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Departure Processes of a Tandem Network

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Abstract Consider a 2-stage single-server tandem queue with a MAP to the first stage and the exponential service times. Using the DREB scheme, we formulate the joint queue length process into a single-dimensional level-dependent quasi-birth-death (LDQBD) process with expanding blocks. This allows us to show that the departure process from stage 1 is a MAP with infinite phases or IMAP and that the departure process of any $IMAP/M/1$ is still an IMAP.

Keywords Tandem network; Level-dependent QBD; Departure process; IMAP.

1 Introduction

We consider a two-stage tandem queueing network. Each stage has a single exponential server with an unlimited input buffer. Customers arrive to stage 1 following a Markovian arrival process (MAP). Using the DREB formulation scheme, Lian and Liu [1] construct a single-dimensional level-dependent quasi-birth-death (LDQBD) process with expanding blocks to study the queue length processes and the system sojourn time process. In this paper, we study the busy period and idle period in stage 2. We also study the departure process from each stage. To do this, we introduce a useful arrival process, the IMAP, which stands for infinite dimensional Markov arrival process. MAP with an infinite dimension has not been formally studied in the literature, although it has appeared in a number of works including Sadre and Havercourt [2], Miyazawa [3], Miyazawa and Zhao [4], and Zhang, Heindl, and Smirni [5]. One interesting work is by Green [6] on the output process from a $MAP/M/1$ queue. He finds a set of conditions on the input MAP under which the output process is still a MAP (with a finite dimension). Though in general, the departure process of an $MAP/M/1$ queue is not a finite MAP, Green restricts his study to the finite cases only. We pick up from where Green left untouched. We found that an IMAP arises naturally as the departure process of a $MAP/M/1$ queue (Theorem 1), and certain properties of the IMAP are important and are very helpful to queueing analysis. For example, the fact that the departure process from an $IMAP/M/1$ queue is still an IMAP (Theorem 2) shows that IMAP has good closure properties. As such, it is natural and important to define and discuss IMAP formally.

This paper is organized as follows. In Section 2, we define the 2-station tandem network. In Section 3, we define IMAP and show that the departure process from $IMAP/M/1$ is still an IMAP. We conclude the paper in Section 4.

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2 System Description

External customers arrive to stage 1 of a two-stage tandem network only following a MAP with an infinitesimal generator $D = D^0 + D^1$ in state space $\{1, \cdots, m\}$, where $D^0 = \left(D^0_{i,j}\right)_{m\times m}$ and $D^1 = \left(D^1_{i,j}\right)_{m\times m}$. All the off-diagonal elements of D^0 and all the elements of D^1 are nonnegative. The transitions associated with D^1 are called type-1 transitions. A customer arrives only at a type-1 transition epoch. Assume irreducibility of the underlying Markov chain D and let z denote its stationary probability vector. Then, z is uniquely determined by $z(D^0 + D^1) = 0$ and $z\mathbf{1} = 1$, and the mean arrival rate of the MAP is $\lambda = zD^1\mathbf{1}$, where $\mathbf{1}$ is a vector with all elements being equal to 1. To avoid trivialities, we assume $D^1 \neq O$, where O denotes a matrix of zeros, so that $\lambda > 0$. Each of the two stages has a single exponential server with service rates μ_1 and μ_2, respectively.

Let $N_i(t)$ be the number of customers in station i, $i = 1,\ 2$, at time t, including the customer being serviced by the server. Let $\rho = \lambda / \min\{\mu_1, \mu_2\}$ denote the traffic intensity of the network. Throughout the paper we assume that the stability condition $\rho < 1$ holds. Let $J(t)$ be the phase of the MAP at time t. Then $\{N_1(t), N_2(t), J(t), t \geq 0\}$ is a Markov chain with a state space $\{(n_1, n_2, j), n_1, n_2 = 0, 1, 2, \cdots, \text{and } 1 \leq j \leq m\}$. Let $N_1(t) + N_2(t)$ be the **level** and arrange the states as follows

