Global stability of the SEIR epidemic model with infectivityin both latent period and infected period

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***Abstract*—An epidemic model with infectivity and recovery in both latent and infected period is introduced. Utilizing the LaSalle invariance principle and Bendixson criterion,the basic reproduction number is found, we prove that the disease-free equilibrium is globally asymptotically stable when the basic reproduction number is less than one. The disease-free equilibrium is unstable and the unique positive equilibrium is globally asymptotically stable when the basic reproduction number is greater than one.Numerical simulations support our conclusions.**

Keywords—epidemic model; latent period; equilibrium; the second compound matrix; global stability

I. INTRODUCTION

Pestilences have a profound impact on human civilization. Human has a long history of struggling with infectious disease,thereinto epidemic dynamics is an important method of studying the spread of epidemic qualitatively and quantitatively. The research results are helpful to predict the developing tendency of the epidemic and to seek the optimum strategies of preventing the spread of the epidemic.In recent years,several studies have been devoted to the epidemic models with exposed (latent) period[1-7] .[1-4] demonstrate the models without infectivity in latent period. [5-7] discuss the models with infectivity ,but without recovery in latent period. However, many infectious diseases,such as SARS, tuberculosis, etc., not only can be transmitted, but also can be recovered in latent period.In this paper,we consider an SEIR epidemic model with infectivity and recovery in both latent period and infected period ,which is given by the following form:

$$\begin{cases} S' = A - \beta_1(N)\dfrac{SE}{N} - \beta_2(N)\dfrac{SI}{N} - dS \\ E' = \beta_1(N)\dfrac{SE}{N} + \beta_2(N)\dfrac{SI}{N} - (d+\varepsilon+\gamma_1)E \\ I' = \varepsilon E - (d+d_1+\gamma_2)I \\ R' = \gamma_1 E + \gamma_2 I - dR \\ N(t) = S(t) + E(t) + I(t) + R(t) \end{cases} \quad (1)$$

Where A is the recruitment rate of the population, d is the natural death rate, d_1 is the disease-related death rate, ε is the transformation rate from the exposed individuals to the infected individuals, the recovery rate of the exposed group is γ_1 , the recovery rate of the infected group is γ_2 , $\beta_1(N)$ and $\beta_2(N)$ are the adequate contact rates,which satisfy the following assumptions[8]:

(A_1) $\beta_i(N)$ is a nonnegative nondecreasing continuous function and is continuously differentiable as $N \geq 0$, $\beta_i(0) = 0$ $i = 1,2$.

(A_2) $\eta_i(N) = \dfrac{\beta_i(N)}{N}$ is a nonincreasing continuously differentiable function as $N > 0$, $i = 1,2$

II. THE EXISTENCE AND STABILITY OF THE EQUILIBRIUM

Let $S(t) = N(t) - E(t) - I(t) - R(t)$, $\delta = d + \varepsilon + \gamma_1$ and $\mu = d + d_1 + \gamma_2$ then the system (1) becomes

$$\begin{cases} E' = [\eta_1(N)E + \eta_2(N)I](N - E - I - R) - \delta E \\ I' = \varepsilon E - \mu I \\ R' = \gamma_1 E + \gamma_2 I - dR \\ N' = A - dN - d_1 I \end{cases} \quad (2)$$

From biological considerations, system (2) is studied in the positive invariant set.

$$\Omega_1 = \left\{ (E, I, R, N) \mid E \geq 0, I \geq 0, R \geq 0, E + I + R \leq N \leq \frac{A}{d} \right\}$$

By setting the right sides of the four differential equations of system (2) to zero, we obtain

$$F(N)I = 0$$

Where

$$F(N) = [\mu\eta_1(N) + \varepsilon\eta_2(N)][\frac{\delta\mu}{\varepsilon d_1} N - \frac{A\mu(d+\gamma_1) + A\varepsilon(d+\gamma_2)}{\varepsilon d d_1}] - \delta\mu$$

