The Sixth International Symposium on Operations Research and Its Applications (ISORA'06)
Xinjiang, China, August 8–12, 2006

E-Bayesian Method to Estimate Failure Rate*

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Abstract This paper develops a new method, named *E-Bayesian estimation*, to estimate reliability parameter. In the paper, the E-Bayesian estimation method of failure rate is derived for testing data from products with exponential distribution. Relations between E-Bayesian estimation and hierarchical Bayesian estimation are discussed. Finally, the new method is applied to a real testing data set, and as can be seen, it is both efficient and easy to operate.

Keywords reliability; failure rate; exponential distribution; E-Bayesian estimation; hierarchical Bayesian estimation

1 Introduction

Development of science and technology always claims efficiently improving the reliability of industrial products. Life-testing for some products often deals with truncated data. In some practical occasion, especially when sample size is relatively small or the concerned product is of high reliability, engineers are confronted with the testing data observed from the type I censored life testing. To a certain extent, it is quite difficult to estimate parameters by using classical statistical techniques. In literature, Lindley and Smith (1972) firstly introduced the idea of hierarchical prior distribution. Han (1997) developed the methods to construct hierarchical prior distribution. Recently, some other results have been made on hierarchical Bayesian method to deal with lifetime data. But those results obtained by means of hierarchical Bayesian methods involve doing complicated integrations. Though some computing methods such as MCMC (Markov Chain Monte Carlo) are available (Brooks (1998)), doing integration is still very inconvenient for practical problems.

This paper develops a new method, named *E-Bayesian estimation,* to estimate reliability parameter. The definition of E-Bayesian estimation is described in Section 2. In Section 3, the method of E-Bayesian estimation of failure rate is derived for testing data from products with exponential distribution. In Section 4, the formulas of hierarchical Bayesian estimation of failure rate are provided. In Section 5, the

*This work was supported partly by the Fujian Province Natural Science Foundation of China, and partly by the Fujian University of Technology of China.

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property of E-Bayesian estimation is discussed. In Section 6 and section 7, a simulation example and an application example are given respectively. Section 8 is a conclusion.

2 Definition of E-Bayesian estimation

Carry out type I censored life testing m times, the censored times are denoted by $t_i(t_1 < t_2 < \cdots < t_m)$, and the corresponding sample numbers are n_i, $i = 1,2,\cdots,m$. If $r_i(r_i = 0,1,\cdots,n_i)$ failure samples are observed in the testing process, then data (n_i,r_i,t_i) is called as the testing data from the products $(i = 1,2,\cdots,m)$.

Suppose that the life of a product is subject to an exponential distribution with probability density function

$$f(t) = \lambda \exp\{-t\lambda\}, \quad t > 0, \tag{1}$$

where $\lambda > 0$, λ is the failure rate of the exponential distribution.

Let the prior distribution of λ be its conjugated distribution——Gamma distribution with density function

$$\pi(\lambda|a,b) = b^a\lambda^{a-1}\exp(-b\lambda)/\Gamma(a),$$

where $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dx$ is Gamma function, $a > 0$, $b > 0$, and both a and b are hyper parameters.

According to Han (1997), a and b should be selected to guarantee that $\pi(\lambda|a,b)$ is a decreasing function of λ. Take derivative of $\pi(\lambda|a,b)$ over λ, we get

$$\frac{d\pi(\lambda|a,b)}{d\lambda} = [b^a\lambda^{a-2}\exp(-b\lambda)/\Gamma(a)][(a-1)-b\lambda].$$

Since $\lambda > 0$, $a > 0$, and $b > 0$, then $0 < a \leq 1$ and $b > 0$ will result in $\frac{d\pi(\lambda|a,b)}{d\lambda} < 0$, that is, $\pi(\lambda|a,b)$ is a decreasing function of λ. When $a = 1$, the density function $\pi(\lambda|a,b)$ of λ is a decreasing function. In view of the robustness of Bayesian estimation (Berger 1985), the narrower tailed prior distribution often leads to worse robustness. Accordingly, b should not be too big while $a = 1$. It is better to choose b below some given upper bound c(c is a positive constant). Thereby, the scope of hyper parameter b may be considered as $0 < b < c$ ($c = 1$ for instance).