$$\begin{aligned}
&\text{Level } 0: \ (0,0,1), \cdots, (0,0,m)\\
&\text{Level } 1: \ (1,0,1), \cdots, (1,0,m); (0,1,1), \cdots, (0,1,m)\\
&\text{Level } 2: \ (2,0,1), \cdots, (2,0,m); (1,1,1), \cdots, (1,1,m); (0,2,1), \cdots, (0,2,m)\\
&\qquad\qquad \cdots, \cdots, \cdots.
\end{aligned} \tag{1}$$

We call this listing **LASD Sequencing [1]:** (1) All the states with the same level are listed in the same row; (2) Rows are listed ascending in the level; and (3) States in a row are listed descending in the order.

For any level $N \geq 1$, the only possible transitions among the levels are $N \to N$, $N \to N+1$, and $N+1 \to N$; and we can write the corresponding infinitesimal generator Q as follows,

$$Q = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2}$$

in which, for all $N \geq 0$, C_{N+1} is the block of transitions from level $N+1$ to level N, B_N is the block of transitions from level N to level N, and A_N is the block of transitions from level N to level $N+1$. The number of states in level N is finite and increasing in N. Thus, the blocks in Q are level-dependent, finite, and expanding with N. Thus, with LASD Sequencing, our orignal QBD process with infinite blocks, and for this matter, a general multi-dimensional QBD process, can be reformulated into a single-dimensional LDQBD process with finite and expanding blocks. We call this formulation method the DREB Scheme (see [1]).

One can easily obtain C_N, B_N and A_N where C_N is of $m(N+1) \times mN$, B_N is of $m(N+$

$1) \times m(N+1)$ and A_N is of $m(N+1) \times m(N+2)$. For $N \in \mathbf{E}$,

$$A_N = \begin{pmatrix} D^1 & & & O_{m\times m} \\ & \ddots & & \vdots \\ & & D^1 & O_{m\times m} \end{pmatrix}, \tag{3}$$

$$C_{N+1} = \begin{pmatrix} O_{m\times m} & \cdots & O_{m\times m} \\ \mu_2 I & & \\ & \ddots & \\ & & \mu_2 I \end{pmatrix} \tag{4}$$

and

$$B_0 = D^0, \tag{5}$$

$$B_1 = \begin{pmatrix} D^0 - \mu_1 I & \mu_1 I \\ 0 & D^0 - \mu_2 I \end{pmatrix}. \tag{6}$$

For $N \geq 2$,

$$B_N = \begin{pmatrix} D^0 - \mu_1 I & \mu_1 I & & & \\ & D^0 - \mu_1 I - \mu_2 I & \mu_1 I & & \\ & & \ddots & \ddots & \\ & & & D^0 - \mu_1 I - \mu_2 I & \mu_1 I \\ & & & & D^0 - \mu_2 I \end{pmatrix}. \tag{7}$$

3 Departure Processes and IMAP

In this section, we study the departure processes from the two stages, respectively. They turn out to be infinite-phase Markovian arrival processes (IMAP). Let us first define an IMAP.

Consider a continuous-time irreducible Markov chain with the state space E=$\{0,1,2,\cdots\}$ and an infinitesimal generator $\tilde{D} = \tilde{D}^0 + \tilde{D}^1$, where all the off-diagonal elements of $\tilde{D}^0$ and all the elements of $\tilde{D}^1$ are nonnegative. Let $\tilde{D}^1_{i,j} \leq \tilde{D}_{i,j},\ i \neq j \in \mathbf{E}$ and $\tilde{D}^1_{i,i} \geq 0,\ \forall i \in \mathbf{E}$. The transitions associated with $\tilde{D}^1$ are called type-1 transitions. A customer arrives only at a type-1 transition epoch. Counting the observed transitions of this Markov chain, we have a point process similar to a MAP. The essential difference is that, unlike D^0, D^1 and D of the MAP defined earlier, the matrices $\tilde{D}^0$, $\tilde{D}^1$ and $\tilde{D}$ are not required to be finite. We call this counting process IMAP. As we will see later, IMAP is a very useful extension of the well known (finite) MAP. Specifically, we call an IMAP a QBD-IMAP if the underlying Markov chain is a QBD process.

