The 7th International Symposium on Operations Research and Its Applications (ISORA'08)
Lijiang, China, October 31–Novemver 3, 2008

Coordinated Pricing and Inventory in A System with Minimum and Maximum Production Constraints

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Abstract Most inventory researches assume that the production level can be fluctuated arbitrarily. However, in many firms, large fluctuation in production levels may be very costly. This paper addresses the coordinated pricing and inventory control problem in a smoothing production system, in which the production level is constrained between a maximum and a minimum level in each period and the price can be adjusted dynamically. We show that the optimal policy is modified-modified base stock list price policy.

Keywords Smoothing production system; pricing, inventory control; supply chain

1 Introduction

This research is based on the following observations: On one hand, the demand fluctuates more and more largely because of development of sciences and technologies and individualization of demands. On the other hand, production changes from period to period may incur large cost. In order to overcome such conflict, firms often exploit dynamic pricing and promotion to manage and smooth demand. Meanwhile, firms also stock the excess products when the demand is low and backlog the excess demand when the demand is high. These methods can make the production as smooth as possible. However, dynamic pricing and promotion are decided by the marketing unit while production plan is decided by the manufacture unit. It is very difficult to coordinate these decisions. We note that demand fluctuation become larger and larger because of improper pricing and promotion in reality. In this paper, we study the coordinated decision problem on pricing and inventory control with production smooth constraints to maximize the total profit in a firm. The goal is to characterize the structure of the optimal policy and to get some more management insights.

The research on production smoothing can be dated back to1950s. Modiglianli and Hohn (1955)is the first one to propose a production smoothing model. Arrow and Karlin (1958), Dobos (1990), Holt et. al (1960), Bergstrom and Smith (1970)studied this problem also. A common characteristic of these researches is that they assume that the production function is increasing and convex function in the production volume. This assumption can prevent the production from too large or too small. Bowman (1956), McGarrah (1963)studied this problem by using linear programming. Recently, Chan and Muckstadt (1999)propose a new model, which assumes that the production in each period is larger than l and less than u. They showed that the optimal policy is the modified—modified base stock policy. That is, in each period n, there exists a critical point y_n^* such that when the initial inventory x satisfies $y_n^* - u \leq x \leq y_n^* - l$, product to bring the inventory level to y_n^*; otherwise if $x \leq y_n^* - u$, production u is and if $x \geq y_n^* - l$, production is l. However, these researches do not consider the impact of the price on the demand.

Since Whitin (1955) first studied newsboy problem with price-dependent demand, there are numerous literature on the coordinated decision on pricing and inventory control. Yano and Gilbert (2003), Elmaghraby and Keskinocak (2003) gave excellent review. Recently, Federgruen and Heching(1999) studied multi-period joint decision problem on pricing and inventory control. For the case with zero fixed cost, they proved that the optimal policy is base-stock list-price policy. Chen and Simchi—levi(2004) considered the case with positive fixed-cost. They show that the optimal policy is (s,S,p) policy for the additive demand function and (s,S,A,p) policy for general demand function. However, these researches do not consider the production constraint and smoothing.

This paper studies the joint decision problem on pricing and inventory control under production constraint. In each period, the manager checks the initial inventory, then decides the production quantity and the sale price in this period. The production belongs to $[l,u]$, where u is the maximum production volume(the production capacity is finite) and l is the minimum production volume. We show that the optimal policy is modified—modified base stock list price policy(which is defined in the following section).

2 Finite-period Problem

Consider $n-$periodic review coordinated pricing and inventory control problem. In period $n = 1,2,...,N$, the demand function is

$$d(p,\varepsilon) = \alpha - \beta p + \varepsilon \tag{1.1}$$

where $\alpha > 0$ is the market size,$\beta > 0$ is price elasticity and ε is a random variable with $E(\varepsilon) = 0$.This is the linear demand function, which has been used extensively in economic and management literature (Petruzzi and Dada(1993)). (Here we only consider linear demand function. We can use the technique in Chen and Simchi-levi (2004) to prove that the results in this paper hold for general demand function. Let $\underline{p}$ and $\overline{p}$ be the minimum and maximum price respectively.