While $a = 1$, the density function of λ becomes

$$\pi(\lambda|b) = b\exp(-b\lambda) \tag{2}$$

Definition 1. Let $\widehat{\lambda}(x)$ be continuous, if

$$\int_D \widehat{\lambda}(x)\pi(x)dx < \infty,$$

then

$$\widehat{\lambda}_{EB} = \int_D \widehat{\lambda}(b)\pi(b)db$$

is called the E-Bayesian estimation (expected Bayesian estimation) of λ, where D is the set of all the possible value of b, $\widehat{\lambda}(b)$ is Bayesian estimation of λ with hyper parameter b, and $\pi(b)$ is the density function of b over D.

In the definition 1, it is assumed, of course, that the integral $\int_D \widehat{\lambda}(x)\pi(x)dx$ exists. As a matter of fact, we have

$$\widehat{\lambda}_{EB} = \int_D \widehat{\lambda}(b)\pi(b)db = E[\widehat{\lambda}(b)],$$

that is, the E-Bayesian estimation of λ is the expectation of Bayesian estimation of λ for the hyper parameter.

3 E-Bayesian estimation

E-Bayesian estimation based on three different prior distributions of parameter λ is used in this section to investigate the influence of different prior distributions on the E-Bayesian estimation of λ.

Theorem 1. *For the testing data set (n_i, r_i, t_i) from m times type I censored testings of life distribution in (1), $i = 1, 2, \cdots, m$, let $M = \sum_{i=1}^{m}(n_i - r_i)t_i$, $r = \sum_{i=1}^{m} r_i$. If the prior density function $\pi(\lambda|b)$ of λ is given by (2), then, we have the following two conclusions.*

(i) Using the quadratic loss function, the Bayesian estimation of λ is $\widehat{\lambda}(b) = \frac{r+1}{M+b}$;

(ii) For the following prior densities function of b

$$\pi_1(b) = 2(1-b),\ 0 < b < 1 \tag{3}$$

$$\pi_2(b) = 1,\ 0 < b < 1 \tag{4}$$

$$\pi_3(b) = 2b,\ 0 < b < 1 \tag{5}$$

the corresponding E-Bayesian estimation of λ are respectively

$$\widehat{\lambda}_{EB1} = 2(r+1)\left\{(M+1)\ln\left(\frac{M+1}{M}\right) - 1\right\},$$

$$\widehat{\lambda}_{EB2} = (r+1)\ln\left(\frac{M+1}{M}\right),$$

$$\widehat{\lambda}_{EB3} = 2(r+1)\left\{1 - M\ln\left(\frac{M+1}{M}\right)\right\}.$$

Proof. (i) Carry out type I censored life testing m times, where the censored times are denoted by $t_i (t_1 < t_2 < \cdots < t_m)$ and the corresponding sample numbers are n_i, $i = 1, 2, \cdots, m$. If corresponding failure samples are X_i in the testing process, according to Lawless(1982), X_i is subject to a Poisson distribution with parameter $(n_i - r_i)t_i\lambda$, that is

$$P\{X_i = r_i\} = \frac{[(n_i - r_i)t_i\lambda]^{r_i}}{(r_i)!} \exp\{-(n_i - r_i)t_i\lambda\}$$

$r_i = 0, 1, 2, \cdots, n_i$, $i = 1, 2, \cdots, m$.

Then the likelihood function of λ is

$$L(r|\lambda) = \prod_{i=1}^{m} P\{X_i = r_i\} = \left\{\prod_{i=1}^{m} \frac{[(n_i - r_i)t_i]^{r_i}}{(r_i)!}\right\} \lambda^r \exp\{-M\lambda\}$$

where $M = \sum_{i=1}^{m}(n_i - r_i)t_i$, $r = \sum_{i=1}^{m} r_i$.

If the prior density function $\pi(\lambda|b)$ of λ is given by (2), then, by means of the Bayesian theorem, the posterior density function of λ will be

$$\begin{aligned} h(\lambda|r) &= \frac{\pi(\lambda|b)L(r|\lambda)}{\int_0^\infty \pi(\lambda|b)L(r|\lambda)d\lambda} \\ &= \frac{\lambda^r \exp\{-(M+b)\lambda\}}{\int_0^\infty \lambda^r \exp\{-(M+b)\lambda\}d\lambda} \\ &= \frac{(M+b)^{r+1}}{\Gamma(r+1)} \lambda^r \exp\{-(M+b)\lambda\},\ \lambda > 0. \end{aligned}$$

