The 7th International Symposium on Operations Research and Its Applications (ISORA'08)
Lijiang, China, October 31–Novemver 3, 2008

Acyclic Edge Colorings of Planar Graphs Without Short Cycles*

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Abstract A proper edge coloring of a graph G is called acyclic if there is no 2-colored cycle in G. The acyclic edge chromatic number of G is the least number of colors in an acyclic edge coloring of G. In this paper, it is proved that the acyclic edge chromatic number of a planar graph G is at most $\Delta(G)+2$ if G contains no i-cycles, $4 \leq i \leq 8$, or any two 3-cycles are not incident with a common vertex and G contains no i-cycles, $i = 4$ and 5.

Keywords acyclic edge coloring; girth; planar graph; cycle.

1 Introduction

In this paper, all graphs are finite, simple and undirected. Let $G = (V,E)$ be a graph, where $V(G)$ and $E(G)$ are the vertex set and the edge set of G, respectively. If $uv \in E(G)$, then u is said to be the *neighbor* of v, and $N(v)$ is the set of neighbors of v. The *degree* $d(v) = |N(v)|$, $\delta(G)$ is the minimum degree and $\Delta(G)$ is the maximum degree of G. A *k-vertex* is a vertex of degree k. Similarly, a *$(\geq k)$-vertex* is a vertex of degree at least k, and a *$(\leq k)$-vertex* is of degree at most k.

A *proper k-edge-coloring* of a graph G is a mapping $\phi : E(G) \to \{1,2,\cdots k\}$ such that no two adjacent edges receive the same color. A proper edge coloring of a graph G is called *acyclic* if there is no 2-colored cycle in G. The *acyclic edge chromatic number* $\chi'_a(G)$ is the smallest integer k such that G has an acyclic edge coloring. The acyclic edge coloring was introduced by Alon et al. in [1], and they proved that $\chi'_a(G) \leq 64\Delta(G)$. Molloy and Reed [5] showed that $\chi'_a(G) \leq 16\Delta(G)$ using the same method. In 2001, Alon, Sudakov and Zaks [2] gave the following conjecture.

Conjecture 1. $\Delta(G) \leq \chi'_a(G) \leq \Delta(G)+2$ for all graphs G.

They proved in the same paper that this conjecture was true for almost all $\Delta(G)$-regular graphs G, and all $\Delta(G)$-regular graphs, whose girth (length of shortest cycle) is at least $c\Delta(G)\log\Delta(G)$ for some constant c. Alon and Zaks [3] proved that determining the acyclic edge chromatic number of an arbitrary graph is an NP-complete problem, even determining whether $\chi'_a(G) \leq 3$ for an arbitrary graph G.

For planar graphs, it is proved in [4] that $\chi'_a(G) \leq \Delta(G)+2$ if $g(G) \geq 5$. In this paper, we prove that $\chi'_a(G) \leq \Delta(G)+2$ if a planar graph G contains no i-cycles, $4 \leq i \leq 8$, or

*This work was partially supported by National Natural Science Foundation of China(10631070, 60673059).
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any two 3-cycles are not incident with a common vertex, and G contains no i-cycles, $i=4$ and 5.

2 Main Result and its Proof

In the section, we always assume that any graph G is planar and is embedded in the plane. We use $F(G)$ to denote the face set of G. The degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A $k(\geq k$, or $\leq k)$-*face* is a face of degree (at least, or at most) k. A $(i,\leq j)$-edge $uv \in E(G)$ is the edge such that $d(u)=i$ and $d(v)\leq j$. A $(i,j,\geq k)$-face uvw is a 3-face such that $d(u)=i$, $d(v)=j$, $d(w)\geq k$ $(i\leq j\leq k)$. For an edge coloring ϕ of G and $v\in V(G)$, let $\Phi(v)=\{\phi(uv)|u\in N(v)\}$.

Theorem 1.
Let G be a planar graph. Then $\chi'_a(G)\leq \Delta(G)+2$ if one of the following conditions holds.

1. *G contains no i-cycles, $4\leq i\leq 8$.*
2. *Any two 3-cycles are not incident with a common vertex, and G contains no i-cycles, $i=4$ and 5.*

Proof. Let G be a minimal counterexample to the theorem. Similar to the proper edge coloring, G is 2-connected and $\delta(G)\geq 2$. Let $k=\Delta(G)+2$ and let L be the color set $\{1,2,\cdots,k\}$ for simplicity. First, we shall prove some results.