In each period, the sequence of the events is as follows: we check the initial inventory x first, then determine the inventory level y and price p. We assume that the lead time is 0 and the production is finished immediately. The demand realizes and is satisfied by the available inventory. The excess inventory incurs an inventory cost h and the unsatisfied demand is backlogged and incurs a penalty cost s. Note that the maximum production is u and the minimum production is l and the price $p \in [\overline{p}, \underline{p}]$, the decision space in period n is

$$Y_n(x,p) = \left\{ y : x + l \leq y \leq x + u, p \in [\underline{p}, \bar{p}] \right\} \tag{1.2}$$

Let $G(y)$ be the one-period holding-penalty cost, then

$$G(y,p) = hE(y-a+bp-\varepsilon)^+ + sE(a-bp+\varepsilon-y)^+ \tag{1.3}$$

where $x^+ = \max\{0,x\}$. Then the one-period profit function is

$$\Pi(y,p) = p(\alpha-\beta p) - c(y-x) - G(y,p)\, eqno(1.4)$$

Let $v_n(x)$ be the maximal profit function from period n to period N with the initial inventory x in period n, then $v_n(x)$ satisfies the following dynamic equation

$$v_n(x) = \max_{(y,p)\in Y_n(x,p)} J_n(y,p) \tag{1.5}$$

where

$$J_n(y,p) = \Pi(y,p) + \gamma E v_{n+1}(y-\alpha+\beta p-\varepsilon) \tag{1.6}$$

and

$$v_{N+1}(\bullet) = J_{N+1}(\bullet,\bullet) = 0 \tag{1.7}$$

where $0 < \gamma \leq 1$ is the discounted factor.

We will show that the optimal policy of dynamic programming (1.5)-(1.7) has the following structure:there exists a base-stock y_n^* and a list price p_n^* ((y_n^*, p_n^*) depends on l and u) such that when the initial inventory x is less than $y_n^* - u$, production quantity is u and charge a price p_n^* which is higher than; If $x \in [y_n^* - u, y_n^* - l]$, then product up to y_n^* and charge the price p_n^*; If x is larger than $y_n^* - l$, production quantity is l and charge a discounted price p_n^*. This policy is called modified-modified base stock list price policy.

Lemma 2.1 *$J_n(y,p)$ is joint concave in y andp, $v_n(x)$ is concave in x*

Proof Since $g(x) = hx^+ + s(-x)^+$ is convex and $y-\alpha+\beta p-\varepsilon$ is linear in y and p for any ε, $G(y,p) = Eg(y-\alpha+\beta p-\varepsilon)$ is joint convex in y, p. Note that $p(\alpha-\beta p)$ is concave in p and $c(y-x)$ is linear in y, $\Pi(y,p)$ is a joint concave function in y and p. Because $v_{N+1}(\bullet) = 0$ is a concave function, we only need to prove that $J_t(y,p)$ is joint concave in y, p, and $v_t(x)$ is concave in x if $v_{t+1}(\bullet)$ is a concave function.

Suppose that $v_{t+1}(\bullet)$ is a concave function, then $E[v_{t-1}(y-\alpha+\beta p-\varepsilon)]$ is a joint concave function in y and p since $y-\alpha+\beta p-\varepsilon$ is linear function in y, p. It follows from (1.6)and that $\Pi(y,p)$ is joint concave in y and p that $J_t(y,p)$ is joint concave in y, p.

In order to prove that $v_t(\bullet)$ is concave, let $a_t(y) = \max\limits_{\underline{p}\leq p\leq \bar{p}} J_t(y,p)$. Form the concave preservation of maximization (Heyman and Sobel(1984)), $\alpha_t(y)$ is concave. From the definition of $v_t(x)$, $v_t(x) = \max\limits_{x+l\leq y\leq x+u} a_t(y)$. Since $\alpha_t(y)$ is concave function, there exists a y_t^* maximizer such that

$$v_t(x) = \begin{cases} a_t(x+u) & x \leq y_t^* - u \\ a_t(y_t^*) & y_t^* - u \leq x \leq y_t^* - l \\ a_t(x+l) & y_t^* - l \leq x \end{cases}$$

Taking derivative in x, we have

$$v_t'(x) = \begin{cases} a_t'(x+u) \geq 0 & x \leq y_t^* - u \\ 0 & y_t^* - u \leq x \leq y_t^* - l \\ a_t'(x+l) \leq 0 & y_t^* - l \leq x \end{cases}$$