Thus, with the quadratic loss function, the Bayesian estimation of λ will be

$$\begin{aligned} \widehat{\lambda}(b) &= \int_0^\infty \lambda h(\lambda|r)d\lambda \\ &= \frac{(M+b)^{r+1}}{\Gamma(r+1)} \int_0^\infty \lambda^{(r+2)-1} \exp\{-(M+b)\lambda\}d\lambda \\ &= \frac{\Gamma(r+2)(M+b)^{r+1}}{\Gamma(r+1)(M+b)^{r+2}} \\ &= \frac{r+1}{M+b}. \end{aligned}$$

(ii) If the prior distribution of b is given by (3), then, by definition 1, the E-Bayesian estimation of λ will be

$$\widehat{\lambda}_{EB1} = \int_D \widehat{\lambda}(b)\pi_1(b)db = 2(r+1)\int_0^1 \frac{1-b}{M+b}db = 2(r+1)\left\{(M+1)\ln\left(\frac{M+1}{M}\right) - 1\right\}.$$

If the prior distribution of b is given by (4), then, by definition 1, the E-Bayesian estimation of λ will be

$$\widehat{\lambda}_{EB2} = \int_D \widehat{\lambda}(b)\pi_2(b)db = (r+1)\int_0^1 \frac{1}{M+b}db = (r+1)\ln\left(\frac{M+1}{M}\right).$$

If the prior distribution of b is given by (5), then, by definition 1, the E-Bayesian estimation of λ will be

$$\widehat{\lambda}_{EB3} = \int_D \widehat{\lambda}(b)\pi_3(b)db = 2(r+1)\int_0^1 \frac{b}{M+b}db = 2(r+1)\left\{1 - M\ln\left(\frac{M+1}{M}\right)\right\}.$$

Thus, the proof is completed. □

4 Hierarchical Bayesian estimation

If the prior density function $\pi(\lambda|b)$ of λ is given by (2), how can the value of hyper parameter b be determined? Lindley and Smith (1972) addressed an idea of hierarchical prior distribution, which suggested that one prior distribution may be adapted to the hyper parameters while the prior distribution includes hyper parameters.

If the prior density function $\pi(\lambda|b)$ of λ is given by (2), and the prior density function of hyper parameter b are given by (3), (4) and (5), then the corresponding hierarchical prior density function of λ will respectively be

$$\pi_4(\lambda) = \int_0^1 \pi(\lambda|b)\pi_1(b)db = 2\int_0^1 b(1-b)\exp(-b\lambda)db, \tag{6}$$

$$\pi_5(\lambda) = \int_0^1 \pi(\lambda|b)\pi_2(b)db = \int_0^1 b\exp(-b\lambda)db, \tag{7}$$

$$\pi_6(\lambda) = \int_0^1 \pi(\lambda|b)\pi_3(b)db = 2\int_0^1 b^2\exp(-b\lambda)db. \tag{8}$$

Theorem 2. *For the testing data set (n_i, r_i, t_i) from m times of type I censored testings of life distribution in (1), $i = 1, 2, \cdots, m$, let $M = \sum_{i=1}^{m}(n_i - r_i)t_i$, $r = \sum_{i=1}^{m} r_i$. If the hierarchical prior density function of λ are given by (6), (7) and (8), then, using the quadratic loss function, the corresponding hierarchical Bayesian estimation of λ are respectively*

$$\widehat{\lambda}_{HB1} = (r+1)\frac{\int_0^1 \frac{b(1-b)}{(M+b)^{r+2}}db}{\int_0^1 \frac{b(1-b)}{(M+b)^{r+1}}db},$$

$$\widehat{\lambda}_{HB2} = (r+1)\frac{\int_0^1 \frac{b}{(M+b)^{r+2}}db}{\int_0^1 \frac{b}{(M+b)^{r+1}}db},$$

$$\widehat{\lambda}_{HB3} = (r+1)\frac{\int_0^1 \frac{b^2}{(M+b)^{r+2}}db}{\int_0^1 \frac{b^2}{(M+b)^{r+1}}db}.$$

Proof. According to the proof of Theorem 1, the likelihood function of λ is

$$L(r|\lambda) = \prod_{i=1}^{m} P\{X_i = r_i\} = \left\{\prod_{i=1}^{m}\frac{[(n_i - r_i)t_i]^{r_i}}{(r_i)!}\right\}\lambda^r\exp\{-M\lambda\}$$

where $M=\sum_{i=1}^{m}(n_i-r_i)t_i$, $r=\sum_{i=1}^{m}r_i$.

