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Dynamic Lot-Sizing Model With a Class of Multi-Breakpoint Discount Cost Structures*

Yu-Zhong Zhang† Jianteng Xu‡ Qingguo Bai§

College of Operations Research and Management, Qufu Normal University,
Rizhao, Shandong 276826, China

Abstract In this paper, we consider a class of single-item dynamic lot-sizing problem with quantity discount cost structure. We present an optimal polynomial algorithm for the case of multi-breakpoint N_i, $i = 1,2,\ldots,m$. The complexity of our algorithm is $O(n^3 + mn^2)$, where n is the number of periods in finite planning horizon.

Keywords discount structure; inventory replenishment; lot-sizing

1 Introduction

The single-item multi-period dynamic lot-sizing problem has received extensive attention in the literature. Immense interest in the dynamic lot-sizing problem is due to the fact that the problem arises in many practical situations, often as subproblems of some production and inventory planning problems. The interest is also due to the fact that the problem is efficiently solvable.

Quantity discount for inventory purchasing systems and production planning systems have attracted much attention. Federgruen and Lee (1990) consider two versions of the ELS problem: the all units discount cost structure, in which the price of all units is discounted when the order size exceeds a critical level, and the incremental discount cost structure in which the discount price is applied only to the units in excess of the critical level. They solve the problem by dynamic programming algorithms of complexity $O(T^3)$ and $O(T^2)$ respectively, where T is the number of periods in the planning horizon. However, Xu and Lu (1998) show that their algorithm fails to find the optimal solution for some special cases.

Chen *et al.* (2002a) consider an ELS problem with a modified all-unit discount freight cost structure. Such an ordering cost function represents transportation costs charged by many FTL carries. They demonstrate the NP-hardness and analyze the

*This work was supported by NSFC (10271054) and NSF of Shandong, China (Y2005A04).
†E-mail: yuzhongrz@163.com
‡E-mail: qingniaojt@163.com
§E-mail: bqg1979@163.com

worst case ratio for an easy-to-implement approximation solution. Their solution is the minimum cost solution that satisfies the ZIO policy. Chan et al (2002b) extend to a single-warehouse multi-retailer setting. In both papers, they show that the cost of the best ZIO policy is no more than $\frac{4}{3}$ ($\frac{5.6}{4.6}$ if costs are stationary) times the optimal one.

In our paper, we extend the model of Lee (1990) and Xu (1998) to multi-breakpoint discount structure, and develop a dynamic programming algorithm to solve our model in $O(n^3+mn^2)$ where m is the number of breakpoints. It is also an extension of the model of Lee (2004).

The paper is organized as follows: in section 2, we describe our model and some notations which will be used throughout the whole paper. In section 3 we present the optimality properties for dynamic lot size model with multi-breakpoint all-unit discount structure. In section 4, we give an exact dynamic programming algorithm with running time in $O(mn^2+n^3)$ and illustrate it with an example.

2 Descriptions and notations

Consider an ELS problem with n periods. For $1 \le t \le n$, we define the following notations:

d_t the demand at period t;

x_t the replenishment quantity at period t;

I_t the on-hand inventory level at the end of period t. We assume that no backlogging is allowed and hence $I_t \ge 0$;

h_t the cost of holding one unit production at period t;

A_t the fixed cost of ordering a cargo at period t;

c_t the cost of replenishing one unit production without discounting at period t;

r_i the discount rate which satisfies $0 < r_i < 1$, $i \in [1,m]$. The discount rate is r_i if and only if the replenishment quantity is in $[N_i, N_{i+1})$. Let $N_{m+1} = +\infty$;

$p_t(x_t)$ the cost of replenishing x_t units at period t

$$p_t(x_t) = \begin{cases} 0, & x_t = 0 \\ A_t + c_t x_t, & 0 < x_t < N_1 \\ A_t + c_t(1-r_1)x_t, & N_1 \le x_t < N_2 \\ A_t + c_t(1-r_2)x_t, & N_2 \le x_t < N_3 \\ \cdots & \\ A_t + c_t(1-r_m)x_t, & N_m \le x_t \end{cases}$$

where $N_1, N_2, \cdots, N_m$ are quantity breakpoints.

It is practically that the more quantities you order, the more discount rate will be. That is, $r_{i+1} > r_i$, $i = 1, 2, \ldots, m-1$. It is also popular in practice that the fixed cost will decrease if you order productions from the same supplier in the next period. So we can assume that

$$A_t \geq A_{t+1}, \quad t = 1, 2, \ldots, n-1$$

In addition, we will suppose the following inequality holding throughout the whole paper:

$$N_{i+1}(1-r_{i+1}) \geq N_i(1-r_i), \quad i = 1, 2, \ldots, m-1$$

Considering that for some commodities, which attach great importance to the fashion, ordering a new one is better than holding it. We can suppose that the cost of holding one unit in period $t-1$ is not less than ordering one in period t, then the following inequality holds:

$$h_{t-1} \geq c_t, \quad t = 2, 3, \ldots, n$$

We assume that all acquisition and demand fulfillment occur instantaneously at the beginning of the period.

