Four Examples of Phase Synchronization in Oscillator Networks

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Abstract This paper presents four kinds of phase synchronization in oscillator networks by constructing the matrix of coupling strength. Numerical simulations are implemented to verify the results.

1 Introduction:

Considerable attention has been devoted to the different synchronization phenomena [1] in biology. Such as complete synchronization, cluster synchronization, anti-synchronization, and so on [2, 3, 4, 5, 6]. During recent twenty years, phase synchronization has attracted much interest. Many dynamical behaviors in biological systems can be explained by phase synchronization finally [8]. This synchronization is different from the classical concept of synchronization. In the research of phase synchronization, one of the simplest model is Kuramoto model [7].

The Kuramoto model describe the coupling dynamics of a set of $n$ oscillators. The Kuramoto model is given by:

$$\dot{\phi}_i = \omega_i + \sum_{j=1}^{n} c_{ij} \sin(\phi_j - \phi_i), \quad i = 1, 2, \ldots, n.$$  \hspace{1cm} (1)

where $c_{ij}$ is the coupling strength, a key parameter in the problem and $\omega_i$ is natural frequency.

In this article we accommodate the coupling strength $c_{ij}$ to simulate the different phase synchronization including frequency synchronization, complete phase synchronization, cluster phase synchronization and anti-phase synchronization. Finally we approve the feasibility of different phase synchronization using Kuramoto model.

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2 Four kinds of phase synchronization

2.1 Frequency synchronization

First, we study frequency synchronization. For the oscillator network (1), frequency synchronization means that all oscillators rotate with the same frequency i.e.:
\[
\lim_{t \to \infty} |\dot{\phi}_i(t) - \dot{\phi}_j(t)| = 0, \quad i, j = 1, 2, \ldots, n.
\]

where \(\dot{\phi}_i(t)\) is the frequency of \(i\)th oscillator.

Here the following example shows that the frequency synchronization can be achieved. Let \(n = 8\) in network (1) and the matrix of coupling strength \(C = (c_{ij})\) is:
\[
C = \begin{pmatrix}
0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & \frac{4}{\sqrt{3}} \\
\frac{2}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{4}{\sqrt{3}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{4}{\sqrt{3}} & 0 & \frac{2}{\sqrt{3}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 & \frac{4}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{4}{\sqrt{3}} & 0 & \frac{2}{\sqrt{3}} \\
\frac{4}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0
\end{pmatrix},
\]
and the frequency of the eight oscillators: \(w_1 = w_3 = w_5 = w_7 = 3\sqrt{3}, w_2 = w_4 = w_6 = w_8 = \sqrt{3},\)

Then the classical numerical results are carried out using MATLAB in the following figures:

The above figures present that oscillators with different nature frequency can achieve frequency synchronization by constructing the matrix of coupling strength in Kuramoto model.

2.2 Complete phase synchronization

Next we consider complete phase synchronization. Complete phase synchronization means the phase of all oscillators converge to the same. i.e. the following equation holds:
\[
\lim_{t \to \infty} |\phi_i(t) - \phi_j(t)| = 0, \quad i, j = 1, 2, \ldots, n.
\]

where \(\phi_i(t)\) is the phase of \(i\)th oscillator.

Here we give the following example to illustrate the existence of complete phase synchronization. Let \(n = 8\) in network (1) and the matrix of coupling strength \(C = \)
(\mathbf{c}_{ij}) is:
\[
C = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\
1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

here all the oscillators have the same frequency \( w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = w_7 = w_8 = 1 \).

And initial phase array is \([0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6]\).

Then we get the following numerical results as shown in Fig5,6,7,8:

We get the similar results for various initial values. So complete phase synchronization can be achieved by constructing the matrix of coupling strength in Kuramoto model.

### 2.3 Cluster phase synchronization

Further we study cluster phase synchronization\(^{[10]}\). Cluster phase synchronization is defined as a spatiotemporal pattern\(^{[9]}\). In this pattern, all the oscillators are
divided into $m$ clusters where the difference of the phase of the oscillators in the
same cluster converge to zero but not in different cluster, i.e,
\[
\lim_{t \to \infty} |\phi_i(t) - \phi_j(t)| = 0, \quad i \in n_l, j \in n_l, l = 1, \cdots, m,
\]
\[
\lim_{t \to \infty} |\phi_i(t) - \phi_j(t)| \neq 0, \quad i \in n_k, j \in n_k, l = 1, \cdots, m, l \neq k,
\]
where $\phi_i(t)$ is the phase of $i$th oscillator.