If the hierarchical prior density function of λ is given by (6), then, by means of the Bayesian theorem, the hierarchical posterior density function of λ will be

$$h_1(\lambda|r)=\frac{\pi_4(\lambda)L(r|\lambda)}{\int_0^\infty \pi_4(\lambda)L(r|\lambda)d\lambda}=\frac{\int_0^1 b(1-b)\lambda^r\exp[-(M+b)\lambda]db}{\int_0^1 \frac{b(1-b)\Gamma(r+1)}{(M+b)^{r+1}}db}$$

where $0<\lambda<\infty$.

Using quadratic loss function, the hierarchical Bayesian estimation of λ will be

$$\begin{aligned}\widehat{\lambda}_{HB1} &= \int_0^\infty \lambda h_1(\lambda|r)d\lambda\\ &= \frac{\int_0^1 b(1-b)\left\{\int_0^\infty \lambda^{(r+2)-1}\exp[-(M+b)\lambda]d\lambda\right\}db}{\int_0^1 \frac{b(1-b)\Gamma(r+1)}{(M+b)^{r+1}}db}\\ &= \frac{\int_0^1 \frac{b(1-b)\Gamma(r+2)}{(M+b)^{r+2}}db}{\int_0^1 \frac{b(1-b)\Gamma(r+1)}{(M+b)^{r+1}}db}\\ &= (r+1)\frac{\int_0^1 \frac{b(1-b)}{(M+b)^{r+2}}db}{\int_0^1 \frac{b(1-b)}{(M+b)^{r+1}}db}.\end{aligned}$$

If the hierarchical prior density function of λ is given by (7), then, by means of the Bayesian theorem, the hierarchical posterior density function of λ will be

$$h_2(\lambda|r)=\frac{\pi_5(\lambda)L(r|\lambda)}{\int_0^\infty \pi_5(\lambda)L(r|\lambda)d\lambda}=\frac{\int_0^1 b\lambda^r\exp[-(M+b)\lambda]db}{\int_0^1 \frac{b\Gamma(r+1)}{(M+b)^{r+1}}db}$$

where $0<\lambda<\infty$.

Using quadratic loss function, the hierarchical Bayesian estimation of λ will be

$$\begin{aligned}\widehat{\lambda}_{HB2} &= \int_0^\infty \lambda h_2(\lambda|r)d\lambda\\ &= \frac{\int_0^1 b\left\{\int_0^\infty \lambda^{(r+2)-1}\exp[-(M+b)\lambda]d\lambda\right\}db}{\int_0^1 \frac{b\Gamma(r+1)}{(M+b)^{r+1}}db}\\ &= \frac{\int_0^1 \frac{b\Gamma(r+2)}{(M+b)^{r+2}}db}{\int_0^1 \frac{b\Gamma(r+1)}{(M+b)^{r+1}}db}\\ &= (r+1)\frac{\int_0^1 \frac{b}{(M+b)^{r+2}}db}{\int_0^1 \frac{b}{(M+b)^{r+1}}db}.\end{aligned}$$

If the hierarchical prior density function of λ is given by (8), then, by means of the Bayesian theorem, the hierarchical posterior density function of λ will be

$$h_3(\lambda|r) = \frac{\pi_6(\lambda)L(r|\lambda)}{\int_0^\infty \pi_6(\lambda)L(r|\lambda)d\lambda} = \frac{\int_0^1 b^2\lambda^r \exp[-(M+b)\lambda]db}{\int_0^1 \frac{b^2\Gamma(r+1)}{(M+b)^{r+1}}db}$$

where $0 < \lambda < \infty$.

Similarly , the hierarchical Bayesian estimation of λ will be

$$\begin{aligned}
\widehat{\lambda}_{HB3} &= \int_0^\infty \lambda h_3(\lambda|r)d\lambda \\
&= \frac{\int_0^1 b^2\left\{\int_0^\infty \lambda^{(r+2)-1}\exp[-(M+b)\lambda]d\lambda\right\}db}{\int_0^1 \frac{b^2\Gamma(r+1)}{(M+b)^{r+1}}db} \\
&= \frac{\int_0^1 \frac{b^2\Gamma(r+2)}{(M+b)^{r+2}}db}{\int_0^1 \frac{b^2\Gamma(r+1)}{(M+b)^{r+1}}db} \\
&= (r+1)\frac{\int_0^1 \frac{b^2}{(M+b)^{r+2}}db}{\int_0^1 \frac{b^2}{(M+b)^{r+1}}db}.
\end{aligned}$$

Thus, we complete the proof. □

5 Property of E-Bayesian estimation

Now we discuss the relations between E-Bayesian estimation and E-Bayesian estimation for different prior distributions, and the relations between E-Bayesian estimation and hierarchical Bayesian estimation.