In the above model, $\tilde{D}$ is a generator matrix, hence $\tilde{D} = (\tilde{D}^0 + \tilde{D}^1)\mathbf{1} = 0$. Denoted by $\tilde{z}$ the stationary probability distribution. The rate of the IMAP is $\tilde{\lambda} = \tilde{z}D\mathbf{1}$. We now show that the departure process from a $MAP/M/1$ queue is a QBD-IMAP.

Theorem 1.
The departure process of the first station is a QBD-IMAP.

Proof. With an infinite buffer between the two stations, the departure process of the first station is independent of the second station, hence the first station can be treated as regular $MAP/M/1$ queue. It is easy to see that $\{N_1(t), J(t), t \geq 0\}$ is a QBD process with the generator matrix

$$\tilde{D} = \begin{pmatrix} D^0 & D^1 & & & \\ \mu_1 I & D^0 - \mu_1 I & D^1 & & \\ & \mu_1 I & D^0 - \mu_1 I & D^1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \quad (8)$$

Decompose Q into filtration matrices

$$\tilde{D}^0 = \begin{pmatrix} D^0 & D^1 & & & \\ O & D^0 - \mu_1 I & D^1 & & \\ & O & D^0 - \mu_1 I & D^1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad (9)$$

and

$$\tilde{D}^1 = \begin{pmatrix} O & & & \\ \mu_1 I & O & & \\ & \mu_1 I & O & \\ & & \ddots & \ddots \end{pmatrix}. \quad (10)$$

We can see that the transitions in $\tilde{D}^1$ correspond to the departures from station 1 while those in $\tilde{D}^0$ do not. By definition, $\tilde{D}^0$ and $\tilde{D}^1$ define a QBD-IMAP. □

Denoted by L_1 the inter-departure time from station 1. Let $\overline{L}_{n_1,j}(x) = P\{L_1 > x \mid N_1(0) = n_1, J(0) = j\}$, where $n_1 \in \mathbf{E}$ and $j = 1, \cdots, m$. Define $\overline{\mathbf{L}}_{n_1}(x) = (\overline{L}_{n_1,1}(x), \cdots, \overline{L}_{n_1,m}(x))^T$ and $\overline{\mathscr{L}}_1(x) = (\overline{\mathbf{L}}_0(x)^T, \overline{\mathbf{L}}_1(x)^T, \overline{\mathbf{L}}_2(x)^T, \cdots)^T$. The tail distribution of L_1 is given below.

Corollary 1.
For any $x \geq 0$,

$$\overline{\mathscr{L}}_1(x) = \xi_0 e^{\tilde{D}^0 x} \mathbf{1}, \quad (11)$$

where ξ_0 is the initial-state probability vector.

In the tandem network, the arrival process to station 2 is exactly the departure process from station 1. Thus station 2 can be seen as an $IMAP/M/1$ queue.

In the following, we will show that the departure process from an $IMAP/M/1$ is still an IMAP. In particular, the departure process from any station of a K-station tandem network is an IMAP process.

Theorem 2.
The departure process of an $IMAP/M/1$ queue is an IMAP. Particularly, the departure process of LDQBD-IMAP$/M/1$ is an LDQBD-IMAP.

