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A Posteriori Error Estimator for Linear Elliptic Problem*

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Abstract This paper concerns with a new error estimator for finite element approximation to the linear elliptic problem. A posteriori error estimator employing both a residual and a recovery based estimator is introduced. The error estimator is constructed by employing the recovery gradient method to obtain the approximated solutions of the linear elliptic problem. These solutions are combined with the residual method to produce the error estimator. Numerical results for selected test problems are demonstrated for the resulting error estimators and discussed.

Keywords A posteriori error estimator; recovery gradient method; residual method; adaptive finite element method.

1 Introduction

The adaptive finite element scheme based on a posteriori error estimates has become an important tool in scientific and engineering problem. One class of a posteriori error estimator is the gradient recovery, see e.g.[5], [7]–[9]. Zienkiewicz *et al.* [10]–[11] constructed a superconvergent patch recovery which converge at a rate one order higher than σ_h and yield asymptotically exact error estimates. Babuska *et al.* [1] present the accuracy of the derivatives which are recovered by local averaging for complex finite element meshes. Bank *et al.* [2] illustrated three posteriori error estimators based on the norm of the residual of the elliptic equation, see [3],[4] for more example.

The main focus of this paper will be to provide the new error estimator for the finite element approximation to a linear elliptic problem. In section 2, we summarize the basic finite element formulation for the linear elliptic problem. In section 3, we present the residual method for obtaining the new error estimator. The recovery gradient method is described in section 4. In section 5, we give some numerical results illustrating the performance of the error estimators. Finally, some concluding remarks are presented.

2 Preliminaries

For simplicity, we focus our attention on the following model problem:

$$\begin{aligned} -\triangle u &= f \quad \in \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

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where $\Omega \subset \mathbb{R}^2$ is an open bounded with Lipschitz boundary $\partial\Omega$, $f \in L_2(\Omega)$. The weak formulation of Eq (1) is as follows: find $u \in H_0^1$ such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \;\; d\mathbf{x} \quad \text{and} \quad (f,v) = \int_\Omega fv \;\; d\mathbf{x}. \tag{3}$$

3 The residual error estimator

In this section we shall applied the residual technique to produce a new error estimator for the model problem (1). Let the finite element error for each element Ω_i be

$$\triangle e_h^i = (\triangle u - \triangle u_h)|_{\Omega_i} = (-f - \triangle u_h)|_{\Omega_i} \equiv -r_h^i \tag{4}$$

where r_h^i is called the *residual* in Ω_i. The aim of this method is to determine the error in energy norm which is given by

$$a(e_h,e_h) = \|e_h\|_E^2 = \sum_{i=1}^{m} \int_{\Omega_i} (\triangledown e_h^i)^2 \;\; d\mathbf{x}. \tag{5}$$

Applying the vector identity and the divergence theorem in Eq (5), we get

$$\int_{\Omega_i} (\triangledown e_h^i)^2 \;\; d\mathbf{x} = \int_{\Omega_i} e_h^i r_h^i \;\; d\mathbf{x} + \int_{\partial\Omega_i} e_h^i \frac{\partial e_h^i}{\partial \mathbf{n}_j^i} \;\; ds, \tag{6}$$

where $\mathbf{n}_j^i$ is the outward normal direction from element Ω_i across the edge Γ_j^i, j=1, 2, 3. Integrating twice over a common element edge is solved by taking an average value for the boundary integral. Then

$$\sum_{i=1}^{m} \int_{\partial\Omega_i} e_h^i \frac{\partial e_h^i}{\partial \mathbf{n}_j^i} \;\; ds = \sum_{i=1}^{m}\sum_{j=1}^{3} \int_{\Gamma_j^i} \frac{e_h^i}{2} \Big(\frac{\partial e_h^i}{\partial \mathbf{n}_j^i} - \frac{\partial e_h^j}{\partial \mathbf{n}_j^i} \Big) \;\; ds, \tag{7}$$

where e_h^j, $j = 1,2,3$ are the *error contributors* from the neighborly element Ω_i which have the common edges Γ_j^i. Now,

$$\frac{\partial e_h^i}{\partial \mathbf{n}_j^i} - \frac{\partial e_h^j}{\partial \mathbf{n}_j^i} = \frac{\partial}{\partial \mathbf{n}_j^i}(u-u_h)|_{\Omega_i} - \frac{\partial}{\partial \mathbf{n}_j^i}(u-u_h)|_{\Omega_j} \equiv J_j,$$

where J_j is called *jump* of $\dfrac{\partial u_h}{\partial \mathbf{n}_j^i}$, across the common edge j with unit outward normal vector $\mathbf{n}_j^i$. For the piecewise linear case, the Eq (6) becomes

$$\|e_h\|_E^2 = \sum_{i=1}^{m} \int_{\Omega_i} e_h^i f^i \;\; d\mathbf{x} + \sum_{i=1}^{m}\sum_{j=1}^{3} \int_{\Gamma_j^i} \frac{e_h^i}{2} J_j \;\; ds, \tag{8}$$

Unfortunately, e_h^i can not determine because the exact solution does not know. One way to solve this problem, the exact solution may be replaced by u^* which will be described in the next section.