According to the assumptions (A_1) and (A_2), we have

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$$F'(N) = -\frac{A\mu(d+\gamma_1)+A\varepsilon(d+\gamma_2)}{\varepsilon d d_1}[\mu\eta_1'(N)+\varepsilon\eta_2'(N)] + \frac{\delta\mu}{\varepsilon d_1}[\mu\beta_1'(N)+\varepsilon\beta_2'(N)] \geq 0$$

The function $F(N)$ is nondecreasing,so the equation $F(N)=0$ has a unique root at most . And because

$$F(0) = -\frac{A\mu(d+\gamma_1)+A\varepsilon(d+\gamma_2)}{\varepsilon d d_1}[\mu\eta_1(N)+\varepsilon\eta_2(N)] - \delta\mu < 0$$

Let $R_0 = \frac{A[\mu\eta_1(\frac{A}{d})+\varepsilon\eta_2(\frac{A}{d})]}{d\delta\mu}$, $F(\frac{A}{d}) = \delta\mu(R_0-1)$

R_0 is called the basic reproduction number.The equation $F(N)=0$ has a unique positive root in the interval $(0,\frac{A}{d})$ When $R_0>1$,then the equation $F(N)=0$ has no positive root When $R_0<1$ .Based on the above analysis,we obtain the following results.

Theorem2.1 The system (2) always has the disease-free disease-free equilibrium $P_0(0,0,0,\frac{A}{d})$,then it has a unique positive equilibrium $P_1(E^*,I^*,R^*,N^*)$ if $R_0>1$.

Theorem2.2 The disease-free equilibrium P_0 of the system (2) is globally asymptotically stable in Ω_1 when $R_0<1$ and it is unstable when $R_0>1$.

Proof The Jacobian matrix of the system (2) at the disease-free equilibrium P_0 is

$$J(P_0) = \begin{pmatrix} \frac{A}{d}\eta_1(\frac{A}{d})-\delta & \frac{A}{d}\eta_2(\frac{A}{d}) & 0 & 0 \\ \varepsilon & -\mu & 0 & 0 \\ \gamma_1 & \gamma_2 & -d & 0 \\ 0 & -d_1 & 0 & -d \end{pmatrix}$$

Its characteristic equation is

$$(\lambda+d)^2\{\lambda^2+[\delta+\mu-\frac{A}{d}\eta_1(\frac{A}{d})]\lambda+\mu[\delta-\frac{A}{d}\eta_1(\frac{A}{d})]-\frac{A\varepsilon}{d}\eta_2(\frac{A}{d})\}=0$$

So $\lambda_1=\lambda_2=-d$,

$$\lambda_3+\lambda_4 = \frac{A}{d}\eta_1(\frac{A}{d})-\delta-\mu < \frac{A}{d}\eta_1(\frac{A}{d})+\frac{A\varepsilon}{d\mu}\eta_2(\frac{A}{d})-\delta = \delta(R_0-1)$$

$$\lambda_3\cdot\lambda_4 = \mu[\delta-\frac{A}{d}\eta_1(\frac{A}{d})]-\frac{A\varepsilon}{d}\eta_2(\frac{A}{d}) = \delta\mu(1-R_0)$$

Therefore,the characteristic roots of Jacobian matrix $J(P_0)$ all have negative real parts.It follows that the disease-free equilibrium P_0 is locally asymptotically stable when $R_0<1$ and it is unstable when $R_0>1$.

