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A Trust Region Algorithm Model With Radius Bounded Below for Minimization of Locally Lipschitzian Functions*

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Abstract The classical trust region algorithm was extended to the nonsmooth minimization problem successful by Qi and Sun. Combining the trust region algorithm of Qi and Sun and the trust region algorithm with radius bounded below of Jiang for solving generalized complementarity problems, this paper present a new trust region algorithm with radius bounded below for the unconstrained nonsmooth optimization problems where the objective function is locally Lipschitzian, the global convergence results are established.

Keywords Nonsmooth optimization; trust region algorithm; critical point.

1 Introduction

Consider the nonsmooth programming:

$$(P) \qquad \min_{x\in R^n} f(x), \tag{1.1}$$

where $f: R^n \to R^n$ is a locally Lipschitzian function.

Qi and Sun [1] extended the classical trust region algorithm to the nonsmooth case where the objective function is locally Lipschitzian, and proved that their algorithm is globally convergent. Their convergence result extends the results of Powell [5], Yuan [10], and Dennis, Li and Tapia [6] for minimization of various functions.

In this paper, we give a new trust region algorithm with radius bounded below for the unconstrained optimization problems (P) and prove that our trust region algorithm is globally convergent. The idea of the algorithm is proposed by H.Jiang in [2] for solving generalized complementarity problems.

Given $x_0 \in R^n, \triangle_{min} > 0$. Let c_0, c_1, c_2, c_3 and c_4 be positive constants satisfying $0 < c_0 \leq 1, c_2 < c_1 < 1, c_3 < 1 < c_4$. The trust region algorithm modal with radius bounded below can be described as follows:

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Algorithm TR: At the kth iteration, given x_k, B_k and $\triangle_k$, solve the subproblem:

$$(SP) \qquad \min_{||d|| \leq \triangle_k} Q_k(d) =: f(x_k) + \phi(x_k, d) + \frac{1}{2} d^T B_k d, \tag{1.2}$$

where $\phi : R^n \times R^n \to R$ is a given iteration function. Assume the exact solution of (1.2) is d_k^*. Suppose that d_k is an inexact solution of (1.2) in the sense that d_k satisfies

$$f(x_k) - Q_k(d_k) \geq c_0[f(x_k) - Q_k(d_k^*)] \tag{1.3}$$

and

$$||d_k|| \leq \triangle_k.$$

If $d_k = 0$, then stop.

Let

$$r_k = \frac{f(x_k) - f(x_k + d_k)}{f(x_k) - Q_k(d_k)}. \tag{1.4}$$

Set

$$x_{k+1} = \begin{cases} x_k + d_k, & \text{if } r_k > c_2, \\ x_k, & \text{otherwise,} \end{cases}$$

Update $\triangle_k$

$$\triangle_{k+1} = \begin{cases} c_3 \triangle_k, & \text{if } r_k \leq c_2, \\ \max\{\triangle_{min}, \triangle_k\}, & \text{if } c_2 < r_k \leq c_1, \\ \max\{\triangle_{min}, c_4 \triangle_k\}, & \text{otherwise.} \end{cases}$$

We denote

$$\psi(x, \triangle) = \sup\{-\phi(x, d) : \ ||d|| \leq \triangle\}, \tag{1.5}$$

if for some $\triangle > 0, \psi(x, \triangle) = 0$, then x is said to be a critical point of (1.1).

2 Basic assumptions

The following basic assumptions on f, ϕ and B_k were given in [1].

A1. For all $x \in R^n, \phi(x, 0) = 0$ and $\phi(x, \cdot)$ is lower semicontinuous.

A2. For any convergent subsequence $\{x_k : k \in K \subseteq J\}$, if $d_k \to 0$, then

$$f(x_k + d_k) - f(x_k) \leq \phi(x_k, d_k) + o(||d||). \tag{2.6}$$

Where $J = \{0, 1, 2, 3, \cdots\}$.

A3. There exists $\bar{\triangle} > 0$ such that for all $||d|| \leq \bar{\triangle}, -\phi(\cdot, d)$ is lower semicontinuous.

A4. $\phi(x, \alpha d) \leq \alpha \phi(x, d), \forall x \in L_0, 0 \leq \alpha \leq 1.$

A5. $||B_k|| \leq c_9$, where c_9 is a constant.

A6. The level set $L(x_0) = \{x \in R^n \mid f(x) \leq f(x_0)\}$ is bounded.

3 The globally convergence of Algorithm TR

In this section, we establish the convergence results of our algorithm given in the previous section.

