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Minimizing the Total Late Work on an Unbounded Batch Machine*

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Abstract We consider the problem of minimizing the total late work ($\sum_{j=1}^{n} V_j$) on an unbounded batch processing machine, where $V_j = min\{T_j, p_j\}$ and $T_j = max\{C_j - d_j, 0\}$. The batch processing machine can process up to B ($B \geq n$) jobs simultaneously. The jobs that are processed together form a batch, and all jobs in a batch start and complete at the same time, respectively. For a batch of jobs, the processing time of the batch is equal to the largest processing time among the jobs in this batch. In this paper, we prove that the problem $1|B \geq n|\sum_{j=1}^{n} V_j$ is NP-hard.

Keywords Scheduling; Batching machine; Late work; NP-hardness.

1 Introduction

The scheduling model that we study is as follows. There are n independent jobs $J_1, J_2, \cdots, J_n$ that have to be scheduled on an unbounded batch machine. Each job J_j $(j = 1, 2, \cdots, n)$ has a processing time p_j and a due date d_j. All jobs are available for processing at time 0. The goal is to schedule the jobs without preemption on the unbounded batch machine such that the total late work is minimized.

A batch machine is a machine that can process up to B jobs simultaneously. The jobs that are processed together form a batch. This model is motivated by the problem of scheduling burn-in operations for large-scale integrated circuit (IC) chips manufacturing (see Lee et al. [1] for detail). There are two variants: the unbounded model, where $B \geq n$; and the bounded model, where $b < n$. In this paper, we study the problem of scheduling n independent jobs on an unbounded batch machine to minimize the total late work. A schedule σ is a sequence of batches $\sigma = (\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_m)$, where each batch $\mathscr{B}_k$ $(k = 1, \cdots, m)$ is a set of jobs that are processed together. The processing time of batch $\mathscr{B}_k$ is $p(\mathscr{B}_k) = max_{J_j \in \mathscr{B}_k}\{p_j\}$ and its completion time is $C(\mathscr{B}_k) = \sum_{q=1}^{k} p(\mathscr{B}_q)$. Note that

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the completion time of job J_j in σ, for each $J_j \in \mathscr{B}_k$ and $k = 1, \cdots, m$, is $C_j(\sigma) = C(\mathscr{B}_k)$. When there is no ambiguity, we abbreviate $C_j(\sigma)$ to C_j. The tardiness and late work of job J_j are defined as $T_j = max\{C_j - d_j, 0\}$ and $V_j = min\{T_j, p_j\}$, respectively. Using the notation of Graham et al. [2], we denote this problem as $1|B \geq n| \sum_{j=1}^{n} V_j$.

Previous related work: The scheduling of batch processing machines is an important research topic and has attracted a lot of attention recently ([4,5]). In [6], Brucker et al. design a dynamic programming algorithm that solves the problem of minimizing an arbitrary regular cost function in pseudopolynomial time. When the jobs have different release times, there has been a lot of research work ([7,8]). As the objective is to minimize the total weighted late work, Zhang and Wang (2005) [3] prove that $1|B \geq n| \sum_{j=1}^{n} w_j V_j$ is NP-hard in the ordinary sense. But it is open whether $1|B \geq n| \sum_{j=1}^{n} V_j$ is polynomially solvable or binary NP-hard.

Our contribution: In this paper, we prove the binary NP-hardness of $1|B \geq n| \sum_{j=1}^{n} V_j$. This answers the open question posed in [3]. Our work's obtaining heavily depends on the reference [9]. As the two scheduling problems are different models (such as the different objectives need different analysis) and the problem $1|B \geq n| \sum_{j=1}^{n} V_j$ has been considered as very difficult by Zhang et al. [3]. It, therefore, is a different work from [9].

2 NP-hardness proof

In this section, we prove the NP-hardness of the problem $1|B \geq n| \sum_{j=1}^{n} V_j$ by a reduction from the NP-complete PARTITION problem.

PARTITION. Given t positive integers $a_1, a_2, \cdots, a_t$ with $\sum_{i=1}^{t} a_i = 2B$, decide if there exists a partition of the index set $\{1, 2, \cdots, t\}$ into two disjoint subsets X and Y such that $\sum_{i \in X} a_i = \sum_{i \in Y} a_i = B$.