$v_t'(x)$ is a decreasing function since $a_t(y)$ is concave function. Then $v_t(x)$ is a concave function.■

Now we prove that the optimal price is decreasing in the inventory level, that is , for any inventory level y, if we set $p_t(y) = \arg\max_{p\in[\underline{p},\bar{p}]} J_t(y,p)$, then $p_t(y)$ is decreasing in y. In order to do this, we only need to prove that $J_t(y,p)$ is a submodular function, i.e., for any given $y_1 < y_2$, $J_t(y_2,p) - J_t(y_1,p)$ is decreasing in p (Topkis (1998)). Since the sum of submodular functions is also a submodular function, we need to prove that $-g(y,p)$ and $Ev_{t+1}(y-\alpha+\beta p-\varepsilon)$ are submodular functions. Since $-g(\bullet)$ and $v_{t+1}(\bullet)$ are concave, from Lemma 2.6.2 (a) in Topkis (1998) we have that $-G(y,p) = E[-g(y-\alpha+\beta p-\varepsilon)]$ and $Ev_{t+1}(y-\alpha+\beta p-\varepsilon)$ are submodular functions. Then $J_t(y,p)$ is a submodular function. From Theorem 8-4 in Heyman and Sobel (1984) we have that $p_t(y)$ is decreasing in y. Summarize above, we have

Lemma 2.2 *The optimal price* $p_t(y)$ *decreases in the inventory level y*

Combine Lemma 2.1and Lemma 2.2, we have

Theorem 2.1 *The optimal policy of (1.5)-(1.7)is the modified-modified base stock list price policy*

Proof From Lemma 2.1, the optimal production policy is

$$y_t^{opt} = \begin{cases} x+u & x \leq y_t^* - u \\ y_t^* & y_t^* - u \leq x \leq y_t^* - l \\ x+l & y_t^* - l \leq x \end{cases}$$

Once we determine the optimal inventory level, the optimal price is determined by $p_t(y) = \arg\max_{p\in[\underline{p},\bar{p}]} J_t(y,p)$. From Lemma2.2, $p_t(y)$ is decreasing in y. The conclusion holds.■

Note that the optimal inventory level y_t^{opt} is increasing in the minimum and maximum production levels. Theorem 2.1 shows that the optimal price is decreasing in the minimum and maximum production levels. This is so because if the minimum production level is large, the production volume is large in each period and the inventory increases, we need to reduce the price to incentive the demand. If the maximum production level is large, the capacity is abundance; we need to reduce the sale price to increase profit.

If $l = 0$, our model is reduced to Federgruen and Heching's model (Federgruen and Heching (1999)), the modified-modified base-stock list-price is reduced to the modified base-stock list-price policy. The results in Federgruen and Heching (1999) are the special cases of our model.

3 Infinite-period Problem

In this section, we address the infinite period problem. In this case, we assume that $0 < \lambda < 1$. From the proof of Lemma 2.1, $\Pi(y,p)$ is joint concave in y and p and the decision space $Y_n(x,p)$ is compact. Then $\Pi(y,p)$is bounded above in $Y_n(x,p)$. Let the upper bound be M. Substituting $\Pi(y,p)-M$ for $\Pi(y,p)$, we obtain a equivalent dynamic programming in which the one period profit is non-positive. We can apply the results on non-positive one-period profit Markov decision to the infinite period problem. Note that

$$\Lambda^{t.x}(\lambda) = \left\{(y,p) : \Pi(y,p) + \alpha E v_{t+1}(y-\alpha+\beta p-\varepsilon) \geq \lambda, x+u \leq y \leq x+l, \underline{p} \leq p \leq \bar{p}\right\}$$
$$\subset \left\{(y,p) : x+u \leq y \leq x+l, \underline{p} \leq p \leq \bar{p}\right\}$$

is compact, it follows from Theorem 9.12, 9.16 and 9.17 in Bertsekas and Shreve(1978) that when $t \to \infty$ we have

Theorem 3.1 $(a)\, v_t(x) \to v(x)$

$(b)\, v(x)$ *satisfies the following optimal equation*

$$v(x) = \max_{(y,p)\in Y(x,p)} J(y,p)$$

where $J(y,p) = \Pi(y,p) + \alpha E v(y-\alpha+\beta p-\varepsilon)$

(c) The stationary policy (y^*, p^*) *maximizing the right hand of (1.8) is the optimal policy for the infinite period problem.*

