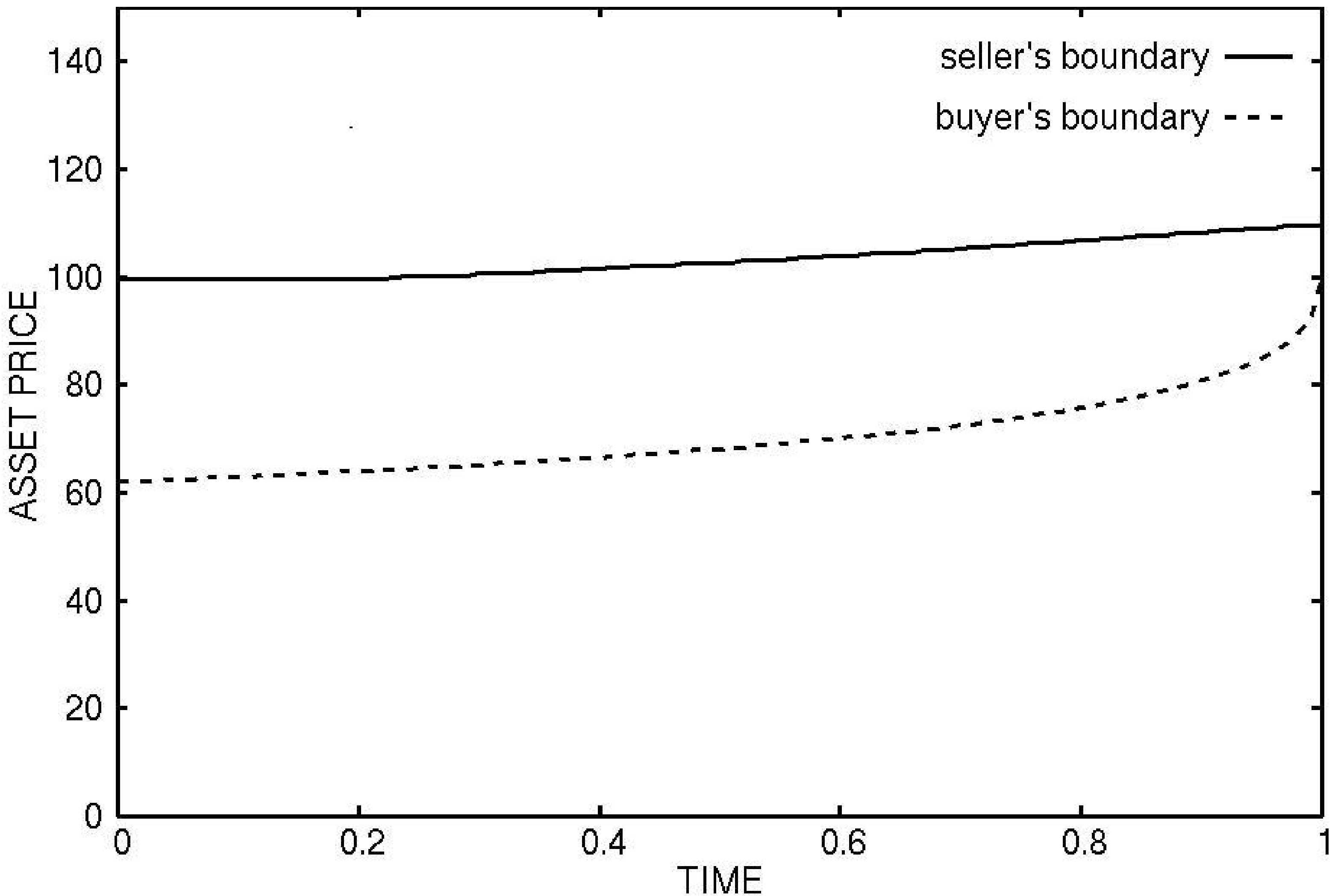
$$K = 100,\ r = 0.05,\ \delta = 0.04,\ \kappa = 0.3,\ T = 1$$
$$p = 1, 5, 10,\ respectively$$