Then we analyse the global stability of the disease-free equilibrium P_0 and construct Lyapunov function.

$$L(E,I,R,N) = R_0E + \frac{A\eta_2(\frac{A}{d})}{d\mu}I$$

We obtain

$$\begin{aligned}\frac{dL}{dt}|_{(2)} &= R_0[\eta_1(N)E+\eta_2(N)I]S-\frac{A}{d}[\eta_1(\frac{A}{d})E+\eta_2(\frac{A}{d})I] \\ &\leq R_0[\eta_1(N)E+\eta_2(N)I]N-\frac{A}{d}[\eta_1(\frac{A}{d})E+\eta_2(\frac{A}{d})I] \\ &\leq \frac{A}{d}[\eta_1(\frac{A}{d})E+\eta_2(\frac{A}{d})I](R_0-1)\end{aligned}$$

and

$$\{(E,I,R,N)\,|\,\frac{dL}{dt}|_{(2)}=0\} = \{(E,I,R,N)\,|\,E=I=R=0, N=\frac{\Lambda}{d}\}$$

It shows that the disease-free equilibrium P_0 of the system (2) is globally asymptotically stable when $R_0<1$ in accordance with Lyapunov-LaSalle invariance principle.

Assume that

(B_1) there is a compact absorbing set $K\subset D$;

(B_2) $\bar{x}$ is the only equilibrium in D

Lemma 2.1[9] Suppose that D is simply connected domain, Under assumptions (B_1) and (B_2) ,the unique equilibrium $\bar{x}$ is globally asymptotically stable in D if $q<0$. The quantity q is defined:

$$q = \limsup_{t\to\infty}\sup_{x_0\in K}\frac{1}{t}\int_0^t \mu(B(x(s,x_0)))ds$$

Where $B = P_f P^{-1} + P\frac{\partial f^{[2]}}{\partial x}P^{-1}$ and $x\to P(x)$ is a $\binom{n}{2}\times\binom{n}{2}$ matrix-valued function,

which is C^1 in D . P_f is the matrix obtained by replacing each entry p_{ij} in P by its directional derivative in the direction of f , $\frac{\partial f^{[2]}}{\partial x}$ is the second additive compound matrix of the Jacobian matrix of f .Then $\mu(B)$ is the Lozinski $\breve{l}$ measure, which is defined by

$$\mu(B) = \lim_{h\to 0^+}\frac{\|I+hB\|-1}{h}$$

When $d_1=0$,that is, $\underset{t\to\infty}{N}(t)\to\frac{A}{d}=a$, the limit equations of system (1) is

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$$\begin{cases} S' = A - \eta_1(a)SE - \eta_2(a)SI - dS \\ E' = \eta_1(a)SE + \eta_2(a)SI - \delta E \\ I' = \varepsilon E - (d+\gamma_2)I \\ R' = \gamma_1 E + \gamma_2 I - dR \end{cases} \quad (3)$$

We need only to consider the system consisted of the first three equations,denote it by (4)

$$\begin{cases} S' = A - \eta_1(a)SE - \eta_2(a)SI - dS \\ E' = \eta_1(a)SE + \eta_2(a)SI - \delta E \\ I' = \varepsilon E - (d+\gamma_2)I \end{cases} \quad (4)$$

system (4) is studied in the positive invariant set.

$$\Omega_2 = \left\{ (S,E,I) \mid S \ge 0, E \ge 0, I \ge 0, S+E+I \le \frac{A}{d} \right\}$$

Obviously system (4) always has the disease-free equilibrium $Q_0(a,0,0)$,

then it has a unique positive equilibrium $Q_1(S^*,E^*,I^*)$ if $R_1 > 1$

where $S^* = \dfrac{\delta(d+\gamma_2)}{(d+\gamma_2)\eta_1(a) + \varepsilon\eta_2(a)}$, $E^* = \dfrac{A - dS^*}{\delta}$,

$$I^* = \frac{\varepsilon(A - dS^*)}{\delta(d+\gamma_2)}, R_1 = \frac{A[(d+\gamma_2)\eta_1(a) + \varepsilon\eta_2(a)]}{d\delta(d+\gamma_2)}$$

And the disease-free equilibrium Q_0 of the system (4) is globally asymptotically stable in Ω_2 when $R_1 < 1$ and it is unstable when $R_1 > 1$.

Theorem2.3 The positive equilibriu Q_1 of the system (4) is locally asymptotically stable in Ω_2 if $R_1 > 1$.

