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The Equilibrium Equivalent Representation for Variational Inequalities Problems with Its Application in Mixed Traffic Flows

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Abstract In this paper, we first introduce the relationship between the Hartman-Stampacchia variational inequalities (VI) problems and equilibrium programming (EP), and we present EP equivalent representation for Hartman-Stampacchia VI problems. After analyze and discuss the condition for optimization of EP, and the relation with the model transformation equivalent applied in mixed traffic assignment problems (MTAP). Finally we give an optimal model and effective algorithm for MTAP.

Keywords Optimization; Hartman-Stampacchia variational inequalities; equilibrium programming; mixed traffic assignment

1 Introduction

Variation inequalities (VI) theory is a very powerful tool of the current mathematical technology. In recent years, the classical VI problems and the parametric optimization (PO) problems have been extended and generalized to study a large variety of problems arising in economics mathematics, game theory, optimization and network equilibrium [1-3]. Equilibrium programming (EP) is one of the most important PO problems [4].

Hartman-Stampacchia VI problem is to determine a vector $x^* \in K$, such that

$$F(x^*)^T(x - x^*) \geq 0, \quad \forall x \in K, \tag{1}$$

Where $K \subseteq R^n$ is a nonempty closed convex set, $F(x) : K \to R^n$ is a continuous mapping. VI problem (1) is the first proposed by Hartman and Stampacehia in 1960's at the early age of VI problem. It is called Hartman-Stampacehia VI problem. On the one hand, since is of closed convexity and it is a continuous set, so VI problem (1) is an infinitive dimensional inequalities which is an essentially expansion of the traditional inequalities. On the other hand, the relation between VI problem (1) and optimization as well as EP has been received much attention by researchers. If $F(x)$ is a grade of the differentiable

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convex function, VI problem (1) is equivalent to a differentiable convex programming problem [5]. In general, VI problem (1) can be transferred into an EP [6].

EP problem is to find $(x^*, y^*) \in K$, $K \subseteq R^{n+m}$, such that

$$\begin{cases} F(x^*, y^*) \leq F(x^*, y), \\ G(x^*, y^*) \leq G(x, y^*), \\ \forall (x, y) \in K. \end{cases} \tag{2}$$

Where $K \subseteq R^{n+m}$ is a nonempty closed convex set and $F(x,y), G(x,y) : K \to R^{n+m}$ are continuous mappings.

EP has many applications in equilibrium situations of science, engineering and economics. And traffic assignment problem (TAP) is often model as an EP, which is equivalent to VI problem. The mixed traffic assignment problem (MTAP) is naturally model as an EP and it is usually solved by the traditional diagonal method [7,8]. The relation between VI problem (1) and mathematical programming (MP) has been received much attention from the beginning. If is a gradient of the differentiable convex function, obviously VI problem (1) can be equivalently transformed into a differentiable mathematical programming. Ref.[9] and Ref.[10] detailedly discussed its application in economic equilibrium problems and the traffic equilibrium problem. Under this condition, the above VI problem (1) can be written as follows:

$$\min_x G(x) \tag{3}$$

$$s.t.\ x \in K \tag{4}$$

where $G(x)$: $R^n \to R$ is the differentiable convex function, and satisfies the following equation:

$$F(x) = (\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \cdots, \frac{\partial G}{\partial x_n}) \tag{5}$$

Under asymmetric condition, VI problem (1) generally cannot be transformed into the optimization equivalent representation in traditional sense. Fukushima (1992) proposed an equivalent differentiable optimization representation by introducing a projection operator. And Larsson and Patriksson (1994) proposed a class of equivalent differentiable optimization representation in more general condition, hence theoretically proved the relation between VI problem (1) and differentiable mathematical programming. However, the above transfor-mation methods require the restriction of strong mathematical conditions or lose the intuitional characteristics of optimization modelling. Therefore, it is unconducive to modelling and solving of practical problems.

Researchers have recognized that EP is an optimization representation for Nash equilibrium problem, and has deep relation with VI problem. Zuhovickii (1969) studied the relation between Nash equilibrium and Saddle point: x^* is a solution of VI problem (1) if and only if (x^*, x^*) is a saddle point of $F(x)^T(y-x)$. In this work, we find that the relation between VI and EP is useful for modelling and solving practical problems, and the characteristic is much in evidence to analyze network equilibrium problems.

This paper is organized as follows. Section 2 discusses the relationship between VI problem and EP, and gives the equivalent optimization representation. Section 3 gives an optimal model for MTAP and analyzes the application of the model equivalent transformation to traffic network equilibrium. Finally, we give an effective algorithm for the model.