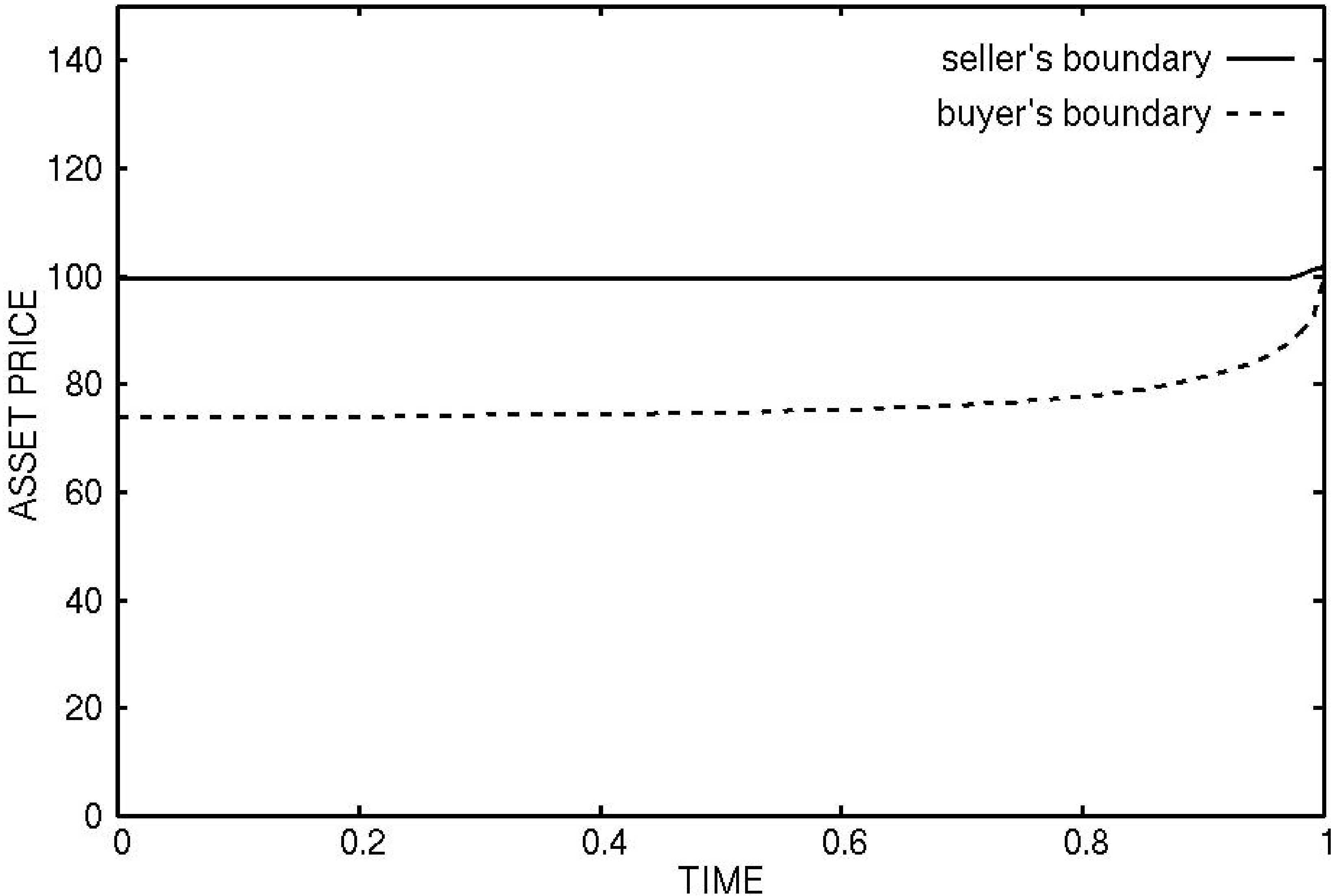
Optimal boundaries of the seller and the buyer;

Parameters;