Proof The Jacobian matrix of the system (4) at the positive equilibrium Q_1 is

$$J(Q_1) = \begin{pmatrix} -\eta_1(a)E^* - \eta_2(a)I^* - d & -\eta_1(a)S^* & -\eta_2(a)S^* \\ \eta_1(a)E^* + \eta_2(a)I^* & \eta_1(a)S^* - \delta & \eta_2(a)S^* \\ 0 & \varepsilon & -(d+\gamma_2) \end{pmatrix}$$

Its characteristic equation is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

Where

$$\begin{aligned} a_1 &= \eta_1(a)E^* + \eta_2(a)I^* + 2d + \delta + \gamma_2 - \eta_1(a)S^* \\ &\ge \eta_1(a)E^* + \eta_2(a)I^* + 2d + \gamma_2 > 0 \end{aligned}$$

$$\begin{aligned} a_2 &= (\delta + d + \gamma_2)[\eta_1(a)E^* + \eta_2(a)I^*] + d(d+\delta+\gamma_2) - d\eta_1(a)S^* \\ &\ge (\delta + d + \gamma_2)[\eta_1(a)E^* + \eta_2(a)I^*] + d(d+\gamma_2) > 0 \end{aligned}$$

$$a_3 = \delta(d+\gamma_2)[\eta_1(a)E^* + \eta_2(a)I^*] > 0$$

Further

$$\begin{aligned} a_1 \cdot a_2 - a_3 &= (\delta + d + \gamma_2)[\eta_1(a)E^* + \eta_2(a)I^*]^2 + [d(d+\gamma_2) \\ &\quad + (2d+\delta+\gamma_2)(d+\delta) + \gamma_2(2d+\gamma_2)][\eta_1(a)E^* \\ &\quad + \eta_2(a)I^*] + d(d+\delta+\gamma_2)(2d+\delta+\gamma_2) + d\eta_1^2(a)S^{*2} \\ &\quad - (2d+\delta+\gamma_2)\eta_1(a)[\eta_1(a)E^* + \eta_2(a)I^*]S^* \\ &\quad - d(3d+2\delta+2\gamma_2)\eta_1(a)S^* \end{aligned}$$

$$\begin{aligned} &\ge (\delta + d + \gamma_2)[\eta_1(a)E^* + \eta_2(a)I^*]^2 \\ &\quad + [d(3d+2\gamma_2+\delta) + \gamma_2(2d+\gamma_2)][\eta_1(a)E^* + \eta_2(a)I^*] \\ &\quad + d(\gamma_2^2 + 2d^2 + 3d\gamma_2) - d\delta\eta_1(a)S^* \\ &\quad + \frac{d\eta_1^2(a)\delta(d+\gamma_2)}{(d+\gamma_2)\eta_1(a) + \varepsilon\eta_2(a)}S^* + \frac{d\varepsilon\eta_2(a)}{(d+\gamma_2)}S^* \\ &\ge (\delta + d + \gamma_2)[\eta_1(a)E^* + \eta_2(a)I^*]^2 \\ &\quad + [d(3d+2\gamma_2+\delta) + \gamma_2(2d+\gamma_2)][\eta_1(a)E^* + \eta_2(a)I^*] \\ &\quad + d(\gamma_2^2 + 2d^2 + 3d\gamma_2) > 0 \end{aligned}$$

Therefore,the Routh-Hurwitz stability conditions are satisfied.It follows that the positive equilibrium Q_1 is locally asymptotically stable.

Theorem2.4 Suppose $2a\eta_1(a) \le \varepsilon + \gamma_1$, the positive equilibrium Q_1 of the system (4) is globally asymptotically stable in Ω_2 if $R_1 > 1$.

Proof The uniform persistence of system (4) equals to the unstability of Q_0 [10] . So there is acompact absorbing set $K \subset \Omega_2$ .In the set K the system (4) satisfies the assumptions (B_1) and (B_2).By Lemma 2.1,we just need to prove $q < 0$.