5.1 Relations Between $\widehat{\lambda}_{EB1}$, $\widehat{\lambda}_{EB2}$ and $\widehat{\lambda}_{EB3}$

Theorem 3. *In Theorem 1, when $M > 2$, $\widehat{\lambda}_{EBi}(i = 1,2,3)$ satisfy:* (i) $\widehat{\lambda}_{EB3} < \widehat{\lambda}_{EB2} < \widehat{\lambda}_{EB1}$, (ii) $\lim_{M\to\infty} \widehat{\lambda}_{EB1} = \lim_{M\to\infty} \widehat{\lambda}_{EB2} = \lim_{M\to\infty} \widehat{\lambda}_{EB3}$.

Proof. (i). According to Theorem 1, it suffices for us to prove that

$$2\left\{1 - M\ln\left(\frac{M+1}{M}\right)\right\} < \ln\left(\frac{M+1}{M}\right) < 2\left\{(M+1)\ln\left(\frac{M+1}{M}\right) - 1\right\}.$$

Since $-1 < x < 1$, we have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{i=1}^\infty (-1)^{i-1}\frac{x^i}{i}.$$

Let $x = \frac{1}{M}$, when $M > 2$, $0 < \frac{1}{M} < 1$, we get

$$\begin{aligned}
&\ln\left(\frac{M+1}{M}\right) - 2\left[1 - M\ln\left(\frac{M+1}{M}\right)\right] \\
&= (1+2M)\ln\left(\frac{M+1}{M}\right) - 2 \\
&= (1+2M)\left(\frac{1}{M} - \frac{1}{2M^2} + \frac{1}{3M^3} - \frac{1}{4M^4} + \frac{1}{5M^5} - \frac{1}{6M^6} + \cdots\right) - 2 \\
&= \left[(1+2M)\left(\frac{1}{M} - \frac{1}{2M^2} + \frac{1}{3M^3} - \frac{1}{4M^4}\right) - 2\right] \\
&\quad +(1+2M)\left[\left(\frac{1}{5M^5} - \frac{1}{6M^6}\right) + \left(\frac{1}{7M^7} - \frac{1}{8M^8}\right) + \cdots\right]
\end{aligned}$$

Notice that

$$(1+2M)\left(\frac{1}{M} - \frac{1}{2M^2} + \frac{1}{3M^3} - \frac{1}{4M^4}\right) - 2 = \frac{1}{12M^4}(2M^2 - 2M - 3),$$

$2M^2 - 2M - 3 > 0$ as $M > 2$, thus

$$\ln\left(\frac{M+1}{M}\right) > 2\left[1 - M\ln\left(\frac{M+1}{M}\right)\right],$$

therefor $\widehat{\lambda}_{EB3} < \widehat{\lambda}_{EB2}$.

Since

$$2\left[(1+M)\ln\left(\frac{M+1}{M}\right) - 1\right] - \ln\left(\frac{M+1}{M}\right) = (1+2M)\ln\left(\frac{M+1}{M}\right) - 2$$

and notice that $(1+2M)\ln(\frac{M+1}{M}) - 2 > 0$ as $M > 2$, hence we have $\widehat{\lambda}_{EB2} < \widehat{\lambda}_{EB1}$.

(ii) Notice that

$$\begin{aligned}
&\widehat{\lambda}_{EB2} - \widehat{\lambda}_{EB3} \\
&= \widehat{\lambda}_{EB1} - \widehat{\lambda}_{EB2} \\
&= (r+1)\left[(1+2M)\ln\left(\frac{M+1}{M}\right) - 2\right] \\
&= (r+1)\left[\frac{1}{12M^4}\left(2M^2 - 2M - 3\right)\right] \\
&\quad +(r+1)(1+2M)\left[\left(\frac{1}{5M^5} - \frac{1}{6M^6}\right) + \left(\frac{1}{7M^7} - \frac{1}{8M^8}\right) + \cdots\right],
\end{aligned}$$

then

$$
\begin{aligned}
&\lim_{M\to\infty}(\widehat{\lambda}_{EB2}-\widehat{\lambda}_{EB3}) \\
&=\lim_{M\to\infty}(\widehat{\lambda}_{EB1}-\widehat{\lambda}_{EB2}) \\
&=(r+1)\lim_{M\to\infty}\left[\frac{1}{12M^4}(2M^2-2M-3)\right] \\
&\quad+(r+1)\lim_{M\to\infty}(1+2M)\left[\left(\frac{1}{5M^5}-\frac{1}{6M^6}\right)+\left(\frac{1}{7M^7}-\frac{1}{8M^8}\right)+\cdots\right] \\
&=0,
\end{aligned}
$$

and hence we have $\lim_{M\to\infty}\widehat{\lambda}_{EB3}=\lim_{M\to\infty}\widehat{\lambda}_{EB2}=\lim_{M\to\infty}\widehat{\lambda}_{EB1}$.