4 The Zienkiewicz-Zhu error estimator

Let u_h be the finite element approximation and u be the analytical solution. The discretization error are defined as

$$e = u - u_h, \;\; \nabla e = \nabla u - \nabla u_h. \tag{9}$$

If the recovered solution u^* can be obtained by some suitable recovery process then the errors in Eq (9) become

$$e^* = u^* - u_h, \;\; \nabla e^* = \nabla u^* - \nabla u_h. \tag{10}$$

The energy norm of the exact and the estimate errors can be written as

$$\|\nabla e\|_{E(\Omega)} = \Big(\int_\Omega \nabla e^T \nabla e \; d\mathbf{x}\Big)^{1/2}, \;\; \|\nabla \tilde{e}\|_{E(\Omega)} = \Big(\int_\Omega \nabla e^{*T} \nabla e^{*} \; d\mathbf{x}\Big)^{1/2}. \tag{11}$$

The quality of the error estimator $\|\tilde{e}\|_{E(\Omega)}$ is measured by

$$\theta_E = \frac{\|\nabla \tilde{e}\|_{E(\Omega)}}{\|\nabla e\|_{E(\Omega)}}. \tag{12}$$

If θ_E approaches unity as the true error $\|\nabla e\|_{E(\Omega)}$ tends to zero, this imply that the estimators converges to the exact error([11]). Wiberg *et al.* [7] expressed the polynomial expansion for u^* as

$$u^* = \mathbf{Pa}, \tag{13}$$

where **P** contains one or two higher order term than those the original finite element solution and **a** is a set of unknown parameters. We then apply a least square fit of expression (13) for u^*. Thus we minimize

$$F(\mathbf{a}) = \sum_{i=1}^{NS} \Big(u_h(x_i, y_i) - \mathbf{P}(x_i, y_i)\, \mathbf{a} \Big)^2. \tag{14}$$

This implies

$$\mathbf{a} = \mathbf{A}^{-1}\, \mathbf{b}, \tag{15}$$

where

$$\mathbf{A} = \sum_{i=1}^{NS} \mathbf{P}^T(x_i, y_i)\, \mathbf{P}(x_i, y_i) \quad \text{and} \quad \mathbf{b} = \sum_{i=1}^{NS} \mathbf{P}^T(x_i, y_i)\, u_h(x_i, y_i). \tag{16}$$

5 Numerical experiment

In this section we determine finite element solution to the tested problems for which the exact solutions are known. In addition, the tolerance for uniform and adaptive refinement are computed as

$$Tol \;= \Big(\frac{R}{100}\Big)\Big(\| \; u_h \; \|^2_{E(\Omega)} + (\| \; e \; \|^2_{E(\Omega)})/NE\Big)^{1/2},$$

where R is the percentage of relative error [6].

5.1 Numerical example 1

Consider the two dimensional boundary value problem

$$\begin{aligned} -\triangle u &= f \quad \text{in } \Omega = (-1,1) \times (-1,1) \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The problem is solved on a sequence of uniform mesh using the triangular element. The function f is chosen so that the exact solution is of the form $u(x,y) = e^{-100(x^2+y^2)}$, see Fig. 3 (a). The performance for the set of successive refined uniform meshes can be seen in Table 1. In Fig. 1(a) the curve for the uniform refinement is shown as a solid line with the value at each new mesh point marked with a square whilst the ZZ and the residual estimator curves are illustrated as a solid line marked with plus and star respectively. It is clear that for the coarser meshes the convergence is slow. this is to be expect as the mesh is not able to correctly resolve the spike feature of the displacement. However, it can be seen that the adaptive route converging to the required 10% error requirement. In Fig. 1 (b) the effectivity index versus log(node) for the ZZ and the residual estimators are represented by a solid line marked with plus and diamond respectively. The comparison between the error estimators is illustrated in Fig. 2.

Table 1: Results for example 1: uniform refinement

$Element$	Node	$\|\mathbf{u}-\mathbf{u}_h\|_E$	$R\%$
128	81	1.4797517	73.6581
512	289	1.0488931	62.0644
2048	1089	0.6674518	37.7634
8192	4225	0.3511169	19.8140
32768	16641	0.1779495	10.0399
131072	66049	0.0892811	5.0372

(a) (b)

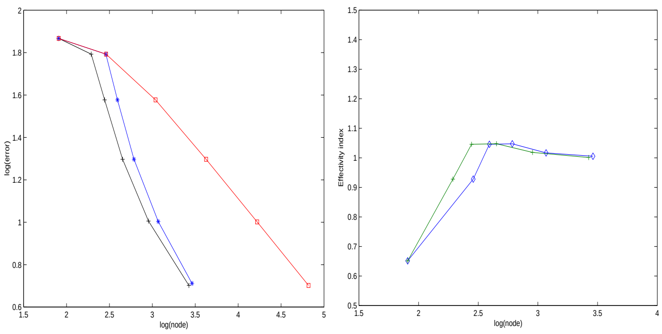

Figure 1: Example 1. (a) Comparison between uniform and adaptive refinement (b) Effectivity indices