(a) *G does not contain an $(2,\leq 3)$-edge.*

Suppose that G does contain such an $(2,\leq 3)$-edge uv such that $d(u)=2$ and $d(v)\leq 3$. Let $N(u)\backslash\{v\}=u_1$ and $G'=G-uv$. By the minimality of G, G' has an acyclic edge coloring ϕ with colors from L. If $\phi(uu_1)\notin\Phi(v)$, then color uv with a color from $L\backslash(\Phi(v)\cup\{\phi(uu_1)\})$. Otherwise, $|\Phi(u_1)\cup\Phi(v)|\leq k-1$ and so we can color uv with a color from $L\backslash(\Phi(u_1)\cup\Phi(v))$. As a result, it is at least 3-colored on any cycle containing the edge uv. Hence we obtain an acyclic edge coloring of G with $\Delta(G)+2$ colors, a contradiction.

(b) *G does not contain a $(2,4,\geq 4)$-face.*

Suppose that G contains such a $(2,4,\geq 4)$-face, say $f=uvwu$, such that $d(u)=2, d(v)=4, d(w)\geq 4$. Let $G'=G-uv$. By the minimality of G, G' have an acyclic edge coloring ϕ with colors from L. If $\phi(uw)\notin\Phi(v)$, then color uv with a color from $L\backslash(\Phi(v)\cup\{\phi(uw)\})$. Otherwise, $|\Phi(v)\cup\Phi(w)|<k$ and it follows that we can color uv with a color from $L\backslash(\Phi(v)\cup\Phi(w))$. Hence we obtain an acyclic edge coloring of G with $\Delta(G)+2$ colors, a contradiction.

(c) *G does not contain a $(3,3,\geq 3)$-face.*

Suppose that G contains such a $(3,3,\geq 3)$-face, say $f=uvwu$, such that $d(u)=3$, $d(v)=3$ and $d(w)\geq 3$. Let G'=$G-uv$, $N(u)\backslash\{w,v\}=\{u_1\}$ and $N(v)\backslash\{w,u\}=\{v_1\}$. By the minimality of G, G' has an acyclic edge coloring ϕ with colors from L. If $\Phi(u)\cap\Phi(v)=\emptyset$, then color edge uv with a color from $L\backslash(\Phi(u)\cup\Phi(v))$. So assume $\Phi(u)\cap\Phi(v)\neq\emptyset$.

If $\phi(uu_1)$=$\phi(vw)$ or $\phi(vv_1)$=$\phi(uw)$, then $|\Phi(w)\cup\{\phi(uu_1),\phi(vv_1)\}|\le k-1$ and it follows that we get a color $i\in L\backslash(\Phi(w)\cup\{\phi(uu_1),\phi(vv_1)\})$ to color uv. Otherwise, we have $\phi(uu_1)$=$\phi(vv_1)$. Without loss of generality, let $\phi(uu_1)=\phi(vv_1)=1$, $\phi(uw)=2$, $\phi(vw)=3$. If there is a color $i\in\{4,5,\cdots k\}\backslash(\Phi(u_1)\cap\Phi(v_1))$, then color uv with i. Otherwise, we have $\{1,4,5,\cdots,k\}=\Phi(u_1)=\Phi(v_1)$ since $|\{1,4,5,\cdots k\}|=\Delta(G)$. Thus we recolor uu_1 with 3, vv_1 with 2, and color uv with 1. Hence we obtain an acyclic edge coloring of G with $\Delta(G)+2$ colors, a contradiction.

(d) *G does not contain a d-vertex adjacent to at least* $(d-2)$ 2*-vertex, where* $d(v)\ge 4$.

Suppose that such a d-vertex, say v, does exists. Let $N(v)=\{u_1,u_2,\cdots,u_d\}$, where $d(u_i)=2$ and $N(u_i)=\{v,w_i\}$, $i=1,2,\cdots,d-2$. By the minimality of G, $G'=G-u_1v$ has an acyclic edge coloring ϕ with colors from L. Without loss of generality, suppose that $\phi(u_iv)=i$ for $i=2,3,\cdots,d$. If $\phi(u_1w_1)\notin\{2,3,\cdots,d\}$, then color u_1v with a color from $L\backslash(\{2,3,\cdots,d\}\cup\{\phi(u_1w_1)\})$. If $\phi(u_1w_1)\in\{2,3,\cdots d-2\}$, without loss of generality, let $\phi(u_1w_1)=2$, then color u_1v with a color from $L\backslash(\{2,3,\cdots,d\}\cup\{\phi(u_2w_2)\})$. So assume that $\phi(u_1w_1)\in\{d-1,d\}$. Without loss of generality, let $\phi(u_1w_1)=d$. If there is a color $i\in\{1,d+1,d+2,\cdots,k\}\backslash\Phi(w_1)$, then color u_1v with color i. Otherwise $\{1,d+1,d+2,\cdots k\}\subseteq\phi(w_1)$. Since $|\Phi(w_1)|\le\Delta(G)$ and $d\in\Phi(w_1)$, there is at least one color $j\in\{2,3,\cdots d-2\}\backslash\Phi(w_1)$. So recolor u_1w_1 with color j, and color u_1v with a color from $L\backslash(\{2,3,\cdots,d\}\cup\{\phi(u_jw_j)\})$. Hence we obtain an acyclic edge coloring of G with $\Delta(G)+2$ colors, a contradiction.