The problem denoted by (*MBP*) can be formulated as follows:

$$\begin{aligned} \min \quad & \sum_{t=1}^{n} (p_t(x_t) + h_t I_t) \\ \text{s.t.} \quad & I_{t-1} + x_t - d_t = I_t, \quad t = 1, \ldots, n \\ & I_0 = 0, \quad I_n = 0 \\ & I_t, \quad x_t \geq 0 \end{aligned}$$

Before analyzing the optimality properties of the problem, let us introduce some more notations. For each i, j with $i \leq j$, we define

$$h(i,j) = h_i + h_{i+1} + \cdots + h_j, \quad d(i,j) = d_i + d_{i+1} + \cdots + d_j$$

We also define $d(i,j) = 0$ and $h(i,j) = 0$ if $i > j$. Note that in the following algorithm we will calculate $d(i,j)$ and $h(i,j)$ for all $1 \leq i \leq j \leq n$ in $O(n^2)$ beforehand.

Definition 1. Period t is called a *regeneration point* if $I_t = 0$, and is called a *replenishment period* if $x_t > 0$.

3 Optimality properties

We now provide some optimality properties that will be used in the following algorithm.

Property 1.
Any optimal solution of problem (MBP) satisfies that $I_{t-1} < d_t$ if and only if $x_t > 0$.

Proof. We prove it by contradiction.

Suppose that there exists an optimal solution such that $I_{t-1} < d_t$ and $x_t = 0$. We can deduce $I_t < 0$ from $I_{t-1} + x_t - d_t = I_t$, which is a contradiction of no shortages.

Suppose that there exists an optimal solution such that there exists a period t with $I_{t-1} \geq d_t,\ \ x_t > 0$. We can delay the replenishment in period t to period $t+1$. This will result in a new feasible solution whose total cost will no more than the optimal one. It is a contradiction. □

Property 2.
Any optimal solution of problem (MBP) satisfies that if there exists a replenishment period t between two consecutive regeneration points $i-1$ and j, then, the replenishment quantity in period t is one of N_k, $k = 1, 2, \ldots, m$ and $d(t, j) - I_{t-1}$.

Proof. Two cases are to be considered.

Case 1. If there exists an optimal solution such that there is no replenishment period between $t+1$ and j, then we can conclude $x_t = d(t, j) - I_{t-1}$ from $I_{t-1} + x_t - d(t, j) = 0$.

Case 2. If there is an optimal solution such that there exists a replenishment period s after period t between two consecutive regeneration points $i-1$ and j, that is, $t < s \leq j$, we can prove the property by contradiction. Suppose $x_t \in (N_{i-1}, N_i)$, $i = 1, 2, \ldots, m$, where $N_0 = 0,\ r_0 = 0$, we can reduce x_t by one unit and increase x_s by one unit and obtain a new feasible solution whose value is denoted as $V^{'}$. The replenishment quantities in period t and s are $x_t^{'} = x_t - 1$ and $x_s^{'} = x_s + 1$.

Let r_v and $r_v^{'}$ be discount rate in period s before and after changed, then $1 - r_v^{'} \leq 1 - r_v$. Hence

$$
\begin{aligned}
V^{'} - V =& p_t(x_t^{'}) - p_t(x_t) + p_s(x_s^{'}) - P_s(x_s) - h(t, s-1) \\
\leq& - c_t(1 - r_{i-1}) + c_s(1 - r_v) - h(t, s-1) \\
<& 0
\end{aligned}
$$

The last inequality holds by $c_s(1 - r_v) \leq c_s \leq h_{s-1} \leq h(t, s-1)$. This is a contradiction. □

Property 3.
Any optimal solution of problem (MBP) satisfies that if there is a period t with $d_t - I_{t-1} \in [N_{u-1}, N_u)$, $u \in [1, m]$, $N_0 = 0$, $i \leq t < j$ between two consecutive regeneration points $i-1$ and j, then period t is a replenishment period with $x_t = N_u$ or $d(t, j) - I_{t-1}$.

Proof. There are two cases to consider:

Case 1. If in the optimal solution there is no replenishment period between $t+1$ and j, then we have $x_t = d(t, j) - I_{t-1}$.

Case 2. If in the optimal solution there exists a replenishment period s after period t, $t < s \leq j$, suppose that the discount rate in period s is r_v. We can prove it by contradiction.

Case 2.1. If $x_t < N_u$, we have $x_t = N_{u-1}$ by property 2. Hence we get $I_t \leq 0$ by $x_t = I_t + d_t - I_{t-1}$ and $d_t - I_{t-1} \in [N_{u-1}, N_u)$, which is a contradiction of the optimality.