(a) (b)

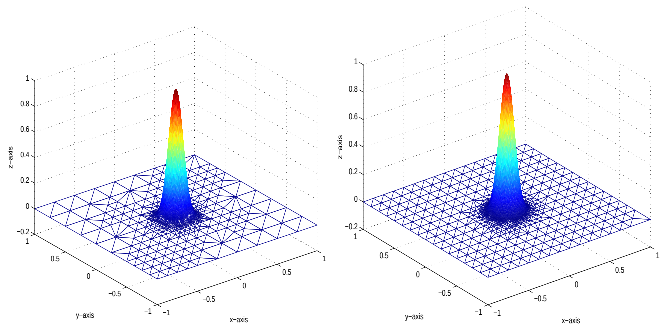

Figure 2: Example 1: Adaptive meshes generated from (a) ZZ method (b) residual method

5.2 Numerical example 2

We consider the two dimensional model problem

$$
\begin{aligned}
-\triangle u &= f \quad \text{in } \Omega = (0,1)\times(0,1) \\
u &= \bar{u} \quad \text{on } \partial\Omega
\end{aligned}
$$

The function f and $\bar{u}$ are chosen so that the theoretical solution to the problem is $u(x,y) = x(2-$

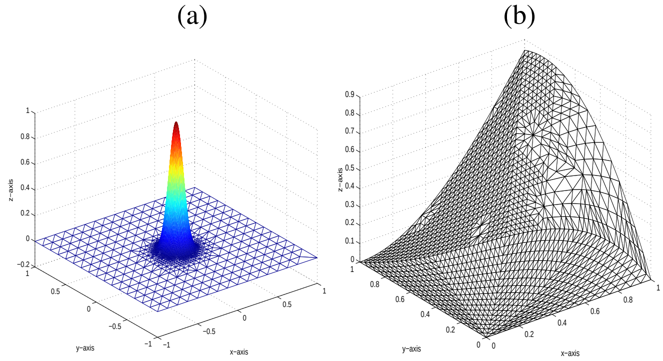

Figure 3: The theoretical solution (a) Example 1 (b) Example 2.

Table 2: Results for example 2: uniform refinement

Element	Node	$\|\mathbf{u}-\mathbf{u}_h\|_E$	*R%*
4	5	0.5804054	61.9514
16	13	0.2878635	31.3248
64	41	0.1439360	15.8107
256	145	0.0719976	7.9308
1024	545	0.0360049	3.9690
4096	2213	0.0180034	1.9850

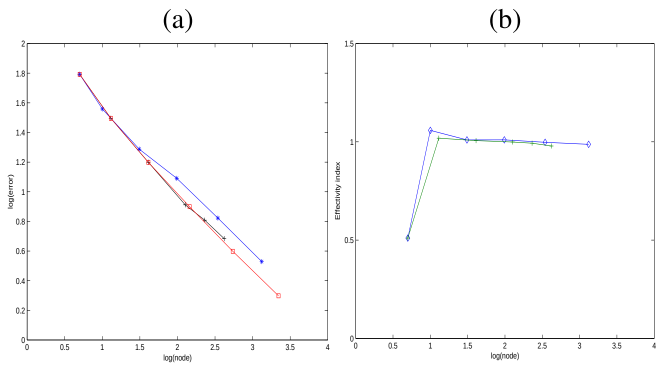

Figure 4: Example 2. (a) Comparison between uniform and adaptive refinement (b) Effectivity indices

$y)sin(xy)$, see Fig. 3 (b). The linear element is used to test the performance of the ZZ and residual method. The numerical results were found to behave in the same way as in the first example. The results obtained for uniform refinement are listed in Table 2. The adaptive and uniform refinement are displayed in Fig. 4 (a). The effectivity indices of the residual and the ZZ method are plotted in Figure 4 (b). In Fig. 5 we compare the performance of the ZZ and residual method for the computation of the error energy norm.

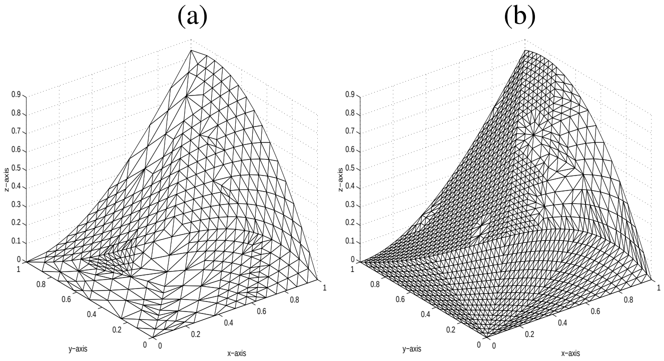

Figure 5: Example 2: Adaptive meshes generated from (a) ZZ method and (b) residual method

6 Conclusion

In this article we introduced the a posteriori error estimator based on the residual and the recovery gradient technique. The ZZ and residual estimators can provide accurate a posteriori error estimators for the linear triangular element in the energy norm. The numerical results show both the residual and the ZZ estimators to be performing well.

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