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Optimal Investment Policy for Real Option Problems with Regime Switching

Naoki Makimoto*

Graduate School of Business Sciences, University of Tsukuba,
3-29-1 Otsuka, Bunkyo-ku, Tokyo 112-0012, Japan

Abstract Real option analysis (ROA) is a way of valuing real assets by invoking option pricing theory. In contrast to the traditional net present value approach, ROA makes it possible to explicitly incorporating management flexibility. Because of such advantages, ROA has become an important tool for project valuation and is widely used in practice.

In this paper, we develop a real option model with regime switching. In addition to stochastic fluctuation over time, the value process also depends on a regime process described by a two-state Markov chain. By extending standard approach based on Bellman equation and smooth pasting condition, we investigate the optimal investment policy and value functions. Unlike the existing real option models with single stochastic factor, two types of solution are possible according to model parameters. We also develop numerical procedures to compute the optimal policy and provide some numerical examples.

Keywords real option; optimal investment; Markovian regime switch; smooth pasting

1 Introduction

Real option analysis (ROA) is a way of valuing real assets such as projects and real estate by invoking financial option pricing theory. In contrast to the traditional approach to valuing investment projects based on the net present value, ROA makes it possible to explicitly incorporating management flexibility in the analysis. Because of this advantage, ROA has been an important and exciting area in the theory and practice of financial engineering.

Many papers have studied the theory and application of real option models. Types of option investigated so far include options to extend, defer, switch and others [4, 7, 8, 9]. However, most literatures assume there is only single source of future uncertainty. This assumption sometimes limits the applicability of ROA since in reality it is often the case where two or more uncertainties could affect the project value. If we assume that the value of the project is given as a product of two variables both of which follow a geometric Brownian motion (GBM), then the problem can be reduced to a single variable case since a product of GBM's is a GBM even though

*E-mail: makimoto@gssm.otsuka.tsukuba.ac.jp

they are correlated. However, if we assume other types of underlying processes, the problem to find optimal policy becomes quite difficult.

In this paper, we propose a real option model with two types of future uncertainty. The first uncertainty is modelled by a GBM as usual, and in addition to this, there is another uncertainty represented by a two-state Markov chain. The state of the Markov chain expresses the circumstance of investment, for example, market condition and perspective of economy. By extending standard approach based on Bellman equation and smooth pasting condition, we investigate the optimal investment policy and the structure of value functions. Unlike the existing real option models which assume only single stochastic factor, it is shown that two types of optimal solution are possible according to model parameters. We then develop numerical procedures which enable us to compute all unknown coefficients of the value functions. Some numerical examples are also provided to see how the optimal policy changes by model parameters.

This paper is organized as follows. In Section 2, we formulate the optimal investment problem. Section 3 is devoted to identifying the optimal investment policy and the structure of value functions. Computational procedures and numerical examples are given in Section 4. Finally in Section 5, we conclude this paper by giving some remarks.

2 Formulation of an optimal investment problem

In this section, we describe an investment decision problem and formulate it as an optimisation problem. Suppose there is a project to which we are thinking about investing some amount of money. At any time point before investment has been executed, we can choose either to invest now or to wait for future investment. If we decide to invest at t, then we will receive payoff $V_t - I$ where V_t is a value of the project and I is an investment cost.

In this paper, we assume that the project value is given by $V_t = R_t U_t$. Conceptually, U_t represents the fundamental value of the project and R_t represents a circumstance of the project such as market condition and perspective of economy. As in standard real option models, we assume that $\{U_t\}$ fluctuates over time according to the stochastic differential equation

$$\frac{dU_t}{U_t} = \mu dt + \sigma dz_t \tag{1}$$

where drift μ and volatility σ are constant and $\{z_t\}$ is a standard Brownian motion. It is well known that the solution of (1) is give as a geometric Brownian motion:

$$U_t = U_0 \exp\{(\mu - \sigma^2/2)t + \sigma z_t\}. \tag{2}$$

On the other hand, $\{R_t\}$ is a continuous time Markov chain on the state space $\mathscr{S} = \{r_1, r_2\}$ whose transition rate matrix is given as

$$Q = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}.$$

We assume without loss of generality that $0 < r_1 < r_2$. Thus, state 1 can be viewed as a *bad* state while state 2 is a *good* state. We also assume $\{U_t\}$ and $\{R_t\}$ are mutually independent. Furthermore, an investment cost I is assumed to be constant regardless of time of investment, as in the most literatures on real options.