First we have the following lemmas:

Lemma 3.1 [1] For any $x \in L_0$, the function $\psi(x,\cdot)$ is nondecreasing and for any $\alpha \in [0,1]$,

$$\psi(x,\alpha\triangle) \geq \alpha\psi(x;\triangle), \tag{3.7}$$

and for any $\triangle > 0$,

$$\psi(x;\triangle) = 0 \Leftrightarrow \psi(x;1) = 0. \tag{3.8}$$

Lemma 3.2 [1] For all $\triangle \geq \triangle_k$,

$$f(x_k) - Q_k(d_k) \geq \frac{c_0}{2\triangle}\psi(x_k,\triangle)\min\{\triangle_k, \psi(x_k,\triangle)/(||B_k||\triangle)\},$$

where the second term in the min notation is understood as ∞ if $B_k = 0$.

In this section, we establish the convergence results of our algorithm given in the previous section.

Proposition 3.2 The Algorithm TR is well-defined.

Proof. Suppose the Algorithm is not well-defined , then there exist $k_0 \in N$, such that $f(x_{k_0}) - Q_k(d_{k_0}) = 0$. By Lemma 3, we have $\psi(x_{k_0},\triangle) = 0$, i.e., x_{k_0} is the critical point of f. Then the Algorithm stop. $\square$

Lemma 3.3 Suppose the Assumption $A_1 - A_6$ are satisfied, and $\{x_k\}$ is generated from Algorithm TR. The subsequence $\{x_k :\in K\}$ convergences to $\bar{x}$, if $\bar{x}$ is not the critical point of f, then we have

$$\lim_{k\to\infty}\inf_{k\in K}\triangle_k > 0. \tag{3.9}$$

Proof. Suppose that (3.9) is not true. Without loss of generality, we assume

$$\lim_{k\to\infty}\inf_{k\in K}\triangle_k = 0. \tag{3.10}$$

By modify rule of trust region radius in Algorithm TR, the trust region radius is bounded below when the iteration is successful. So it's failed in the $k-1$ iteration when $k \in K$ is large enough. Then we have

$$r_{k-1} < c_2, \tag{3.11}$$

i.e., for any $k \in K$ large enough, $x_{k-1} = x_k$, and $\triangle_k = c_3\triangle_{k-1}$. By the assumption, subsequences $\{x_k : k \in K\}$ convergences to $\bar{x}$, so $\{x_{k-1} : k \in K\}$ convergence to $\bar{x}$ too. We have

$$\lim_{k\to\infty}\inf_{k\in K}\triangle_{k-1} = 0. \tag{3.12}$$

on the other hand, because $\bar{x}$ is not the stationary of f, then by Lemma 3, there is a constant $\gamma_1 > 0$, such that for all large enough k, $k \in K$, we have

$$\psi(x_{k-1}, \triangle_0) \geq \gamma_1.$$

From lemma 3, and Assumption A_5, for the enough large k, $k \in K$, we have

$$\begin{aligned} f(x_{k-1}) - Q_{k-1}(d_{k-1}) &\geq \frac{c_0}{2\triangle_0}\psi(x_{k-1}, \triangle_0)\min\{\triangle_{k-1}, \psi(x_{k-1}, \triangle_0)/(\|B_{k-1}\|\triangle_0)\} \\ &\geq \frac{c_0\gamma_1}{2\triangle_0}\min\{\triangle_{k-1}, \frac{\gamma_1}{c_9\triangle_0}\} \\ &= \frac{c_0\gamma_1}{2\triangle_0}\triangle_{k-1} \\ &\geq \frac{c_0\gamma_1}{2\triangle_0}\| d_{k-1} \|. \end{aligned}$$

Then by Lemma 3 and Assumption A_5

$$\begin{aligned} \frac{f(x_{k-1}+d_{k-1}) - Q_{k-1}(d_{k-1})}{f(x_{k-1}) - Q_{k-1}(d_{k-1})} &\leq \frac{o\|d_{k-1}\| - \frac{1}{2}d_{k-1}^T B_{k-1} d_{k-1})}{\frac{c_0\gamma_1}{2\triangle_0}\| d_{k-1} \|} \\ &\leq \frac{o\|d_{k-1}\|}{\frac{c_0\gamma_1}{2\triangle_0}\| d_{k-1} \|}, \end{aligned}$$

combining with (3.12), let k large enough, $k \in K$, we have

$$\frac{f(x_{k-1}+d_{k-1}) - Q_{k-1}(d_{k-1})}{f(x_{k-1}) - Q_{k-1}(d_{k-1})} < 1 - c_2,$$

arrange it over again, we get

$$\frac{f(x_{k-1}) - f(x_{k-1}+d_{k-1})}{f(x_{k-1}) - Q_{k-1}(d_{k-1})} > c_2.$$

which contradicts (3.11). □

Lemma 3.4 Suppose that Assumption A_1 and Assumption A_6 are satisfied, and $\{x_k\}$ is generated from Algorithm TR. Then there are infinite successful iterations or terminated in finite iterations.