$$K = 100,\ r = 0.05,\ \delta = 0.04,\ \kappa = 0.3,\ p = 10$$

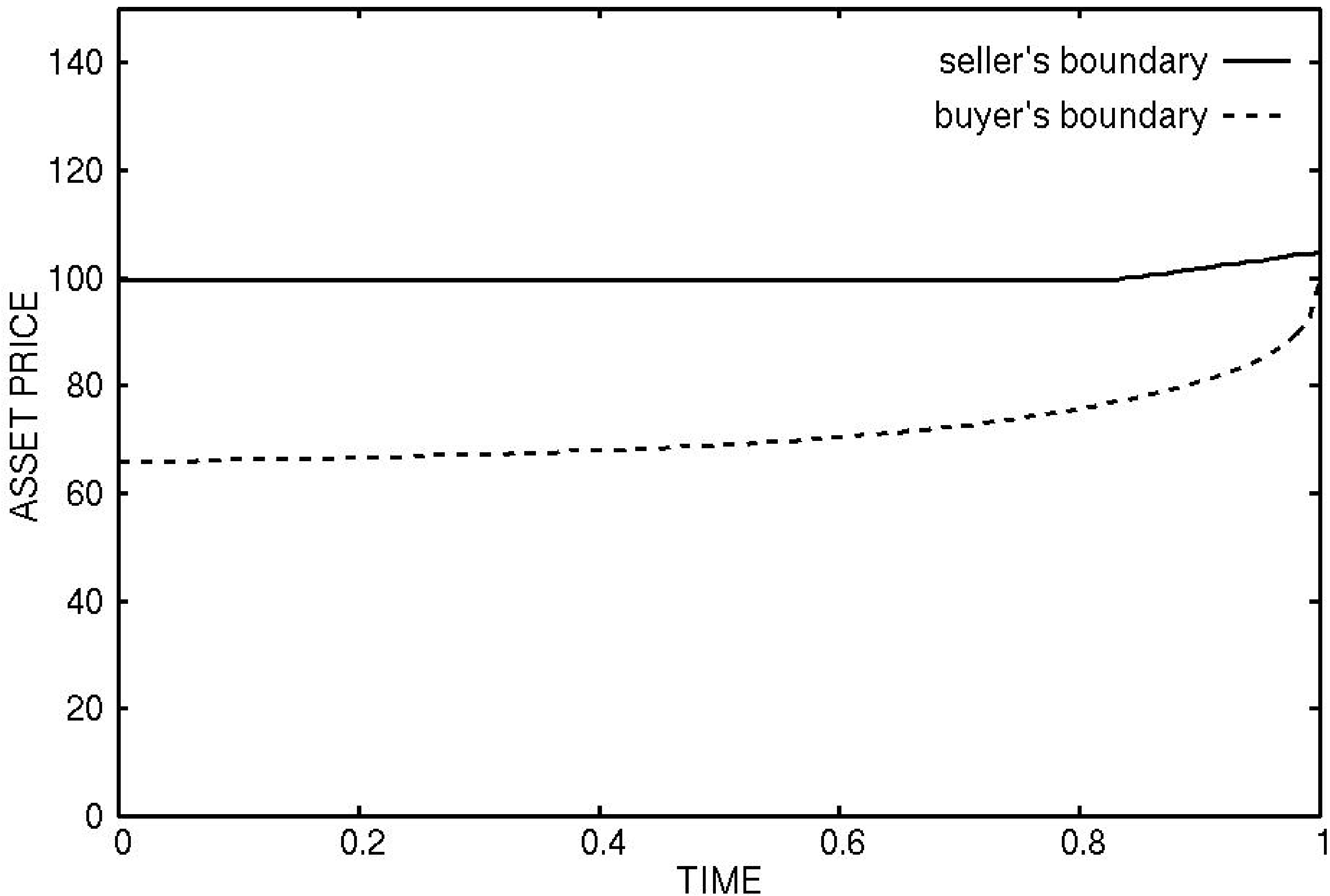


$p = 2$



seller's boundary
buyer's boundary
140
120
100
80
60
40
20
0
ASSET PRICE
0
0.2
0.4
0.6
0.8
1
TIME

$p = 5$



seller's boundary
buyer's boundary
ASSET PRICE
TIME
0
20
40
60
80
100
120
140
0
0.2
0.4
0.6
0.8
1

(B) Penalty costs are constant (no discounted)