Thus, we complete the proof. □

5.2 Relations Between $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}(i=1,2,3)$

Theorem 4. *In Theorem 1 and Theorem 2, $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}(i=1,2,3)$ satisfy:*

$$\lim_{M\to\infty}\widehat{\lambda}_{EBi}=\lim_{M\to\infty}\widehat{\lambda}_{HBi}(i=1,2,3)$$

We cannot prove this theorem. But we will give simulation example In the next section to verify this theorem.

Theorem 4 shows that $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}(i=1,2,3)$ are equal as M approaches infinity, or $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}(i=1,2,3)$ are close to each other when M is sufficiently big.

6 Simulation Example

According to Theorem 1 and Theorem 2, we can obtain the E-Bayesian estimation $\widehat{\lambda}_{EBi}$ of λ and the hierarchical Bayesian estimation $\widehat{\lambda}_{HBi}(i=1,2,3)$ of λ. Some results are listed in Table 1($r=0$), Table 2($r=1$), Table 3($r=2$) and Table 4($r=3$).

From Table 1, 2, 3, 4, we find that for some of M and r, $\widehat{\lambda}_{EBi}(i=1,2,3)$ satisfy Theorem 3, $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}(i=1,2,3)$ are very close to each other and satisfy Theorem 4.

7 Application Example

Consider the testing data of some electronic products in Table 5(time unit: hour), the life of such type of electronic products has an exponential distribution.

By Theorem 1 and Theorem 2, we can obtain the $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}(i=1,2,3)$. All numerical results are listed in Table 6.

From Table 6, we find that $\widehat{\lambda}_{EBi}(i=1,2,3)$ satisfy Theorem 3, $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}(i=1,2,3)$ are very close to each other and satisfy Theorem 4.

Table 1: Results of $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}$ $(r = 0)$

M	i	1	2	3	Range
1000	$\widehat{\lambda}_{EBi}$	9.9967E-04	9.9950E-04	9.9933E-04	3.3301E-07
1000	$\widehat{\lambda}_{HBi}$	9.9995E-04	9.9993E-04	9.9992E-04	7.5001E-07
1000	$\widehat{\lambda}_{-}$	2.8317E-07	4.2967E-07	1.3383E-07	1.4933E-07
5000	$\widehat{\lambda}_{EBi}$	1.9999E-04	1.9998E-04	1.9997E-04	1.3302E-08
5000	$\widehat{\lambda}_{HBi}$	1.9998E-04	1.9997E-04	1.9996E-04	2.0021E-08
5000	$\widehat{\lambda}_{-}$	6.6678E-09	1.0003E-08	1.3338E-08	6.6698E-09
10000	$\widehat{\lambda}_{EBi}$	9.9997E-05	9.9992E-05	9.9993E-05	4.3326E-09
10000	$\widehat{\lambda}_{HBi}$	9.9994E-05	9.9992E-05	9.9991E-05	3.1013E-09
10000	$\widehat{\lambda}_{-}$	2.6666E-09	3.0003E-09	2.2134E-09	8.0667E-08
50000	$\widehat{\lambda}_{EBi}$	1.9999E-05	1.9999E-05	1.9999E-05	1.1513E-10
50000	$\widehat{\lambda}_{HBi}$	1.9999E-05	1.9999E-05	1.9999E-05	2.0123E-10
50000	$\widehat{\lambda}_{-}$	5.5757E-10	6.0001E-10	6.4244E-10	8.4873E-11

Remark: $5.5757E-10 = 5.5757\times10^{-10}$, $\widehat{\lambda}_{-} = |\widehat{\lambda}_{EBi} - \widehat{\lambda}_{HBi}|(i = 1,2,3)$.