From Lemma 2.1, 2.2 and Theorem 2.1, we have

Theorem 3.2 (a) $J(y,p)$ *is joint concave in y, p* and $v(x)$ *is concave in* x;

(b) $y_t^{opt} \to y^*, p_t^{opt} \to p^*$, *the optimal policy of the infinite period problem is the modified-modified base stock list price policy.*

4 Conclusion

In this paper, we study the coordinated decision problem on pricing and inventory control in a smoothing production system, in which the production level is bounded by a minimum and maximum production level. For finite and infinite period problem, we show that the optimal policy is the modified-modified base stock list price policy. That is, in each period, there exist a base stock y_t^* and a list price p_t^*, when the initial inventory $y_t^* - u \leq x \leq y_t^* - l$, product up to y_t^* and charge the price; If $x \geq y_t^* - u$, production quantity is u, and set a price which is larger than p_t^*; If $x \geq y_t^* - l$, production quantity is l, and set a discounted price p_t^* which is less than p_t^*. The simple structure of the optimal policy can help firms manage the production plan and pricing. Moreover, we prove that the optimal price is decreasing in the inventory level as well as the minimum and maximum production level. This shows that if the minimum production level is large, the production volume is large and the inventory increases, we need to reduce the price to incentive the demand. If the maximum production level is large, the capacity is abundance; we need to reduce the sale price to increase profit.

References

[1] Arrow, K., Karlin, S., Production over time with increasing marginal costs, in *Studies in the Mathematical Theory of Inventory and Production,* Stanford, 61-69, Stanford University Press, 1958

[2] Bergstrom, G.L., Smith, B., Multi-item production planning – An extension of the HMMS rules, *Management Science,* 16(10), 614-629, 1970

[3] Bertsekas, D. P., Shreve, S. E., *Stochastic Optimal Control*, Academic Press, New York, 1978. Bowman, E., Production scheduling by the transportation method of linear programming, *Operations Research*, 4(1), 100-103, 1956

[4] Chan, E. W., Muckstadt, J. A., The effects of load smoothing on inventory levels in a capacitated production and inventory system, working paper, Cornell University, Ithaca NY

[5] Chen, X., Simchi-Levi, D. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case, *Operations Research*, Vol. 52, 887-896, 2004.

[6] Dobos, I., The Modigliani-Hohn model with capacity and warehousing constraints, *International Journal of Production Economics,* 24, 49-54, 1990

[7] Elmaghraby, W. and Keskinocak, P., Dynamic pricing in the presence of inventory considerations: research overview, current practices, and future directions, *Management Science*, Vol. 49, No. 10, 1287-1309, 2003.

[8] Federgruen, A. and Heching, A., Combined pricing and inventory control under uncertainty, *Operations Research*, Vol. 47, No.3, 454-475, 1999

[9] Heyman, D. P. and Sobel, M. J., *Stochastic Models in Operations Research Volume II*, McGraw-Hill, New York, 1984.

[10] Holt, C.C., Modigliani, F., Muth, J.F., Simon, H.A., *Planning production, inventories and work force*, Prentice-Hall, Englewood Cliffs, NJ. 1960.

[11] McGarrah, R. E., *Production and Logistics Management: Text and Cases,* New York: John Wiley and Sons, 1963

[12] Modigliani, F., Hohn, F., Production planning over time and the nature of the expectation and planning horizon, *Econometrica*, 23, 46-66, 1955

[13] Petruzzi, N. and Dada. M., Pricing and the newsvendor problem: A review with extensions, *Operations Research*, Vol. 47, 183-194, 1999

[14] Topkis, D. M., *Supermodularity and Complementarity,* Princeton University Press, Princeton, NJ, 1998.

[15] Whitin, T. M., Inventory control and price theory, *Management Science*, Vol. 2, 61-80, 1955

[16] Yano, C.A., Gilbert, S.M., Coordinated pricing and production/procurement decisions: a review. In: *Managing Business Interfaces: Marketing, Engineering and Manufacturing Perspectives*, A. Chakravarty and J. Eliashberg (eds.), Kluwer Academic Publishers, 2003.