$$r = 0,1, \ \ \kappa = 0.3, \ \ K = 100$$

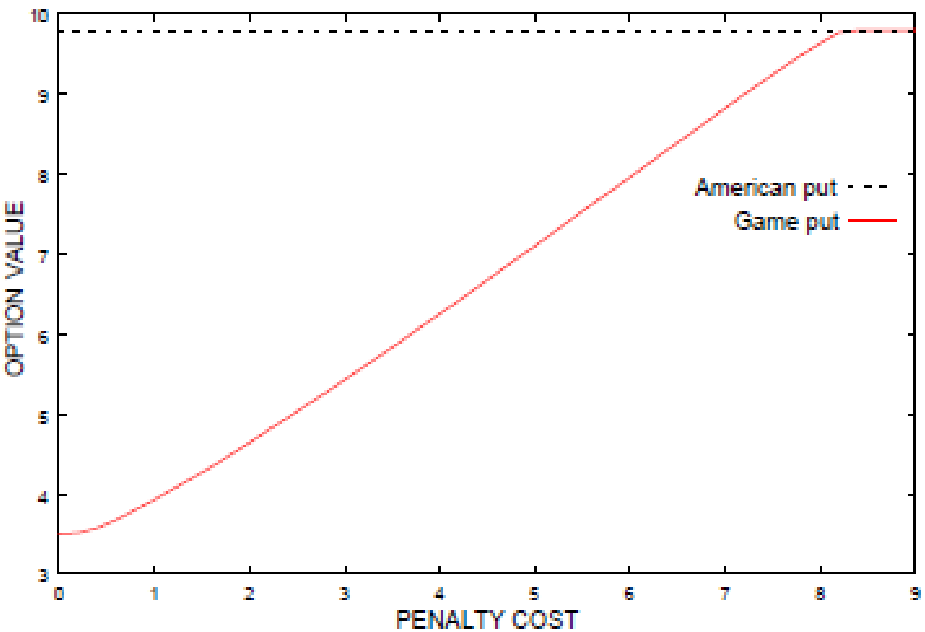


Fig. 1: Behavior of the callable American put price when the penalty cost changes.

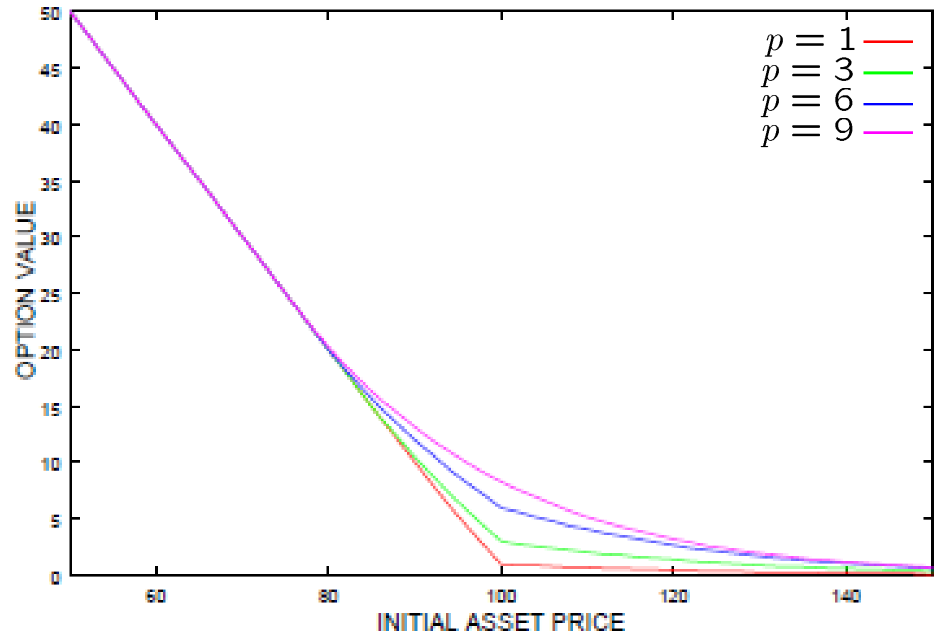


Fig. 2: Behavior of the callable American put price when the initial asset price changes.