2 EP Equivalent Representation for VI Problem

Based on the relation between VI problem and MP, and the relation between EP and MP, we propose the transformation of the relationship between VI problem and EP.

Theorem 1. *Let $K \subseteq R^n$ be a nonempty closed convex set and $F(x): K \to R^n$ be a continuous mapping. Then x^* is a solution of VI problem (1) if and only if (x^*, x^*) is the solution to the following EP.*

$$\begin{cases} \min\limits_{x\in K} F(y)x, \quad \forall y \in K, \\ \\ \min\limits_{y\in K} F(x)y, \quad \forall x \in K. \end{cases} \tag{6}$$

Proof. Let x^* be a solution of VI problem (1), then, for any $x \in K$, we have

$$F(x^*)^T (x - x^*) \geq 0, \tag{7}$$

therefore,

$$\min_{x\in K} F(x^*)^T x, \quad \forall x \in K. \tag{8}$$

Conversely, if (x^*, y^*) is a solution to EP (6), then $x^* = y^*$, and

$$F(x^*)^T y \geq F(x^*)^T x^*, \quad \forall y \in K, \tag{9}$$

therefore, x^* is a solution to VI problem (1). The proof is complete.

According to the existence and uniqueness condition of solutions of VI problem, if K is a bounded closed convex set, and $F(x)$ is continuous on K, then there exists at least one solution to VI problem. If $F(x)$ is strictly monotone function on K, and the solution of VI problem is nonempty, then EP (6) has only one equilibrium solution x^*.

For the case that feasible region K is unbounded, the strong monotonicity of function $F(x)$ can ensure the existence and uniqueness of solution, i.e., if $F(x)$ is strongly monotone or coercive, then VI problem (1) has one and only solution (see Ref.[5]).

The above equilibrium can be regarded a situation in which system optimization is consistent with the forecast. Thus, we have the following theorem.

Theorem 2. *VI problem (1) is equivalent to the following EP.*

$$\begin{cases} \min\limits_{y\in K} F(x)^T y, \\ \min\limits_{x\in K} \|x - y\|^2, \end{cases} \tag{10}$$

where K is the same as before.

Proof. If x^* is a solution to VI problem (1), then x^* is also a solution of $\min_{y\in K} F(x^*)^T y$. So (x^*, x^*) is a solution to EP (10). Conversely, if $(x^*, x^*)$is a solution to EP (10), then for any $y \in K, x^* = y^*$, $F(x^*)^T y \geq F(x^*)^T x^*$. So x^* is a solution to VI problem (1). The proof is complete.

From the above results, we can explain the following traffic flow equilibrium for the VI problem: Regard $F(x)$ as vector function of unit cost, and forecast a situation x of system equilibrium. Given x, to find the system equilibrium situation. When forecast is consistent with user equilibrium, this x corresponds to a solution of VI problem (1). When vector function $F(x)$ satisfies symmetric condition, VI problem (1) has the following optimization form. Specially, when $F(x)$ is the gradient of a differentiable function, we have the following corollary.

Corollary 1. *If $F(x)$ is symmetric, especially, if $F(x)$ is the gradient vector function, then EP (6) is equivalent to the following optimization problem*

$$\min_x G(x), \tag{11}$$

$$s.t.\ x \in K. \tag{12}$$

3 The Application of EP Model in MTAP

The mixed traffic equilibrium problem is a typical one of EP. In the real traffic network, traffic flows usually consist of two or more kinds of vehicles, such as motor vehicle and non-motor vehicle running in a mixed way. Generally, the interaction between different kind vehicles is asymmetric, similar to the conclusion drawn by Smith. In this condition, we cannot establish equivalent optimization model in the traditional sense for this mixed traffic system equilibrium problem. Note that Bechmann equivalent transformation of the single kind system equilibrium model and the characteristics of mixed traffic flow, by the EP theory, we can easily set EP model for mixed traffic equilibrium problem.

3.1 Optimal model

Here, we only discuss deterministic user equilibrium assignment problem for two kind vehicles (*e.g. motor vehicle and non-motor vehicle*).