Let $\{\mathscr{F}_t\}$ be a filtration generated by $\{U_t\}$ and $\{R_t\}$. That is, $\{\mathscr{F}_t\}$ contains all information of $\{U_t\}$ and $\{R_t\}$ available at t. An investment decision problem can now be formulated as

$$F_i(u) = \sup_{\tau\geq 0} E\left[e^{-\rho\tau}(R_\tau U_\tau - I)\,|\,U_0 = u, R_0 = r_i\right] \tag{3}$$

where supremum is taken over all $\{\mathscr{F}_t\}$-stopping times and ρ is a exogenously given discount rate for a future payoff. To assure the existence of the expectation, we assume throughout the paper that $\rho > \mu$.

Before closing this section, we will introduce the solution when $R_t = r$ is constant. The problem in this case is to obtain

$$H(u;r) = \sup_{\tau\geq 0} E[e^{-\rho\tau}(rU_\tau - I)|U_0 = u] \tag{4}$$

(we specify the dependence of $H(u;r)$ on r for later use). Then, as described in Dixit and Pindyck [4], $H(u;r)$ is given by

$$H(u;r) = \begin{cases} \left(\frac{r\upsilon - I}{\upsilon^\xi}\right)u^\xi, & u < \upsilon, \\ ru - I, & u \geq \upsilon \end{cases} \tag{5}$$

where $\upsilon = \frac{\xi I}{(\xi-1)r}$ and

$$\xi = \frac{\sigma^2/2 - \mu + \sqrt{(\sigma^2/2-\mu)^2 + 2\rho\sigma^2}}{\sigma^2} > 1.$$

Remark 1. In real option analysis, it is common to define the value function (3) by

$$\sup_{\tau\geq 0} E\left[e^{-\rho\tau}\max(R_\tau U_\tau - I, 0)\,|\,U_0 = u, R_0 = r_i\right] \tag{6}$$

to clarify analogy of real option with financial call option. It is however obvious that (3) and (6) lead to the same value function since it is never optimal to invest when $R_t U_t < I$ and, starting with any values of U_0 and R_0, $R_t U_t > I$ occurs for some future time t with probability 1.

3 Optimal investment policy

In this section, we identify the characteristics of the optimal investment policy. First, we will show that the optimal policy is of threshold type.

Lemma 1. *The optimal investment policy is of threshold type, i.e., according to the regime state R_t, there exist threshold levels υ_1, υ_2 such that it is optimal to invest at time t if $U_t \geq \upsilon_i$ and $R_t = r_i$ and otherwise it is optimal to continue waiting.*

Proof. Fix $R_0 = i$. Suppose to the contrary that there exist $u < \tilde{u}$ such that it is optimal to invest now when $U_0 = u$ while it is optimal to wait when $U_0 = \tilde{u}$. Specifically,

$$r_i u - I \geq E[e^{-\rho\tau}(R_\tau U_\tau - I) | U_0 = u, R_0 = r_i] \tag{7}$$

holds for any $\{\mathscr{F}_t\}$-stopping time τ, and there exists a $\{\mathscr{F}_t\}$-stopping time $\tilde{\tau}$ satisfying

$$r_i \tilde{u} - I < E[e^{-\rho\tilde{\tau}}(R_{\tilde{\tau}} U_{\tilde{\tau}} - I) | U_0 = \tilde{u}, R_0 = r_i]. \tag{8}$$

We adopt the following stochastic coupling argument. Fix arbitrary sample paths R_t^A and U_t^A starting with $U_0 = \tilde{u}$ and $R_0 = r_i$ (we call it sample path A) and let $\tilde{\tau}$ be the optimal investment time for sample path A. To construct a sample path starting with $U_0 = u$ and $R_0 = r_i$ (we call it sample path B), set $R_t^B = R_t^A$ and $U_t^B = \gamma U_t^A$ for all $t \geq 0$ with $\gamma = u/\tilde{u}$. Note from (2) that U_t^B so constructed follows a GBM with $U_0 = u$. If one uses the policy to invest at $\tilde{\tau}$ for sample path B, we then obtain

$$e^{-\rho\tilde{\tau}}(R_{\tilde{\tau}}^B U_{\tilde{\tau}}^B - I) = \gamma e^{-\rho\tilde{\tau}}(R_{\tilde{\tau}}^A U_{\tilde{\tau}}^A - I) - (1-\gamma)I.$$