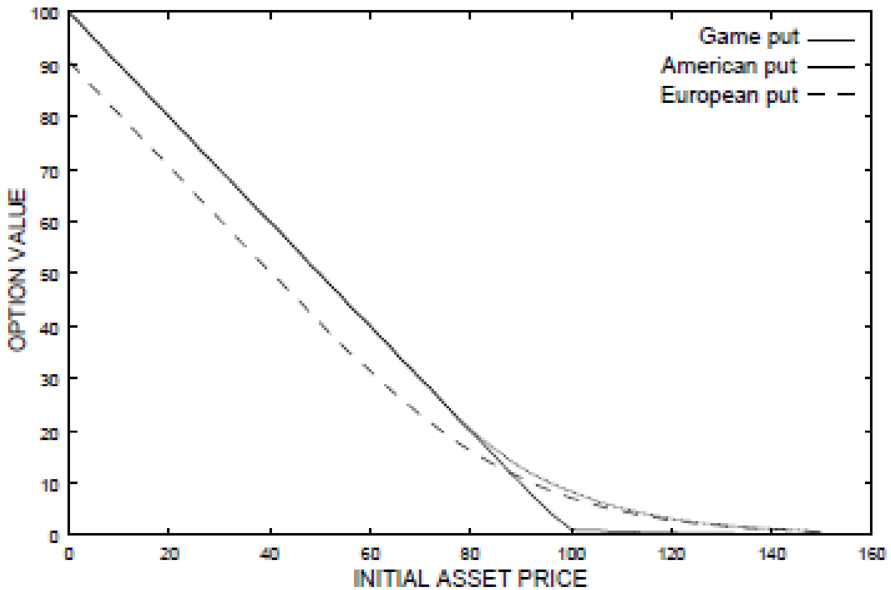


Fig. 3: Comparison of the callable American put price($p = 1$), corresponding American and European put price.

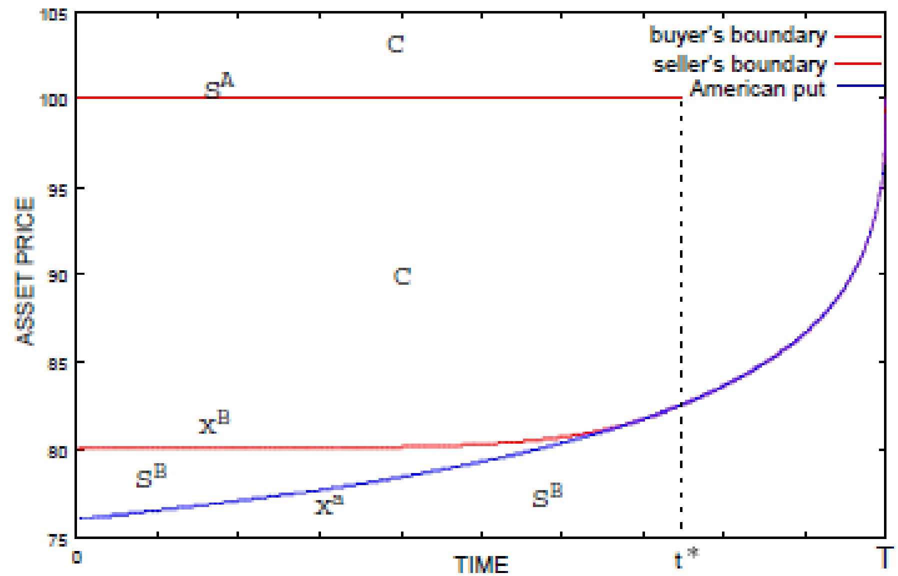
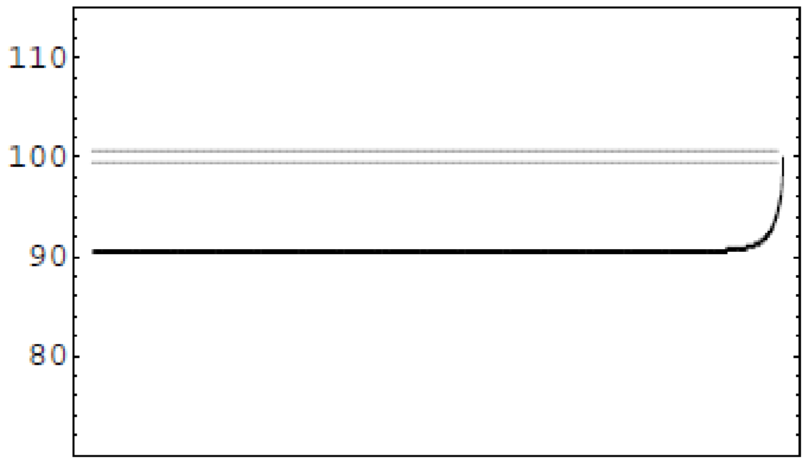
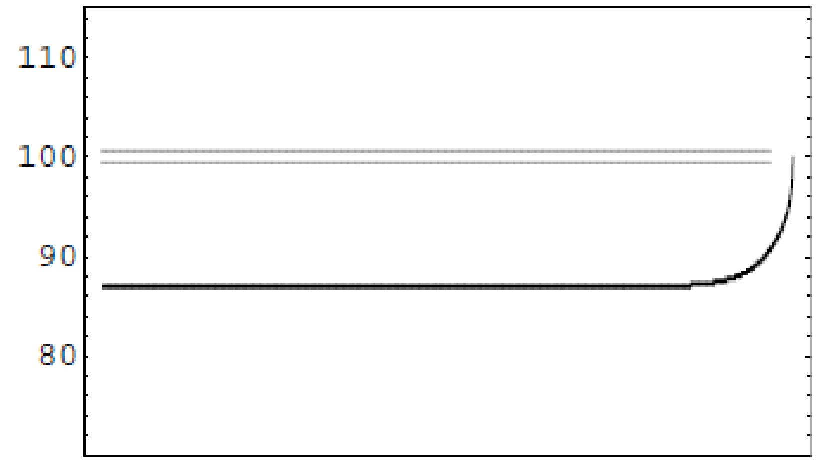