Proof. Let $N_q(t)$ be the number of customers in the system at time t, and $J_q(t)$ be the phase of the IMAP defined by $\tilde{D}^0$ and $\tilde{D}^1$. Obviously, $\{N_q(t), J_q(t), t \geq 0\}$ is a Markov process with a state space $\{(n, j), n, j \in \mathbf{E}\}$. Define $N(t) = N_q(t) + J_q(t)$ as the system level. Similar to (1), we arrange all states in the following order: $(0,0)$; $(1,0)$, $(0,1)$; $(2,0)$, $(1,1)$, $(0,2)$; $\cdots$. We can then construct a QBD-type transition generator matrix Q_D with the following nonzero blocks, for all $n, i, j \in \mathbf{E}$: $\tilde{D}^0_{i,i} - \mu\chi(n)$ for transitions from (n,i) to (n,i); $\tilde{D}^0_{i,j}$ for transitions from (n,i) to (n, j) for $j \neq i$; $\tilde{D}^1_{i,j}$ for transitions from (n,i) to $(n+1, j)$ for $j \neq i$; and $\mu\chi(n)$ for transitions from $(n+1,i)$ to (n,i), where $\chi(n) = 0$ if $n = 0$ and $\chi(n) = 1$ otherwise. All the other elements in Q_D are zero elements. Define

$$Q_D^1 = \begin{pmatrix} O & & & & \\ C_1 & O & & & \\ & C_2 & O & & \\ & & C_3 & O & \\ & & & \ddots & \ddots \end{pmatrix}, \tag{12}$$

where the transitions in C_N are from (n, j) to $(n-1, j)$ all with rate μ, and let $Q_D^0 = Q_D - Q_D^1$. We can see that all the off-diagonal elements of Q_D^0 and all the elements of Q_D^1 are nonnegative. Q_D^0 corresponds to transitions when no customers depart while Q_D^1 corresponds to transitions with departures from the system. By definition, the generator Q_D defines an IMAP.

Furthermore, if the underline Markov process $(\tilde{D}^0 + \tilde{D}^1)$ is a QBD (including LDQBD) process, then Q_D is also an LDQBD process. □

Denoted by L_2 the inter-departure time from station 2. Let $\overline{L}_{n_1,n_2,j}(x) = P\{L_2 > x \mid N_1(0) = n_1,\ N_2(0) = n_2, J(0) = j\}$ where $(n_1, n_2) \in \mathbf{E}^2$, and $j = 1, \cdots, m$, and $\overline{\mathscr{L}}_2(x) = \{(\overline{L}^2_{n_1,n_2,j}(x))^T,\ (n_1,\ n_2) \in \mathbf{E}^2,\ j = 1, \cdots, m\}$. The following corollary gives the tail distribution of L_2.

Corollary 2.
For any $x \geq 0$,

$$\overline{\mathscr{L}}_2(x) = \eta_0 e^{Q_D^0 x}\mathbf{1}, \tag{13}$$

where η_0 is the initial-state probability vector.

Consider a K-station tandem network with a MAP external arrival process (note that the following result still holds when the external arrival process is an IMAP) and with exponential service times for all the K servers. We immediately have the following corollary.

Corollary 3.
The departure process from station i of a K-station tandem network, $i = 1, \cdots, K$, is an LDQBD-IMAP.

With this corollary, we can study any stage of a K-stage system as an $LDQBD-IMAP/M/1$ queue.

4 Concluding Remarks

A 2-station tandem network with a MAP external input process is studied in this paper. This is a generalization of the Jackson network and cannot be easily handled by the standard modeling and solution method. We first show that the departure process from the first station is a MAP with infinitely many phases. This IMAP is a useful extension of the finite MAP. For example, treating this IMAP as the input process, the second station of the tandem network can be studied as an $IMAP/M/1$ queue. Furthermore, we demonstrate that the output process from $IMAP/M/1$ is still an IMAP, and in particular, if the imput is an QBD-IMAP or LDQBA-IMAP, the output is an LDQBD-IMAP. This shows that the IMAP process is preserved by some queueing operations which is not true for the finite MAP.

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