Proof. Assume that there are just finite successful iterations. Then there exists $k_0 \in N$, such that for all $k \in N,\ k > k_0$, we have $x_{k+1} = x_{k_0}$. So $\triangle_k \to 0$ and $x_k \to x_{k_0}$. But x_{k_0} is not the stationary point of f (else the Algorithm will be terminated), which contradicts lemma 3. □

Now we are in the position to give the global convergence theorem.

Theorem 3.5 Suppose the Assumption 1 and Assumption 2 is satisfied, and $\{x_k\}$ is generated from Algorithm TR. Then every accumulation point of the sequence $\{x_k\}$ is stationary point of f.

Proof. Suppose to contrary that any accumulation point $\bar{x}$ of $\{x_k\}$ is not the critical point of f, then there exists a subsequent $\{x_k :\in K\}$ convergences to $\bar{x}$. Then for any integer $\triangle_0$, there exist positive numbers $\varepsilon_0, \varepsilon_1 > 0$, and $N > 0$, such that for all $k > N, k \in K$ and $||x_k - \bar{x}|| \leq \varepsilon_1$, we have

$$\psi(x_k, \triangle_0) \geq \varepsilon_0. \tag{3.13}$$

As we know, if the kth iteration is not a successful iteration, then $x_{k+1} = x_k$. By Lemma 3, there are infinite successful iteration. So, without losing generality, we assume all the iterations are successful for $k \in K$. It means that $r_k > c_2$ for all $k \in K$. Combining with Lemma 3, we have, for all $k \in K$

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\geq c_2[f(x_k) - Q_k(d_k)] \\ &\geq \frac{c_0 c_2}{2\triangle_0} \psi(x_k, \triangle_0) \min\{\triangle_k, \frac{\psi(x_k, \triangle_0)}{(||B_k||\triangle_0)}\} \\ &\geq \frac{c_0 c_2}{2\triangle_0} \varepsilon_0 \min\{\triangle_k, \frac{\varepsilon_0}{c_9 \triangle_0}\}. \end{aligned} \tag{3.14}$$

Since $\{f(x_k)\}$ decreasing and bounded below, we have

$$\begin{aligned} \frac{c_0 c_2}{2\triangle_0} \varepsilon_0 \min\{\triangle_k, \frac{\varepsilon_0}{c_9 \triangle_0}\} &\leq \sum_{k \in K} f(x_k) - f(x_{k+1}) \\ &\leq \sum_{k=1}^{\infty} f(x_k) - f(x_{k+1}) \\ &< \infty, \end{aligned}$$

then, we conclude that $\triangle_k \to 0, k \in K$, which contradicts Lemma 3. $\square$

4 Applications

There are some applications to the above theory, such as Lipschitzian piecewise C^1 optimization and the nonlinear complementarity problem et al [1]. In this section, we discuss the following convex composite programming:

$$(CP) \qquad \min_{x \in R^n} h(g(x)), \tag{4.15}$$

where h is a convex and $g \in C^1$. Qi and Sun [1] have shown that the assumptions A1-A6 are satisfied for $\phi(x,d) = h(g(x) + \nabla g(x)^T d) - h(g(x))$. We can apply our trust region algorithm to solve 4.15.

Algorithm TR:

Let $\triangle_0, c_0, c_1, c_2, c_3$ and c_4 be positive constants satisfying $c_0 \leq 1, c_2 < c_1 < 1, c_3 < 1 < c_4$. At the kth iteration, given x_k, B_k and $\triangle_k$, solve the subproblem:

$$(SCP) \qquad \min_{||d|| \leq \triangle_k} Q_k(d) \quad =: \quad h(g(x_k) + \nabla g(x_k)^T d) + \frac{1}{2} d^T B_k d, \tag{4.16}$$

Assume the exact solution of (4.16) is d_k^*. Suppose that d_k is an inexact solution of (4.16) in the sense that d_k satisfies

$$h(g(x_k)) - Q_k(d_k) \geq c_0[h(g(x_k)) - Q_k(d_k^*)] \tag{4.17}$$

and

$$||d_k|| \leq \triangle_k.$$

If $d_k = 0$, then stop. Otherwise, let

$$r_k = \frac{h(g(x_k)) - h(g(x_k + d_k))}{h(g(x_k)) - Q_k(d_k)}, \tag{4.18}$$

Set

$$x_{k+1} = \begin{cases} x_k + d_k, & \text{if } r_k > c_2, \\ x_k, & \text{otherwise,} \end{cases}$$

Update $\triangle_k$.

$$\triangle_{k+1} = \begin{cases} c_3 \triangle_k, & \text{if } r_k \leq c_2, \\ \max\{\triangle_{min}, \triangle_k\}, & \text{if } c_2 < r_k \leq c_1, \\ \max\{\triangle_{min}, c_4 \triangle_k, \} & \text{otherwise} \end{cases}$$

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