Fig. 4: Optimal exercise boundaries of the callable American put for the seller and the buyer when p= 5 and of the American put.

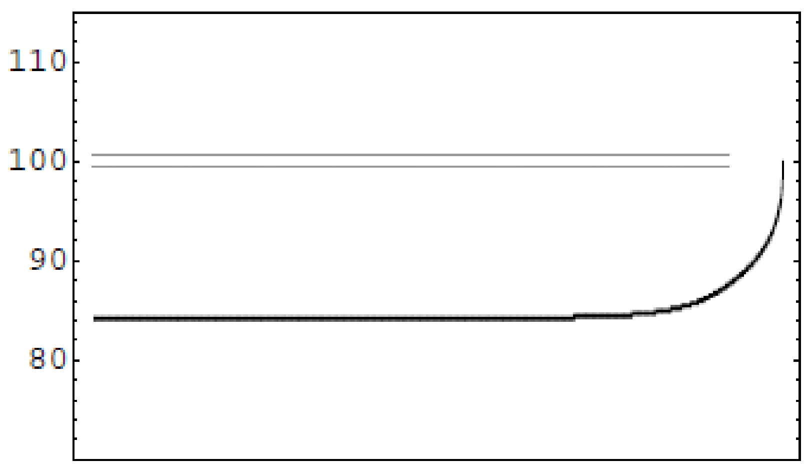
Figure Optimal strategies of the seller and the buyer