The Jacobian matrix of the system (4) is

$$J = \begin{pmatrix} -\eta_1(a)E - \eta_2(a)I - d & -\eta_1(a)S & -\eta_2(a)S \\ \eta_1(a)E + \eta_2(a)I & \eta_1(a)S - \delta & \eta_2(a)S \\ 0 & \varepsilon & -(d+\gamma_2) \end{pmatrix}$$

The second compound matrix $J^{[2]}$ of J is

$$J^{[2]} = \begin{pmatrix} a_{11} & \eta_2(a)S & \eta_2(a)S \\ \varepsilon & a_{22} & -\eta_1(a)S \\ 0 & \eta_1(a)E + \eta_2(a)I & a_{33} \end{pmatrix}$$

where

$$a_{11} = -\eta_1(a)E - \eta_2(a)I + \eta_1(a)S - d - \delta$$

$$a_{22} = -\eta_1(a)E - \eta_2(a)I - 2d - \gamma_2$$

$$a_{33} = \eta_1(a)S - \delta - d - \gamma_2$$

We set $P(S,E,I) = diag(1, \dfrac{E}{I}, \dfrac{E}{I})$,then

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$$P_f P^{-1} = diag(0, \frac{E'}{E} - \frac{I'}{I}, \frac{E'}{E} - \frac{I'}{I}).$$

Thus the matrix $B = P_f P^{-1} + PJ^{[2]}P^{-1}$ can be written in the block form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Where $B_{11} = \eta_1(a)S - \eta_1(a)E - \eta_2(a)I - d - \delta$

$$B_{12} = \left(\frac{I}{E}\eta_2(a)S, \frac{I}{E}\eta_2(a)S\right), B_{21} = \left(\frac{E}{I}\varepsilon, 0\right)^T$$

$$B_{22} = \begin{pmatrix} \frac{E'}{E} - \frac{I'}{I} - \eta_1(a)E - \eta_2(a)I - 2d - \gamma_2 & -\eta_1(a)S \\ \eta_1(a)E + \eta_2(a)I & \frac{E'}{E} - \frac{I'}{I} + \eta_1(a)S - \delta - d - \gamma_2 \end{pmatrix}$$

Defining the vector norm $\|\cdot\|$ in $R^3 \cong R^{\binom{3}{2}}$ as

$\|(u,v,w)\| = \max\{|u|, |v| + |w|\}$, the Lozinski $\breve{I}$ measure $\mu(B)$ is estimated as follows

$$\mu(B) \le \sup\{g_1, g_2\}$$

where $g_1 = \mu(B_{11}) + |B_{12}|$, $g_2 = |B_{21}| + \mu(B_{22})$

Futhermore $\mu(B_{11}) = \eta_1(a)S - \eta_1(a)E - \eta_2(a)I - d - \delta$,

$$|B_{12}| = \frac{I}{E}\eta_2(a)S, |B_{21}| = \frac{E}{I}\varepsilon$$

To compute $\mu(B_{22})$, we add the absolute value of the off-diagonal elements to the diagonal one in each column of B_{22} ,and $\mu(B_{22})$ is the maximum of the diagonal elements.Noticing that $2a\eta_1(a) < \varepsilon + \gamma_1$,we have

$$\mu(B_{22}) = \frac{E'}{E} - \frac{I'}{I} - 2d - \gamma_2$$

Using $\frac{E'}{E} = \eta_1(a)S + \frac{I}{E}\eta_2(a)S - \delta$ and $\frac{I'}{I} = \frac{\varepsilon E}{I} - d - \gamma_2$,we have

$$g_1 = \frac{E'}{E} - \eta_1(a)E - \eta_2(a)I - d \le \frac{E'}{E} - d, g_2 = \frac{E'}{E} - d,$$

$$\mu(B) \le \sup\{g_1, g_2\} = \frac{E'}{E} - d$$

Therefore $\frac{1}{t}\int_0^t \mu(B)d\tau \le \frac{1}{t}\int_0^{t_1} \mu(B)d\tau + \frac{1}{t}\ln\frac{E}{E_1} - d\frac{t - t_1}{t}$

That is $q = \limsup_{t\to\infty} \sup_{x_0 \in K} \frac{1}{t}\int_0^t \mu(B(x(s,x_0)))ds \le -d < 0$,

then the positive equilibrium Q_1 of the system (4) is globally asymptotically stable in Ω_2.