Given an instance of PARTITION, we first define $3t + 1$ integers

$M_t = \sum_{i=1}^{t} (t-i)a_i + 8B$,

$M_k = 2 \sum_{i=k+1}^{t} M_i + \sum_{i=1}^{t} (t-i)a_i + 8B$, $(k = 1, 2, \cdots, t-1)$

$L_1 = 7 \sum_{i=1}^{t} M_i + \sum_{i=1}^{t} (t-i)a_i + 4B$,

$L_k = 2 \sum_{i=1}^{k-1} L_i + 7 \sum_{i=1}^{t} M_i + \sum_{i=1}^{t} (t-i)a_i + 4B$, $(k = 2, 3, \cdots, 2t+1)$.

It is easy to get that

$2B < M_t < M_{t-1} < \cdots < M_1 < L_1 < L_2 < \cdots < L_{2t+1}$.

We define an instance I of $1|B \geq n| \sum_{j=1}^{n} V_j$ as follows.

I consists of $10t+3$ jobs that are classified into $2t+1$ types. Each type $2k-1$ $(1 \le k \le t)$ contains five jobs: $J^1_{2k-1}, J^2_{2k-1}, J^3_{2k-1}$ and two additional copies of J^1_{2k-1}. Their processing times and due dates are given by

$$p^1_{2k-1} = L_{2k-1}, \qquad d^1_{2k-1} = 2\sum_{i=1}^{2k-2} L_i + 5\sum_{i=1}^{k-1} M_i + L_{2k-1} + M_k + 2B,$$

$$p^2_{2k-1} = L_{2k-1} + M_k, \qquad d^2_{2k-1} = 2\sum_{i=1}^{2k-1} L_i + 5\sum_{i=1}^{k-1} M_i,$$

$$p^3_{2k-1} = L_{2k-1} + 2M_k, \qquad d^3_{2k-1} = 2\sum_{i=1}^{2k-1} L_i + 5\sum_{i=1}^{t} M_i + 2B.$$

Each type $2k$ $(1 \le k \le t)$ also contains five jobs: $J^1_{2k}, J^2_{2k}, J^3_{2k}$ and two additional copies of J^1_{2k}. Their processing times and due dates are given by

$$p^1_{2k} = L_{2k}, \qquad d^1_{2k} = 2\sum_{i=1}^{2k-1} L_i + 5\sum_{i=1}^{k-1} M_i + L_{2k} + 3M_k + 2B,$$

$$p^2_{2k} = L_{2k} + M_k + a_k, \qquad d^2_{2k} = 2\sum_{i=1}^{2k} L_i + 5\sum_{i=1}^{k-1} M_i + 3M_k - (t-k+1)a_k,$$

$$p^3_{2k} = L_{2k} + 2M_k, \qquad d^3_{2k} = 2\sum_{i=1}^{2k} L_i + 5\sum_{i=1}^{t} M_i + 2B.$$

Type $2t+1$ contains three copies of job J^1_{2t+1} with

$$p^1_{2t+1} = L_{2t+1}, \qquad d^1_{2t+1} = L_{2t+1} + 2\sum_{i=1}^{2t} L_i + 5\sum_{i=1}^{t} M_i + B.$$

Set the threshold value $V^* = 2\sum_{i=1}^{t} M_i + \sum_{i=1}^{t} (t-i)a_i + B$.

Clearly, the construction of the instance I of $1|B \ge n|\sum_{j=1}^{n} V_j$ takes a polynomial time under the binary coding. Next, we show that the PARTITION instance has a solution if and only if there exists a schedule σ for the corresponding instance I with $V(\sigma) \le V^*$, where $V(\sigma)$ denotes the total late work of σ.

Suppose that I has a schedule $\sigma = (\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_m)$ such that $V(\sigma) \le V^*$, where each $\mathscr{B}_i$ is a batch. The processing time of batch $\mathscr{B}_i$ is denoted by $p(\mathscr{B}_i)$. It is reasonable to require that the processing time of each job in $\mathscr{B}_{i+1}$ is larger than $p(\mathscr{B}_i)$; otherwise, shifting the jobs in $\mathscr{B}_{i+1}$ with processing times no larger than $p(\mathscr{B}_i)$ to $\mathscr{B}_i$ does not increase $V(\sigma)$. Then, we have the following result about the schedule σ.

Lemma 1. (1) The jobs in each $\mathscr{B}_i$ come from a contiguous segment of the SPT sequence, and all $\mathscr{B}_i$'s are arranged in order of increasing $p(\mathscr{B}_i)$;
(2) For each k $(1 \le k \le 2t+1)$, all J^1_k's are scheduled in a batch.