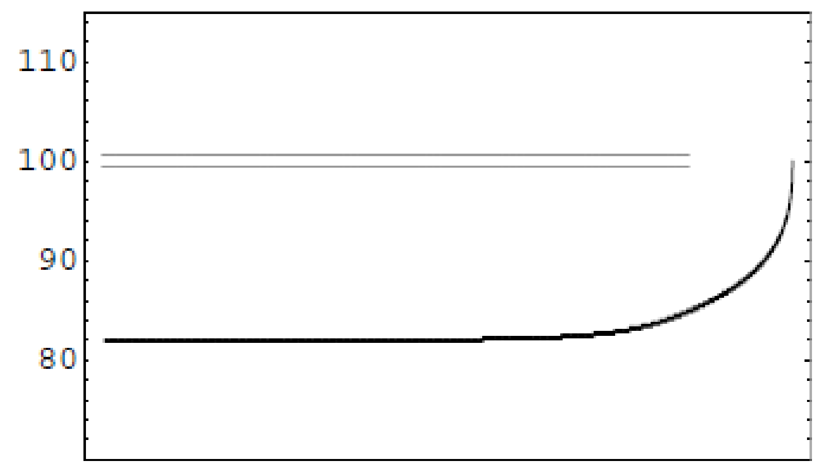
$p = 1$



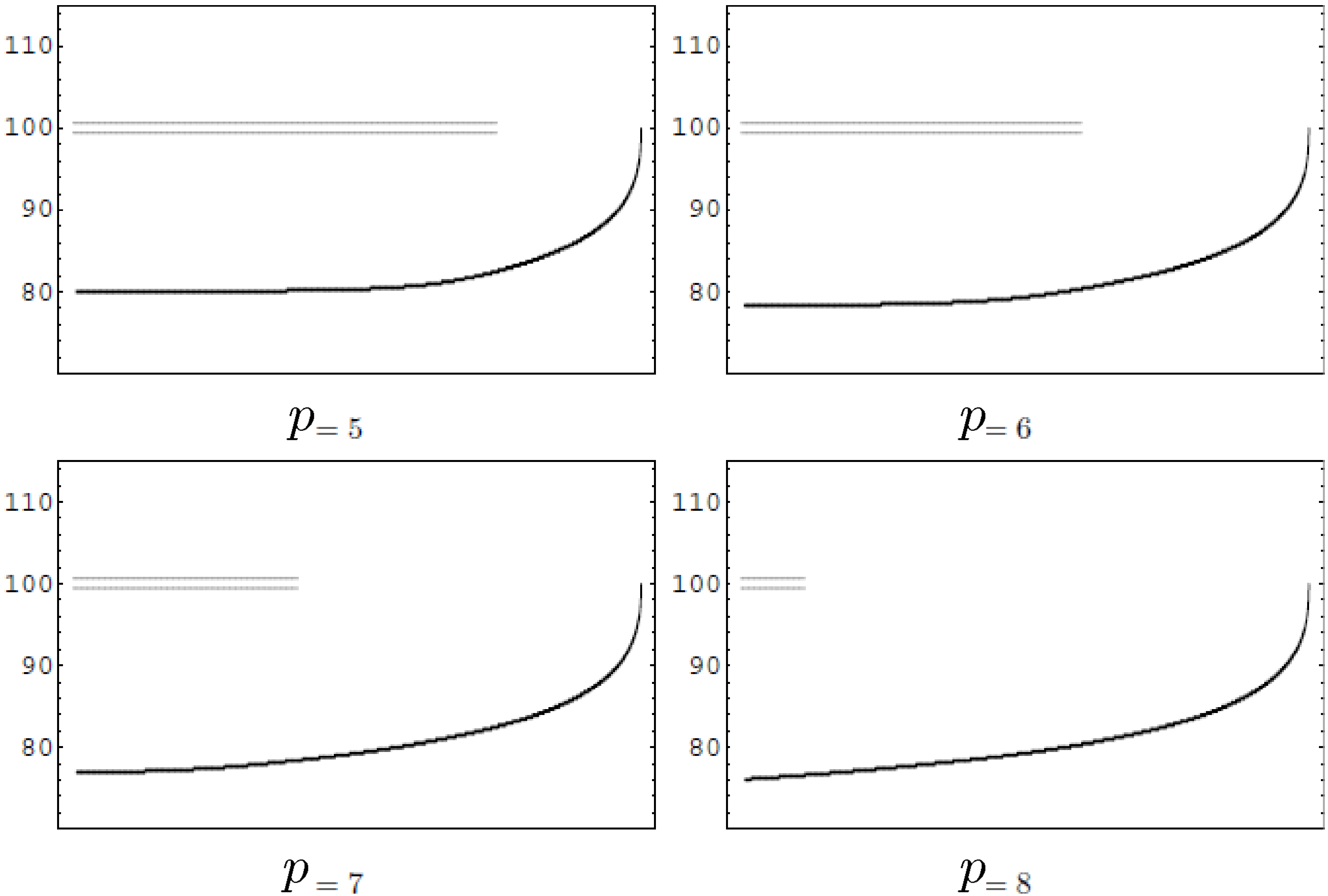
$p = 2$



$p = 3$



$p = 4$


110
100
90
80
$p = 5$
$p = 6$
$p = 7$
$p = 8$

$$If \quad p > V^a(K,0) \doteq 8.337, \ then \ t^* = 0$$

$$V(S,t) \uparrow \ as \ p \uparrow$$

$$If \quad p = q > V^a(K,0), \ V(S,t) = V^a(S,t).$$

$$If \quad t \in [0,t^*], \ S_t^A = \{K\}, \ and \ x_t^B > x_t^a$$

$$If \quad t \in (t^*,T], \ S_t^A = \phi, \ and \ x_t^B = x_t^A$$

6. Conclusion

- Callable security can provide the upper bound for the seller's cost

- Putable security may guarantee the lower bound for the buyer's profit
  - — maximum loss
  - — maximum gain

- The value of such securities lies in between them

- Optimal boundaries for the seller may vanish for p large enough

- What is your risk capacity ?

- New financial commodities can be designed with risk aspects