Denote the flows of the two kind vehicles on link path a as x_a and $\hat{x}_a$ respectively, and the travel time (*or cost*) function as $t_a(x_a, \hat{x}_a)$ and $\hat{t}_a(\hat{x}_a, x_a)$ respectively. When the one mode vehicle's flow pattern can be given, the other mode vehicle's user equilibrium flow can be obtained by solving MP. Therefore, the MTAP can be formulated in terms of the EP as follows:

$$\begin{cases} \min\limits_{x_a} \sum\limits_a \int_0^{x_a} t_a(w, \hat{x}_a) dw, \\ \\ \min\limits_{\hat{x}_a} \sum\limits_a \int_0^{\hat{x}_a} \hat{t}_a(v, x_a) dv, \end{cases} \tag{13}$$

$$s.t. \quad q_{rs} = \sum_k f_k^{rs}, \quad \forall (r,s) \in C, \tag{14}$$

$$\hat{q}_{rs} = \sum_k \hat{f}_k^{rs}, \quad \forall (r,s) \in C, \tag{15}$$

$$x_a = \sum_{r,s} \sum_k f_k^{rs} \delta_{a,k}^{rs}, \quad \forall a \in A, \tag{16}$$

$$\hat{x}_a = \sum_{r,s} \sum_k \hat{f}_k^{rs} \delta_{a,k}^{rs}, \quad \forall a \in A, \tag{17}$$

$$f_k^{rs}, \hat{f}_k^{rs} \geq 0, \quad \forall k \in K_{rs}, \tag{18}$$

where A is the set of directed links, C is the set of centroid points, K_{rs} is the set of paths from r to s, q^{rs} and $\hat{q}^{rs}$ are the travel demand of the two mode vehicles from r to s respectively, f_k^{rs} and $\hat{f}_k^{rs}$ are the flows of the two mode vehicles on path k from r to s respectively, and $\delta_{a,k}^{rs}$ is the incidence function, $a \in K_{rs}$ if ,$\delta_{a,k}^{rs}$ =1, otherwise $\delta_{a,k}^{rs}$=0.

According to EP theory, because the constraints (14)-(18) are linear, therefore, if $\hat{t}_a$ and $t_a$are continuous, and $\hat{x}_a$ and x_a are strictly monotone function, then MTAP exists unique equilibrium solution $(x_a^*, \hat{x}_a^*), \forall a \in A$ [7].

If $(x_a^*, \hat{x}_a^*)$is a solution to MTAP, by the solution of EP, $x_a^*$and $\hat{x}_a^*$ are UE solutions respectively. Therefore, $(x_a^*, \hat{x}_a^*)$ satisfies Wardropian UE principle. Which can be derived from one order condition of EP solution (see Ref.[11]).

Let Jacobi matrix be a positive definite matrix, to insure the traffic flow assignment is unique. The convergence of the solution has been proved in Ref.[11]. The procedure is as follows.

3.2 Algorithm

Step 1: Initialize the feasible solutions $X^{(1)} = (\cdots, x_a^{(1)}, \cdots)$ and $\hat{X}^{(1)} = (\cdots, \hat{x}_a^{(1)}, \cdots)$ to MTAP, let $n = 1$.

Step 2: Given $\hat{X}^{(n)}$ and $X^{(n)}$, to solve the following subproblem

$$min\, z^{(n)}(X^{(n)}, \hat{X}^{(n)}) = \sum_a [\int_0^{x_a} t_a(w, \hat{x}_a^{(n)}) dw + \int_0^{\hat{x}_a} \hat{t}_a(v, x_a^{(n)}) dv], \tag{19}$$

$$s.t. \quad q_{rs} = \sum_k f_k^{rs}, \quad \forall (r,s) \in C, \tag{20}$$

$$\hat{q}_{rs} = \sum_k \hat{f}_k^{rs}, \quad \forall (r,s) \in C, \tag{21}$$

$$x_a = \sum_{r,s} \sum_k f_k^{rs} \delta_{a,k}^{rs}, \quad \forall a \in A, \tag{22}$$

$$\hat{x}_a = \sum_{r,s} \sum_k \hat{f}_k^{rs} \delta_{a,k}^{rs}, \quad \forall a \in A, \tag{23}$$

$$f_k^{rs}, \hat{f}_k^{rs} \geq 0, \quad \forall k \in K_{rs}, \tag{24}$$

and denote the new iteration points as $X^{(n+1)}$ and $\hat{X}^{(n+1)}$.

Step3: Terminate check.

When $\left\|X^{(n)} - X^{(n+1)}\right\| \leq \varepsilon$ and $\left\|\hat{X}^{(n)} - \hat{X}^{(n+1)}\right\| \leq \varepsilon$, to terminate the procedure, otherwise, let $n = n+1$, go to Step 2.

It can be proved that the above output results are equivalent with mixed traffic user equilibrium flows.

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