Table 2: Results of $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}$ $(r = 1)$

M	i	1	2	3	Range
1000	$\widehat{\lambda}_{EBi}$	1.9993E-03	1.9990E-03	1.9986E-03	6.6601E-07
1000	$\widehat{\lambda}_{HBi}$	1.9990E-03	1.9987E-03	1.9985E-03	5.0201E-07
1000	$\widehat{\lambda}_{-}$	3.3367E-07	3.0067E-07	1.6767E-07	1.6601E-07
5000	$\widehat{\lambda}_{EBi}$	3.9997E-04	1.9996E-04	1.9995E-04	2.6661E-08
5000	$\widehat{\lambda}_{HBi}$	3.9995E-04	1.9994E-04	1.9992E-04	1.9531E-08
5000	$\widehat{\lambda}_{-}$	2.3336E-08	2.0005E-08	3.3751E-09	1.6661E-08
10000	$\widehat{\lambda}_{EBi}$	1.9999E-04	1.9999E-04	1.9998E-04	6.6651E-09
10000	$\widehat{\lambda}_{HBi}$	1.9999E-04	1.9999E-04	1.9998E-04	3.1013E-09
10000	$\widehat{\lambda}_{-}$	2.6666E-09	3.6559E-09	3.3319E-09	1.3328E-12
50000	$\widehat{\lambda}_{EBi}$	3.9999E-05	3.9999E-05	3.9999E-05	2.3025E-10
50000	$\widehat{\lambda}_{HBi}$	3.9999E-05	3.9999E-05	3.9999E-05	1.9012E-09
50000	$\widehat{\lambda}_{-}$	1.8487E-10	3.0021E-10	1.4849E-10	1.3001E-09

Table 3: Results of $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}$ $(r = 2)$

M	i	1	2	3	Range
1000	$\widehat{\lambda}_{EBi}$	2.9990E-03	2.9985E-03	2.9980E-03	9.9905E-07
1000	$\widehat{\lambda}_{HBi}$	2.9985E-03	2.9980E-03	2.9978E-03	7.0321E-07
1000	$\widehat{\lambda}_{-}$	5.3050E-07	5.0100E-07	2.0150E-07	3.9901E-07
5000	$\widehat{\lambda}_{EBi}$	5.9996E-04	5.9994E-04	5.9992E-04	3.9991E-08
5000	$\widehat{\lambda}_{HBi}$	5.9995E-04	5.9993E-04	5.9990E-04	5.9633E-08
5000	$\widehat{\lambda}_{-}$	1.3393E-08	1.0208E-08	2.0013E-08	2.0009E-08
10000	$\widehat{\lambda}_{EBi}$	2.9999E-04	2.9999E-04	2.9998E-04	9.9977E-09
10000	$\widehat{\lambda}_{HBi}$	2.9999E-04	1.9998E-04	1.9997E-04	2.1041E-09
10000	$\widehat{\lambda}_{-}$	3.9998E-09	5.0011E-09	1.0102E-08	2.3219E-12
50000	$\widehat{\lambda}_{EBi}$	5.9999E-05	5.9999E-05	5.9999E-05	3.4538E-10
50000	$\widehat{\lambda}_{HBi}$	5.9999E-05	5.9999E-05	5.9999E-05	3.6535E-10
50000	$\widehat{\lambda}_{-}$	4.2730E-10	5.9999E-10	7.7268E-10	3.4538E-10

Table 4: Results of $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}(r = 3)$

M	i	1	2	3	Range
1000	$\widehat{\lambda}_{EBi}$	3.9987E-04	3.9981E-04	3.9973E-04	1.3321E-06
1000	$\widehat{\lambda}_{HBi}$	3.9981E-04	3.9974E-04	3.9970E-04	1.1328E-06
1000	$\widehat{\lambda}_{-}$	5.6733E-07	6.0133E-07	3.3533E-07	2.3200E-07
5000	$\widehat{\lambda}_{EBi}$	7.9995E-04	7.9992E-04	7.9989E-04	5.3321E-08
5000	$\widehat{\lambda}_{HBi}$	7.9992E-04	7.9991E-04	7.9988E-04	4.2373E-08
5000	$\widehat{\lambda}_{-}$	2.6671E-08	1.0011E-08	1.3350E-08	1.3321E-08
10000	$\widehat{\lambda}_{EBi}$	3.9999E-04	3.9998E-04	3.9997E-04	2.0330E-08
10000	$\widehat{\lambda}_{HBi}$	3.9999E-04	3.9998E-04	3.9997E-04	2.2016E-08
10000	$\widehat{\lambda}_{-}$	1.3334E-09	1.9999E-09	2.6664E-09	1.3330E-08
50000	$\widehat{\lambda}_{EBi}$	7.9999E-05	7.9999E-05	7.9999E-05	4.6050E-10
50000	$\widehat{\lambda}_{HBi}$	7.9999E-05	7.9999E-05	7.9999E-05	4.5531E-10
50000	$\widehat{\lambda}_{-}$	4.3026E-10	2.0001E-10	3.0243E-10	4.0002E-10