III. NUMERICAL SIMULATIONS

Example 3.1 We choose the function

$\beta_i(N) = \frac{N}{1 + bN + \sqrt{1 + 2bN}}$ [11], $i = 1,2$. $b = 4, \varepsilon = 0.4, \gamma_1 = 0.2,$ $\gamma_2 = 0.3, d = 0.015, a = 10000, d_1 = 0.001,$ then $R_0 = 0.914 < 1,$

Its phase diagram is illustrated in Figure 1.Numerical simulations show that the solutions approach to the disease-free equilibrium P_0 if $R_0 < 1$

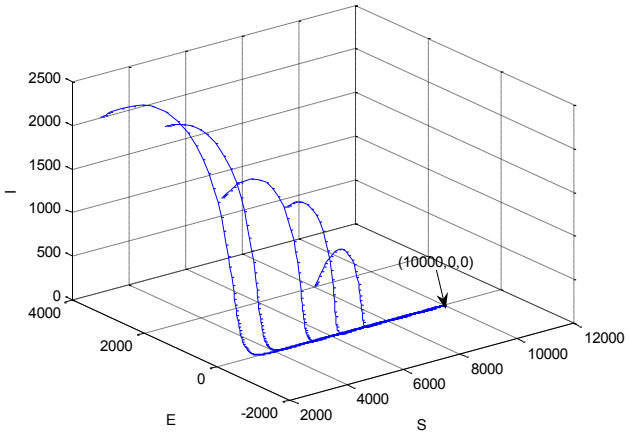

Figure1 The phase diagram of example3.1

Example 3.2 We choose the function

$\beta_i(N) = \frac{N}{1 + bN + \sqrt{1 + 2bN}}$, $i = 1,2$. $b = 3, \varepsilon = 0.4, \gamma_1 = 0.1,$ $\gamma_2 = 0.3, d = 0.015, a = 10000,$ then $R_1 = 1.46 > 1$,Its phase diagram is illustrated in Figure 2.Numerical simulations show that the solutions approach to the positive equilibrium Q_1 if $R_1 > 1$

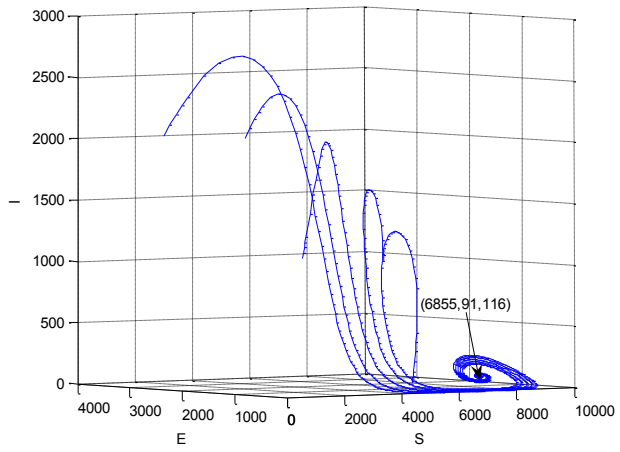

Figure2 The phase diagram of example 3.2

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IV. CONCLUSIONS

This article has illustrated an SEIR epidemic model with infectivity in both latent period and infected period.The basic reproduction number $R_0(R_1)$ is established.When $R_0 < 1$ the disease-free equilibrium is globally asymptotically stable and the disease dies out eventually. When $R_0 > 1$ the disease-free equilibrium is unstable and the positive equilibrium is globally asymptotically stable in certain conditions. Further,the conclusions are verified by numerical simulations .

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