Lemma 2. For the schedule σ.

(1) Each batch contains only jobs of one type;

(2) For each k $(1 \le k \le t)$, the jobs of types $(2k-1)$ and $2k$ are divided into four batches: $\{J^1_{2k-1}, J^2_{2k-1}\}$, $\{J^3_{2k-1}\}$, $\{J^1_{2k}\}$, $\{J^2_{2k}, J^3_{2k}\}$ (pattern one), with total processing time $2(L_{2k-1}+L_{2k})+5M_k$;
or $\{J^1_{2k-1}\}$, $\{J^2_{2k-1}, J^3_{2k-1}\}$, $\{J^1_{2k}, J^2_{2k}\}$, $\{J^3_{2k}\}$ (pattern two), with total processing time $2(L_{2k-1}+L_{2k)}+5M_k+a_k$.

Proof. If (1) does not hold, there must exist some k $(1 \le k \le 2t+1)$ such that J^3_k and J^1_{k+1} are scheduled in a batch. We consider the tardiness of job J^3_k's.

Firstly, we have that

$$
\begin{aligned}
T_k^3 &\geq p_{k+1}^1 - d_k^3 = L_{k+1} - d_k^3 \\
&= (2\sum_{i=1}^{k} L_i + 7\sum_{i=1}^{t} M_i + \sum_{i=1}^{t}(t-i)a_i + 4B) - (2\sum_{i=1}^{k} L_i + 5\sum_{i=1}^{t} M_i + 2B) \\
&= 2\sum_{i=1}^{t} M_i + \sum_{i=1}^{t}(t-i)a_i + 2B \\
&> V^*.
\end{aligned}
$$

Note that

$$p_k^3 \quad > \quad L_k = \sum_{i=1}^{k-1} L_i + 7\sum_{i=1}^{t} M_i + \sum_{i=1}^{t}(t-i)a_i + 4B > V^*.$$

Hence, the job J_k^3's late work is

$$V_k^3 = min\{p_k^3, T_k^3\} \quad > \quad V^*.$$

A contradiction to $V^*(\sigma) \leq V^*$, which implies that each batch contains only jobs of one type.

Next, we prove (2) by induction. Firstly, we prove that (2) is true for $k = 1$ by proving the following **Observation** a_1–**Observation** b_3.

Observation a_1. All jobs of type 1 cannot be scheduled in a batch.

Proof. Suppose that the jobs J_1^1, J_1^2 and J_1^3 are scheduled in a batch $\{J_1^1, J_1^2, J_1^3\}$. Then, the total tardiness of three J_1^1's is

$$
\begin{aligned}
3T_1^1 &= 3(p_1^3 - d_1^1) \\
&= 3[L_1 + 2M_1 - (L_1 + M_1 + 2B)] \\
&= 2M_1 + M_1 - 2B \\
&= V^* + B.
\end{aligned}
$$

Note that

$$p_1^1 = L_1 = 7\sum_{i=1}^{t} M_i + \sum_{i=1}^{t}(t-i)a_i + 4B = V^* + 2B.$$

We obtain

$$V(J_1^1) = min\{\frac{1}{3}[V^* + B], p_1^1\} = \frac{1}{3}[V^* + B].$$

So the total late work of three J_1^1's is

$$3V(J_1^1) = V^* + B > V^*.$$

A contradiction to $V(\sigma) \leq V^*$, which implies that the jobs of type 1 cannot be scheduled in a batch.

Observation b_1. Jobs J_1^1, J_1^2 and J_1^3 cannot be scheduled in three batches.

Proof. Suppose that the jobs J_1^1, J_1^2 and J_1^3 are scheduled in three batches $\{J_1^1\}$, $\{J_1^2\}$ and

$\{J_1^3\}$.
We can easily get that the late work of job J_1^3's is

$$V(J_1^3) = min\{T_1^3, p_1^3\} \geq V^* + B.$$

A contradiction to $V(\sigma) \leq V^*$, which implies that the jobs of type 1 cannot be scheduled in three batches.