In this paper, we present an example with $n = n_1 + n_2 + n_3 = 8$, where $n_1, n_2, n_3$
correspondingly represent the size of the each cluster. Let $n_1 = 3, n_2 = 3, n_3 = 2$. The
matrix of coupling strength $C = (c_{ij})$ is:

\[
C = \begin{pmatrix}
0 & 3 & 0 & \varepsilon_1 & \varepsilon_1 & \varepsilon_1 & \varepsilon_3 & \varepsilon_3 \\
3 & 0 & 2 & \varepsilon_1 & \varepsilon_1 & \varepsilon_1 & \varepsilon_3 & \varepsilon_3 \\
0 & 2 & 0 & \varepsilon_1 & \varepsilon_1 & \varepsilon_1 & \varepsilon_3 & \varepsilon_3 \\
\varepsilon_1 & \varepsilon_1 & \varepsilon_1 & 0 & \sqrt{3} & 0 & \varepsilon_2 & \varepsilon_2 \\
\varepsilon_1 & \varepsilon_1 & \varepsilon_1 & \sqrt{3} & 0 & 1 & \varepsilon_2 & \varepsilon_2 \\
\varepsilon_1 & \varepsilon_1 & \varepsilon_1 & 0 & 1 & 0 & \varepsilon_2 & \varepsilon_2 \\
\varepsilon_3 & \varepsilon_3 & \varepsilon_3 & \varepsilon_2 & \varepsilon_2 & \varepsilon_2 & 0 & 1 \\
\varepsilon_3 & \varepsilon_3 & \varepsilon_3 & \varepsilon_2 & \varepsilon_2 & \varepsilon_2 & 1 & 0
\end{pmatrix},
\]
$\epsilon_1 = \frac{1}{32}, \epsilon_2 = \frac{1}{34}, \epsilon_3 = \frac{1}{35}$, the frequency: $w_1 = w_2 = w_3 = \frac{3}{32} + \frac{3\sqrt{3}}{35}, w_4 = w_5 = w_6 = \frac{1}{32} + \frac{2\sqrt{3}}{35}, w_7 = w_8 = \frac{7}{30}$. The initial phase array is $\phi_0 = [0.5, 1, 3, 4, 2, 1.5, 3, 5]$

The clustering phase synchronization can be shown clearly, as presented in Fig.9,10,11,12:

![Figure 9: The eight oscillators become cluster synchronization](image1)

![Figure 10: The time evolution of $\phi_1 - \phi_2$, $\phi_2 - \phi_3$. They belong to the first cluster.](image2)

![Figure 11: The time evolution of $\phi_4 - \phi_5$, $\phi_5 - \phi_6$ in the second cluster, $\phi_7 - \phi_8$ in the third cluster](image3)

![Figure 12: The time evolution of $\phi_1 - \phi_4$, $\phi_5 - \phi_8$](image4)

We get the similar results for various initial values. Clustering phase synchronization can be achieved by constructing the matrix of coupling strength in Kuramoto model.

### 2.4 Anti-phase synchronization

Finally, we consider anti-phase synchronization\textsuperscript{[11, 12]}. Anti-synchronization is a phenomenon that the state vectors of synchronized systems have the same absolute values but opposite signs. We say that anti-synchronization of two systems $S_1$ and $S_2$ is achieved if the following equation holds:

$$\lim_{t \to \infty} ||\phi_1(t) + \phi_2(t)|| = 0.$$

where $\phi_1(t), \phi_2(t)$ are the state vectors of the system $S_1, S_2$. In this paper, all the oscillators in the network are divided into two clusters. Complete phase synchronization
is reached in each cluster and the phase of the oscillators in different clusters fulfil the above equation. Because the oscillators in the two clusters have the same $2\pi$ cycle, the above equation equals the following:

$$
\lim_{t \to \infty} |\phi_1(t) - \phi_2(t)| = \pi.
$$

where $\phi_i(t)$ is the phase of $i$th oscillator. The following example proves the existence of this anti-phase synchronization. Let $n = 7, n_1 = 3, n_2 = 4$ and the the matrix of coupling strength is:

$$
C = \begin{pmatrix}
0 & \sqrt{3} - 1 & 0 & -2 & 0 & 0 & 0 \\
\sqrt{3} - 1 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & -5 & 2 & 0 & -3 \\
-2 & -2 & -5 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 2 & 0 & 0 & \frac{1}{3} & 6 \\
0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
0 & 0 & -3 & 0 & 6 & 0 & 0
\end{pmatrix},
$$

and the frequency of each oscillator: $w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = w_7 = w_8 = 1$, The initial phase array is $\phi_0 = [1, 2, 3, 2.53, 3.5, 2, 4]$.

Then we get the following numerical results:

We get the similar results for various initial values. The above figures presents that anti-phase synchronization can be achieved by constructing the matrix of coupling strength in Kuramoto model.

### 3 Conclusion

By means of studying the phase synchronization of classical Kuramoto model through some examples, we find that changing the matrix of coupling strength result in different kinds of phase synchronization. We confirm it possible that the existence of different synchronization states in nature.

Our study barely stay on examples. The further theoretical analysis will be provided in addition.

### References


Figure 13: The oscillators become anti-phase synchronization

Figure 14: The time evolution of $\phi_1$, $\phi_2$ and $\phi_3$

Figure 15: The time evolution of $\phi_1$, $\phi_2$, $\phi_3$ and $\phi_4$ being the same

Figure 16: The time evolution of $\phi_1 - \phi_4$. They become anti-phase synchronization