(e) *G does not contain a* 4*-vertex adjacent to a* 2*-vertex and a* 3*-vertex.*

Suppose that there exists a 4-vertex v adjacent to a 2-vertex u and a 3-vertex w. It follows from the above proof that u,v,w are not form a 3-cycle. Let $N(u)\backslash\{v\}=\{u_1\}$, $N(w)\backslash\{v\}=\{w_1,w_2\}$ and $N(v)\backslash\{u,w\}=\{x,y\}$. Then $u_1\notin N(v)$ by (b). Let $G'=G-uv$. By the minimality of G, G' have an acyclic edge coloring ϕ with colors from L. Without loss of generality, suppose that $\phi(vw)=1$, $\phi(vx)=2$ and $\phi(vy)=3$. If $\phi(uu_1)\notin\{1,2,3\}$, then color uv with a color from $L\backslash\{1,2,3,\phi(uu_1)\}$. If $\phi(uu_1)=1$, then color uv with a color from $L\backslash\{1,2,3,\phi(ww_1),\phi(ww_2)\}$. If $\phi(uu_1)\in\{2,3\}$, without loss of generality, let $\phi(uu_1)=2$. If there exists a color $i\in\{4,5,\cdots k\}\backslash\Phi(u_1)$, then color uv with i. Otherwise, $\{2,4,5,\cdots,k\}=\Phi(u_1)$ and then we can recolor uu_1 with 1, and color uv with a color from $L\backslash\{1,2,3,\phi(ww_1),\phi(ww_2)\}$. Hence we obtain an acyclic edge coloring of G with $\Delta(G)+2$ colors, a contradiction.

By Euler's formula $|V|-|E|+|F|=2$, we have

$$\sum_{v\in V(G)}(2d(v)-6)+\sum_{f\in F(G)}(d(f)-6)=-6(|V|-|E|+|F|)=-12<0. \qquad (1)$$

Now we define $w(x)$ to be the initial charge function to each $x\in V(G)\cup F(G)$. Let $w(v)=2d(v)-6$ for $v\in V(G)$ and $w(f)=d(f)-6$ for $f\in F(G)$. In the following, we will reassign a new charge denoted by $w'(x)$ to each $x\in V(G)\cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x\in V(G)\cup F(G)} w'(x) = \sum_{x\in V(G)\cup F(G)} w(x) < 0. \tag{2}$$

If we can show that $w'(x) \geq 0$ for each $x \in V(G) \cup F(G)$, then we obtain a contradiction to (2), completing the proof.

For (I), the discharging rules are defined as follows.

1. R1-1: From each (≥ 4)-vertex to each of its adjacent 2-vertices, transfer 1.
2. R1-2: From each (≥ 4)-vertex v to each of its incident 3-faces, transfer $\frac{w(v)-n_2(v)}{f_3(v)}$, where $n_2(v)$ is the number of 2-vertices adjacent to v, $f_3(v)$ is the number of 3-faces incident with v.
3. R1-3: From each (≥ 9)-face to each of its adjacent 3-faces, transfer $\frac{1}{3}$ through each of its incident edges. (Note: If a (≥ 9)-face and a 3-face are incident with two common edges, then the (≥ 9)-face transfer $\frac{1}{3} \times 2$ to the 3-face.)

Let v be a vertex of G. If $d(v) = 2$, then v is incident with two (≥ 4)-vertices by (a) and it follows by R1-1 that $w'(v) = w(v) + 2 = 0$. If $d(v) = 3$, then $w'(v) = w(v) = 0$. If $d(v) \geq 4$, then $w'(v) \geq w(v) - n_2(v) - \frac{w(v)-n_2(v)}{f_3(v)} \times f_3(v) = 0$.