Table 5: Testing data of the electronic products

i	1	2	3	4	5	6	7
t_i	480	680	880	1080	1280	1480	1680
n_i	3	3	5	5	8	8	8
r_i	0	0	0	1	0	2	1

Table 6: Results of $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}$

i	1	2	3	Range
$\widehat{\lambda}_{EBi}$	8.53238E-05	8.53235E-05	8.53233E-05	4.97347E-10
$\widehat{\lambda}_{HBi}$	8.53241E-05	8.53237E-05	8.53234E-05	7.02103E-10
$\widehat{\lambda}_{-}$	3.57000E-10	2.33000E-10	1.73000E-07	6.50000E-10

Table 7: Results of $\widehat{R}_{EBi}(500)$ and $\widehat{R}_{HBi}(500)$

i	1	2	3	Range
$\widehat{R}_{EBi}(500)$	0.9582353	0.9582354	0.9582356	2.38288E-07
$\widehat{R}_{HBi}(500)$	0.9582351	0.9582353	0.9582354	3.84252E-07
$\widehat{R}_{-}(500)$	2.290E-07	1.070E-07	8.270E-08	1.46300E-07

Remark: $\widehat{R}_{-}(500) = |\widehat{R}_{EBi}(500) - \widehat{R}_{HBi}(500)|(i = 1,2,3)$.

Based on Table 6, we can calculate the E-Bayesian estimation $\widehat{R}_{EBi}(t) = \exp(-\widehat{\lambda}_{EBi}t)$ and the hierarchical Bayesian estimation $\widehat{R}_{HBi}(t) = \exp(-\widehat{\lambda}_{HBi}t)$ of the reliability for these electronic products. Some results are listed in Table 7 (where $\widehat{R}_{-}(t) = |\widehat{R}_{EBi}(t) - \widehat{R}_{HBi}(t)|$).

From Table 7, we find that $\widehat{R}_{EBi}(500)$ and $\widehat{R}_{HBi}(500)(i = 1,2,3)$ are very close to each other.

8 Conclusion

This paper develops a new method, named E-Bayesian estimation to estimate reliability parameters. The author would like to put forward the following two questions for any new parameter estimation method: (1) How much dependence is there between the new method and other already-made ones? (2) In which aspects is the new method superior to the old ones?

For the E-Bayesian estimation method, Theorem 3 and Theorem 4 gave a good answer to the above question (1). To the above question (2), from Theorem 1 and Theorem 2, we find that the expression of the E-Bayesian estimation is the simplest one, whereas the expression of the hierarchical Bayesian estimation relies on integral expression, which is often not easy, that is an answer to the above question (2).

By Theorem 3, $\widehat{\lambda}_{EBi}(i = 1,2,3)$ satisfy: (i) $\widehat{\lambda}_{EB3} < \widehat{\lambda}_{EB2} < \widehat{\lambda}_{EB1}$, (ii) $\lim_{M\to\infty}\widehat{\lambda}_{EB1} = \lim_{M\to\infty}\widehat{\lambda}_{EB2} = \lim_{M\to\infty}\widehat{\lambda}_{EB3}$. By the simulation example and the application example, $\widehat{\lambda}_{EBi}(i = 1,2,3)$ are very close to each other and satisfy Theorem 3; also that $\widehat{\lambda}_{HBi}(i = 1,2,3)$ and $\widehat{\lambda}_{HBi}(i = 1,2,3)$ are very close to each other and satisfy: (i) $\widehat{\lambda}_{HB3} < \widehat{\lambda}_{EB2} < \widehat{\lambda}_{HB1}$, (ii) $\lim_{M\to\infty}\widehat{\lambda}_{HB1} = \lim_{M\to\infty}\widehat{\lambda}_{HB2} = \lim_{M\to\infty}\widehat{\lambda}_{HB3}$; $\widehat{\lambda}_{EBi}$ and $\widehat{\lambda}_{HBi}(i = 1,2,3)$ are very close to each other and satisfy Theorem 4. So, in this paper, the author suggests uniform distribution as prior distribution of the hyper parameter. This might also be the main reason that in some literatures uniform distribution usually serves as prior distribution of the hyper parameter.

Acknowledgements

We wish to thank Professor X. Z. Wu, who has checked the earlier draft of this paper and given us very helpful suggestions.

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