From **Observation** a_1 and **Observation** b_1, we get that jobs J_1^1, J_1^2 and J_1^3 must be scheduled in two batches: $\{J_1^1, J_1^2\}$, $\{J_1^3\}$; or $\{J_1^1\}$, $\{J_1^2, J_1^3\}$.
Next, we show that the jobs of type 2 also must be scheduled in two batches: $\{J_2^1, J_2^2\}$, $\{J_2^3\}$; or $\{J_2^1\}$, $\{J_2^2, J_2^3\}$.
Observation a_2. All jobs of type 2 cannot be scheduled in a batch.
Proof. Suppose that the jobs J_2^1, J_2^2 and J_2^3 are scheduled in a batch $\{J_2^1, J_2^2, J_2^3\}$.
Then the total late work of three J_2^1's is

$$3V(J_2^1) > V^* + B > V^*.$$

A contradiction to $V(\sigma) \leq V^*$, which implies that the jobs of type 2 cannot be scheduled in a batch.
Observation b_2. Jobs J_2^1, J_2^2 and J_2^3 cannot be scheduled in three batches.
Proof. Suppose that the jobs J_2^1, J_2^2 and J_2^3 are scheduled in three batches $\{J_2^1\}$, $\{J_2^2\}$ and $\{J_2^3\}$.
Then, we also can get that the late work of job J_2^3's is

$$V(J_2^3) = min\{T_2^3, p_2^3\} \geq V^* + B.$$

A contradiction to $V(\sigma) \leq V^*$, which implies that the jobs of type 1 cannot be scheduled in three batches.

From **Observation** a_2 and **Observation** b_2, we get that jobs J_2^1, J_2^2 and J_2^3 must be scheduled in two batches: $\{J_2^1, J_2^2\}$, $\{J_2^3\}$; or $\{J_2^1\}$, $\{J_2^2, J_2^3\}$.
Observation a_3. Batches $\{J_1^1, J_1^2\}$ and $\{J_2^1, J_2^2\}$ are not exist in σ simultaneously.
Proof. Suppose that the batches $\{J_1^1, J_1^2\}$ and $\{J_2^1, J_2^2\}$ are both appear in the schedule σ.
Then the total tardiness of three J_2^1's is

$$\begin{aligned} 3T_2^1 &= 3(p_1^2 + p_1^3 + p_2^2 - d_2^1) \\ &\geq V^* + B. \end{aligned}$$

Note that

$$p_2^1 = L_2 > V^* + B.$$

So the total late work of three J_2^1's is

$$3V(J_2^1) > V^* + B > V^*.$$

A contradiction to $V(\sigma) \leq V^*$.
Observation b_3. Batches $\{J_1^2, J_1^3\}$ and $\{J_2^2, J_2^3\}$ are not exist in σ simultaneously.
Proof. Suppose that the batches $\{J_1^2, J_1^3\}$ and $\{J_2^2, J_2^3\}$ are both appear in the schedule σ.
Then the total late work of J_1^2 and J_2^2 is

$$V(J_1^2 \cup J_2^2) = V(J_1^2) + V(J_2^2) = T_1^2 + T_2^2 > V^*.$$

A contradiction to $V(\sigma) \leq V^*$.
From the above **Observation** a_1–**Observation** b_3, the four batches of type 1 and type 2 must be

$$\{J_1^1, J_1^2\}, \{J_1^3\}, \{J_2^1\}, \{J_2^2, J_2^3\}; or \{J_1^1\}, \{J_1^2, J_1^3\}, \{J_2^1, J_2^2\}, \{J_2^3\}.$$

That is the conclusion (2) of **Observation 2** being true for $k = 1$.

Secondly, we assume that the conclusion (2) of **Lemma 2** is true for each $i = 1, 2, \cdots k - 1$. It is similar to prove that the conclusion (2) of **Lemma 2** is also true for $i = k$, i.e., the jobs of types $(2k-1)$ and $2k$ must be divided into the following model four batches: $\{J_{2k-1}^1, J_{2k-1}^2\}$, $\{J_{2k-1}^3\}$, $\{J_{2k}^1\}$, $\{J_{2k}^2, J_{2k}^3\}$ with total processing time $2(L_{2k-1} + L_{2k}) + 5M_k$; or $\{J_{2k-1}^1\}$, $\{J_{2k-1}^2, J_{2k-1}^3\}$, $\{J_{2k}^1, J_{2k}^2\}$, $\{J_{2k}^3\}$, with total processing time $2(L_{2k-1} + L_{2k}) + 5M_k + a_k$.