By taking the expectation on both sides and using (8), we get

$$\begin{aligned} & E[e^{-\rho\tilde{\tau}}(R_{\tilde{\tau}} U_{\tilde{\tau}} - I) | U_0 = u, R_0 = i] \\ &= \gamma E[e^{-\rho\tilde{\tau}}(R_{\tilde{\tau}} U_{\tilde{\tau}} - I) | U_0 = \tilde{u}, R_0 = r_i] - (1-\gamma)I \\ &> \gamma(r_i \tilde{u} - I) - (1-\gamma)I \\ &= r_i u - I. \end{aligned} \tag{9}$$

Since (9) contradicts the fact that (7) holds for all τ, this completes the proof. □

From Lemma 1, we obtain

$$\begin{aligned} F_i(u) &> r_i u - I, \quad 0 < u < \upsilon_i, \\ F_i(u) &= r_i u - I, \quad u \geq \upsilon_i. \end{aligned}$$

The next result implies that an investment occurs at lower levels of U_t if R_t is in a good state while if R_t is in a bad state we need to wait until U_t increases to a higher level. Intuitively, if R_t is in a good state then there is an incentive to invest before R_t drops into a bad state. A reversed argument applies if R_t is in a bad state.

Lemma 2. *$\upsilon_1 > \upsilon_2$ under the assumption of* $0 < r_1 < r_2$.

Proof. Comparing with the value function (4) when $R_t = r$ is constant, it is clear that $H(u;r_1) \leq F_1(u)$ and $F_2(u) \leq H(u;r_2)$ for all $u > 0$. Note that $F_1(u)$ is continuous and connects from above to the straight line $r_1 u - I$ at $u = \upsilon_1$. Since $H(u;r_1)$ connects from above to the same line at $u = \frac{\xi I}{(\xi-1)r_1}$, we have $\upsilon_1 \geq \frac{\xi I}{(\xi-1)r_1}$. By a similar argument, we obtain a reversed inequality $\upsilon_2 \leq \frac{\xi I}{(\xi-1)r_2}$. Thus, we obtain $\upsilon_2 \leq \frac{\xi I}{(\xi-1)r_2} < \frac{\xi I}{(\xi-1)r_1} \leq \upsilon_1$, completing the proof. □

Now we will investigate the finiteness of υ_i's. By the law of iterated logarithm of a Brownian motion, $\lim_{t\to\infty} z_t/\sqrt{2t\ln\ln t} = 1$, w.p.1. Thus, we see that

$$\lim_{t\to\infty} e^{-\rho t} R_t U_t = 0, \qquad \text{w.p.1} \tag{10}$$

for any initial values of R_0 and U_0. If $\upsilon_2 = \infty$, then $\upsilon_1 = \infty$ from Lemma 2 and we never invest. This policy is not optimal since (10) implies the payoff $R_t U_t - I$ eventually goes to zero. Thus, $\upsilon_2 < \infty$.

On the other hand, $\upsilon_1 = \infty$ can be optimal. To see this, suppose $R_0 = r_1, U_0 = u$ and compare 2 policies: (a) invest at $t = 0$, and (b) wait while $R_t = r_1$ and invest when R_t first becomes r_2. The payoff of (a) is $r_1 u - I$ and the expected payoff of (b) is

$$\int_0^\infty \alpha e^{-\alpha t} e^{-\rho t}(r_2 u e^{\mu t} - I)dt = \frac{\alpha r_2 u}{\alpha+\rho-\mu} - \frac{\alpha}{\alpha+\rho} I.$$

Now define two cases A and B as

$$\text{case A:} \quad r_1 > \frac{\alpha r_2}{\alpha+\rho-\mu}, \qquad \text{case B:} \quad r_1 \le \frac{\alpha r_2}{\alpha+\rho-\mu}. \tag{11}$$

Then, in case B, policy (b) is better than (a) for all $u > 0$. In other words, it is optimal not to invest when $R_t = r_1$ in case B. The next table summarises these observations.