Now assume that $d(v) \geq 4$ and $f_3(v) \geq 1$. Since G contains no 4-cycles, any two 3-faces are not adjacent. So v is incident with at most $\lfloor \frac{d(v)}{2} \rfloor$ 3-faces. By (d), v is adjacent to at most ($d(v)$-3) 2-vertices. If $d(v) = 4$ and $n_2(v) = 1$, then $f_3(v) = 1$ by (b). So $\frac{w(v)-n_2(v)}{f_3(v)} = 2 \times 4 - 6 - 1 = 1$. If $d(v) = 4$ and $n_2(v) = 0$, then $f_3(v) \leq 2$ and it follows that $\frac{w(v)-n_2(v)}{f_3(v)} \geq \frac{2\times 4-6}{2} = 1$. If $d(v) \geq 5$, then $\frac{w(v)-n_2(v)}{f_3(v)} \geq \frac{2d(v)-6-(d(v)-3)}{\lfloor \frac{d(v)}{2} \rfloor} \geq 1$. Hence we always have

$$\frac{w(v)-n_2(v)}{f_3(v)} \geq 1. \tag{3}$$

Let f be a face of G. Suppose that $d(f) = 3$. Then f must be a $(2, \geq 5, \geq 5)$-face, or a $(3, \geq 4, \geq 4)$-face, or a $(\geq 4, \geq 4, \geq 4)$-face by (b) and (c). If f is a $(\geq 4, \geq 4, \geq 4)$-face, then f can receive at least 1 from each of its incident 4-vertices by R1-2 and (3). So $w'(f) \geq w(f) + 1 \times 3 = 0$. If f is a $(2, \geq 5, \geq 5)$-face, or a $(3, \geq 4, \geq 4)$-face, then f receives at least 1 from each of its incident (≥ 4)-vertices by R1-2 and (3), and $\frac{1}{3}$ from each of its adjacent (≥ 9)-faces through each of its incident edges by R1-3. So $w'(f) \geq w(f) + 1 \times 2 + \frac{1}{3} + \frac{1}{3} \times 2 = 3 - 6 + 3 = 0$. If $d(f) \geq 9$, then it follows from R1-3 that $w'(f) \geq w(f) - \frac{1}{3} \times d(f) \geq 0$.

For (II), the discharging rules are defined as follows.

1. R2-1: From each (≥ 4)-vertex to each of its adjacent 2-vertices, transfer 1.

2. R2-2: From each (≥ 4)-vertex v to each of its incident 3-faces, transfer $(w(v) - n_2(v))$, where $n_2(v)$ is the number of 2-vertices adjacent to v.

Let v be a vertex of G. Since any two 3-cycles have no the same vertex in common, v is incident with at most one 3-face. If $d(v) = 2$, then v is adjacent to two (≥ 4)-vertices by (a) and it follows by R2-1 that $w'(v) = w(v) + 2 = 0$. If $d(v) = 3$, then $w'(v) = w(v) = 0$. Now assume that $d(v) \geq 4$. It follows by R2-2 that $w'(v) \geq w(v) - n_2(v) - (w(v) - n_2(v)) = 0$. At the same time, we know that v is adjacent to at most $(d(v) - 3)$ 2-vertices by (d). If $d(v) = 4$ and $n_2(v) = 1$, then $w(v) - n_2(v) = 2 \times 4 - 6 - 1 = 1$. If $d(v) = 4$ and $n_2(v) = 0$, then $w(v) - n_2(v) = 2$. If $d(v) \geq 5$, then $(w(v) - n_2(v)) = 2d(v) - 6 - (d(v) - 3) = d(v) - 3 \geq 2$.

Let f be a face of G. Suppose that $d(f) = 3$. Then f must be a $(2, \geq 5, \geq 5)$-face, or a $(3, \geq 4, \geq 4)$-face, or a $(\geq 4, \geq 4, \geq 4)$-face by (b) and (c). If f is a $(\geq 4, \geq 4, \geq 4)$-face, then f can receive at least 1 from each of its incident 4-vertices by R2-2. So $w'(f) \geq w(f) + 1 \times 3 = 0$. If f is a $(2, \geq 5, \geq 5)$-face, or a $(3, \geq 4, \geq 4)$-face, then f receives at least 2 from each of its incident (≥ 4)-vertices by R2-2 and (e). So $w'(f) \geq w(f) + 2 \times 2 = 3 - 6 + 4 > 0$. If $d(f) \geq 6$, then $w'(f) = w(f) \geq 0$.

Hence we have $w'(x) \geq 0$ for each $x \in V(G) \cup F(G)$, a contradiction with (2). □

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