Let X be the set of indices k $(1 \leq k \leq t)$ such that the four batches of types $(2k-1)$ and $2k$ are of pattern one. Let $Y = \Pi \setminus X$, where $\Pi = \{1, 2, \cdots, t\}$. A schedule with properties of **Lemma 1** and **Lemma 2** must contain $4t+1$ batches in the following form.

$$(\mathscr{B}_{4k-3}, \mathscr{B}_{4k-2}, \mathscr{B}_{4k-1}, \mathscr{B}_{4k}) = \begin{cases} \{J_{2k-1}^1, J_{2k-1}^2\}, \{J_{2k-1}^3\}, \{J_{2k}^1\}, \{J_{2k}^2, J_{2k}^3\}, & k \in X, \\ \{J_{2k-1}^1\}, \{J_{2k-1}^2, J_{2k-1}^3\}, \{J_{2k}^1, J_{2k}^2\}, \{J_{2k}^3\}, & k \in Y. \end{cases}$$

$$\mathscr{B}_{4t+1} = \{J_{2t+1}^1\}. \qquad (*)$$

Lemma 3. For each $k \in X$, J_{2k}^2 is the only late work job in $\mathscr{B}_{4k-3}$, $\mathscr{B}_{4k-2}$, $\mathscr{B}_{4k-1}$ and $\mathscr{B}_{4k}$, and its late work is $2M_k + (t-k+1)a_k + \sum_{i<k,\ i \in Y} a_i$. For each $k \in Y$, J_{2k-1}^2 is the only late work job in $\mathscr{B}_{4k-3}$, $\mathscr{B}_{4k-2}$, $\mathscr{B}_{4k-1}$ and $\mathscr{B}_{4k}$, and its late work is $2M_k + \sum_{i<k,\ i \in Y} a_i$.

The lemma can be proved by the method in [9].

Theorem 1. The problem $1|B \geq n| \sum_{j=1}^{n} V_j$ is NP-hard.

Proof. The time it takes to construct the scheduling instance is obviously polynomial. We show that the PARTITION has a solution if and only if there exists a schedule σ for the scheduling instance I such that $V(\sigma) \leq V^*$.

First, suppose that X and Y define a solution to PARTITION. Let σ be the schedule defined by $(*)$. The scheduling σ has properties of **Lemma 1**, **Lemma 2** and **Lemma 3**. By **Lemma 3**

$$V(\sigma) = \sum_{k=1}^{t} (2M_k + \sum_{i<k, i \in Y} a_i) + \sum_{k \in X} (t-k+1)a_k + 3max\{0, \ \sum_{k \in Y} a_k - B\}.$$

Where the third term is the total late work of three J_{2t+1}^1.
As

$$\sum_{k=1}^{t} \sum_{i<k, i \in Y} a_i = \sum_{k=i+1}^{t} \sum_{i<k, i \in Y} a_i = \sum_{i \in Y} (t-i)a_i.$$

So

$$V(\sigma) = 2\sum_{k=1}^{t} M_k + \sum_{k=1}^{t} (t-k)a_k + 3max\{0, \ \sum_{k \in Y} a_k - B\}.$$

Note that

$$\sum_{i\in X} a_i = \sum_{i\in Y} a_i = B.$$

By (2), we have

$$V(\sigma) = 2\sum_{i=1}^{t} M_i + \sum_{i=1}^{t}(t-i)a_i + B = V^*.$$

Conversely, suppose that there exists a schedule σ with $V(\sigma) \leq V^*$. From the proof of **Lemma 1**, **Lemma 2** and **Lemma 3**, the schedule σ must have properties of **Lemma 1**, **Lemma 2** and **Lemma 3**. For the schedule σ, let X be the set of indices k $(1 \leq k \leq t)$ such that the four batches of types $(2k-1)$ and $2k$ are of pattern one and $Y = \Pi \setminus X$, where $\Pi = \{1, 2, \cdots, t\}$.

Note that

$$3max\{0, \sum_{k\in Y} a_k - B\} \geq 3\sum_{k\in Y} a_k - 3B,$$

$$3max\{0, \sum_{k\in Y} a_k - B\} \geq 0.$$

By

$$V(\sigma) \leq V^*.$$

we have

$$\sum_{k\in X} a_k \leq B. \tag{1}$$

$$\sum_{k\in X} a_k + 3\sum_{k\in Y} a_k - 3B \leq B. \tag{2}$$

From (1), we get

$$\sum_{k\in Y} a_k \leq B. \tag{3}$$

Applying (1), (3) and $\sum_{k\in X\cup Y} a_k = 2B$, we get

$$\sum_{k\in X} a_k = \sum_{k\in Y} a_k = B.$$

Which shows that X and Y define a solution to PARTITION.

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