case A

R_t	$0 < u < \upsilon_2$	$\upsilon_2 \le u < \upsilon_1$	$\upsilon_1 \le u$
r_1	wait		invest
r_2	wait	invest	

case B

R_t	$0 < u < \upsilon_2$	$\upsilon_2 \le u$
r_1	wait	
r_2	wait	invest

To proceed further, we derive Bellman equation that the value functions satisfy. Suppose that $R_t = r_1$ and $U_t < \upsilon_i$, i.e., the state is in the continuation region. Since we will wait to invest in this case, we obtain by partitioning by the events in $(t, t+dt)$

$$F_1(u) = e^{-\rho dt}\{(1-\alpha dt)E[F_1(u+dU_t)|U_0 = u, R_0 = r_1] + \alpha dt F_2(u)\}. \tag{12}$$

Note that probability of concurrent transitions of U_t and R_t is $o(dt)$. From Ito's lemma and (1), we have

$$F_1(u+dU_t) = F_1(u) + \mu u F_1'(u)dt + \sigma u F_1'(u)dz_t + \frac{\sigma^2}{2} F_1''(u)dt. \tag{13}$$

Taking expectation of (13) and substituting $e^{-\rho dt} = 1 - \rho dt + o(dt)$ into (12), we obtain Bellman equation of $F_1(u)$ as

$$-(\alpha+\rho)F_1(u) + \mu u F_1'(u) + \frac{\sigma^2}{2} F_1''(u) + \alpha F_2(u) = 0, \qquad u < \upsilon_1. \tag{14}$$

In the same way, $F_2(u)$ satisfies

$$-(\beta+\rho)F_2(u) + \mu u F_2'(u) + \frac{\sigma^2}{2} F_2''(u) + \beta F_1(u) = 0, \qquad u < \upsilon_1. \tag{15}$$

It is noted here that the last terms in (14) and (15) show the effect caused by a regime switch which does not appear when R_t is constant as in the standard real option models.

Using these Bellman equations, we next derive the solution of $F_1(u)$ and $F_2(u)$. As a preliminary, we state the following result whose proof can be found in a standard textbook on differential equations.

Lemma 3. *For $a \leq 0$ and $c \geq 0$, the solution of the ordinary differential equation*

$$af(u) + buf'(u) + cu^2 f''(u) + du + e = 0$$

is given by

$$f(u) = k_1 u^{\xi_1} + k_2 u^{\xi_2} - \frac{d}{a+b} - \frac{e}{a}$$

where

$$\begin{aligned} \xi_1 &= \frac{c-b+\sqrt{(c-b)^2-4ac}}{2c} > 1, \\ \xi_2 &= \frac{c-b-\sqrt{(c-b)^2-4ac}}{2c} < 0. \end{aligned}$$

Coefficients k_1, k_2 are determined by boundary conditions.

For $0 < u < \upsilon_2$, we will wait irrespective of R_t and both (14) and (15) hold. Multiplying (14) and (15) by β and α respectively and summing up them, we obtain

$$-\rho H_1(u) + \mu u H_1'(u) + \frac{\sigma^2}{2} H_1''(u) = 0$$

where

$$H_1(u) = \beta F_1(u) + \alpha F_2(u). \tag{16}$$

From Lemma 3, the solution of (16) is

$$H_1(u) = a_1 u^{\eta_1} + a_2 u^{\eta_2} \tag{17}$$

where

$$\begin{aligned} \eta_1 &= \frac{\sigma^2/2-\mu+\sqrt{(\sigma^2/2-\mu)^2+2\sigma^2\rho}}{\sigma^2} > 1, \\ \eta_2 &= \frac{\sigma^2/2-\mu-\sqrt{(\sigma^2/2-\mu)^2+2\sigma^2\rho}}{\sigma^2} < 0. \end{aligned}$$

Now, since $\lim_{u\to 0} F_i(u) = 0$ for $i = 1, 2$, we obtain $\lim_{u\to 0} H_1(u) = 0$. Thus, the second term in (23) must vanish and

$$H_1(u) = au^{\eta} \tag{18}$$

(we set $a = a_1$ and $\eta = \eta_1$ for simplicity).