Case 2.2. If $x_t > N_u$, without loss of generality, according to property 2 we can let $x_t = N_{u+q}$, $q \neq 0$. In such a case, we can decrease x_t by $N_{u+q} - N_u$ and increase x_s by the same quantities, then we have a new solution with $x_t^{'} = N_u$ and $x_s^{'} = x_s + (N_{u+q} - N_u)$, the corresponding discount rate in period s is $r_v^{'}$. Hence

$$\begin{aligned}
V^{'} - V =& p_t(x_t^{'}) - p_t(x_t) + p_s(x_s^{'}) - p_s(x_s) - (N_{u+q} - N_u)h(t, s-1) \\
=& c_t N_u(1 - r_u) - c_t N_{u+q}(1 - r_{u+q}) + c_s(1 - r_v^{'})(x_s + N_{u+q} - N_u) \\
& - c_s(1 - r_v)x_s - (N_{u+q} - N_u)h(t, s-1) \\
\leq& c_s(1 - r_v)x_s + c_s(1 - r_v)(N_{u+q} - N_u) - c_s(1 - r_v)x_s - (N_{u+q} - N_u)h(t, s-1) \\
\leq& 0
\end{aligned}$$

It is a contradiction of the optimality. □

4 Polynomial algorithm and example

Let $C(i, j)$ be the minimum total cost of satisfying demands from period i to j such that there is no regeneration point in between. Define $F(j)$ as the cost associated with an optimal replenishment plan from period 1 to j. We can solve (MBP) by the following dynamic programming algorithm:

$$F(j) = \min_{1 \leq i \leq j} \{F(i-1) + C(i, j)\}, \quad j \in [1, n]; \quad F(0) = 0 \tag{1}$$

Obviously, the optimal solution of (MBP) is $F(n)$. It is easy to know that we may find the optimal solution in $O(n^2)$ if $C(i, j)$ is known for all i, j.

The remaining question is how to find $C(i, j)$, $1 \leq i \leq j \leq n$. For this purpose, we only consider those solutions that satisfy properties $1-3$ in the remainder of the paper.

Suppose $i-1$ and j are two consecutive regeneration points, let $X(i, j) = (x_i, x_{i+1}, \ldots, x_j)$ be the solution satisfying $d(i, j)$ and $\overline{X}(i, j)$ be the solution of $X(i, j)$ which satisfies the following three conditions:

(1) for period t which satisfies $d_t - I_{t-1} \in [N_{u-1}, N_u), u \in [1, m]$, let $x_t = N_u$;

(2) for period t which satisfies $t = j$ or $d_t - I_{t-1} > N_m$, let $x_t = d(t, j) - I_{t-1}$;

(3) for period t with $d(t, j) - I_{t-1} < 0$, let $x_t = 0$.

Let $P(i, j)$ be the set of the replenishment periods in $\overline{X}(i, j)$. Let $I_{i,t}$ $(i \leq t \leq j)$ be the inventory level of period t when $i-1$ is the regeneration point and the replenishment policy follows $P(i, j)$ and $M(i, t)$ $(t \in [i, j))$ be the cost of satisfying demands from period i to t by replenishment periods in $P(i, j)$.

The following algorithm is used to find $P(i, j)$ and $M(i, t)$ for all $j \in [i, n]$, $t \in [i, j)$.

Algorithm :

Step 0: Set $t := i$, $I_{i,t-1} := 0$, $P(i, t-1) := \emptyset$. $M(i, t-1) := 0$, and go to Step 1.

Step 1: if $I_{i,t-1} - d_t > 0$, go to step 2; otherwise, go to Step 3.

Step 2: if $t = j$, set $P(i,t) := P(i,t-1)$, stop; otherwise, set $I_{i,t} = I_{i,t-1} - d_t$, $P(i,t) := P(i,t-1)$, $M(i,t) := M(i,t-1) + h_t I_{i,t}$, $t := t+1$, go to step 1.

Step 3: set $P(i,t) := P(i,t-1) \cup \{t\}$, $u_0 = \arg\min_{u\in[1,m]}\{u |\ d_t - I_{i,t-1} < N_u\}$, if u_0 doesn't exist or $t = j$, set $P(i,j) := P(i,t)$, stop; otherwise, set $I_{i,t} := I_{i,t-1} + N_{u_0} - d_t$, $M(i,t) := M(i,t-1) + A_t + c_t(1 - r_{u_0})N_{u_0} + h_t I_{i,t}$, $t := t+1$, and go to step 1.

For each $i \in [1,n]$, implementing the algorithm by $j = n$ will obtain all value of $P(i,t)$, $M(i,t)$ and $I_{i,t}$. So for all i The complexity of the algorithm is $O(mn^2)$.

$$C_k(i,j) = M(i,k-1) + p_k(d(k,j) - I_{i,k-1}) + \sum_{l=k}^{j-1} h_l d(l+1,j), \quad M(i,i-1) = 0$$

which means that the cost of replenishment $d(k,j) - I_{i,k-1}$ in period k, and before k, all the replenishment periods are following $P(i,k)$, then we have

$$C(i,j) = \min_{k\in P(i,j)} C_k(i,j).$$

If we have the value of $C_k(i,j)$ for all $k \in P(i,j)$, then we can calculate $C(i,j)$ in time $O(n)$. We calculate $\sum_{l=k}^{j-1} h_l d(l+1,j)$ in $O(n^2)$ in advance. We can find $C_k(i,j)$ for all possible i and j in $O(n^3)$. Totally, we can find $F(j)$ in $O(mn^2 + n^3)$ for all $j \in [1,n]$.

Example omitted.

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