To obtain another solution, we subtract (14) from (15) to get

$$-(\alpha+\beta+\rho)H_2(u)+\mu u H_2'(u)+\frac{\sigma^2}{2}H_2''(u)=0$$

where

$$H_2(u) = -F_1(u)+F_2(u). \tag{19}$$

By a similar argument as above, a valid solution to (19) is

$$H_2(u) = bu^{\theta} \tag{20}$$

where

$$\theta = \frac{\sigma^2/2-\mu+\sqrt{(\sigma^2/2-\mu)^2+2\sigma^2(\alpha+\beta+\rho)}}{\sigma^2} > 1.$$

From (18) and (20), we obtain $F_1(u)$ and $F_2(u)$ for $0 < u < \upsilon_2$ as

$$F_{1a}(u) = au^{\eta} - \alpha b u^{\theta}, \tag{21}$$

$$F_{2a}(u) = au^{\eta} + \beta b u^{\theta} \tag{22}$$

where we redefine a by $\frac{a}{\alpha+\beta}$ and b by $\frac{b}{\alpha+\beta}$ (we use subscript a and b for the solution on $0 < u < \upsilon_2$ and $\upsilon_2 \le u < \upsilon_1$, respectively).

For $\upsilon_2 < u < \upsilon_1$, it is optimal to invest for $R_t = r_2$ and thus $F_{2b}(u) = r_2 u - I$. On the other hand, since waiting is optimal for $R_t = r_1$, $F_1(u)$ satisfies (14). Substituting $F_2(u) = r_2 u - I$, we obtain from Lemma 3

$$F_{1b}(u) = c_1 u^{\zeta_1} + c_2 u^{\zeta_2} + \frac{\alpha r_2}{\alpha+\rho-\mu}u - \frac{\alpha I}{\alpha+\rho} \tag{23}$$

where

$$\zeta_1 = \frac{\sigma^2/2-\mu+\sqrt{(\sigma^2/2-\mu)^2+2\sigma^2(\alpha+\rho)}}{\sigma^2} > 1,$$

$$\zeta_2 = \frac{\sigma^2/2-\mu-\sqrt{(\sigma^2/2-\mu)^2+2\sigma^2(\alpha+\rho)}}{\sigma^2} < 0.$$

(21)–(23) determines the form of the value functions in case A.

In case B, we need further investigation. Since it is optimal not to invest when $R_t = r_1$, we can show

$$F_1(u) = \int_0^{\infty} \alpha e^{-\alpha t} e^{-\rho t} E[F_2(U_t)|U_0 = u, R_0 = r_1]dt. \tag{24}$$

As $u \to \infty$ in (24), $E[F_2(U_t)|U_0 = u, R_0 = r_1] \to r_2 u e^{\mu t} - I$ since, starting with large u, U_t tends to exceed υ_2 when R_t first becomes r_2. Substituting this into (24), we see that

$$F_1(u) \to \frac{\alpha r_2}{\alpha+\rho-\mu}u - \frac{\alpha I}{\alpha+\rho} \qquad (u \to \infty). \tag{25}$$

To satisfy (25), c_1 in (23) must be 0 and we have

$$F_{1c}(u) = c_2 u^{\zeta_2} + \frac{\alpha r_2}{\alpha+\rho-\mu} u - \frac{\alpha I}{\alpha+\rho}$$

for $u > \upsilon_2$ in case B. Thus, we obtain the following.

Theorem 4. *The value function is given as follows:*

case A

	$0 < u \le \upsilon_2$	$\upsilon_2 < u \le \upsilon_1$	$\upsilon_1 < u$
$F_1(u)$	$F_{1a}(u)$	$F_{1b}(u)$	$r_1 u - I$
$F_2(u)$	$F_{2a}(u)$	$r_2 u - I$	

case B

	$0 < u \le \upsilon_2$	$\upsilon_2 < u$
$F_1(u)$	$F_{1a}(u)$	$F_{1c}(u)$
$F_2(u)$	$F_{2a}(u)$	$r_2 u - I$

To completely identify the value functions, we need to determine unknown coefficients of $F_1(u), F_2(u)$ and threshold levels. For this purpose, we will use value matching and smooth pasting conditions which are known as the optimality conditions for a class of stopping problems and widely used in real option analysis [3, 4, 6, 9].

Case A: There are 6 unknown parameters in case A, i.e., 2 threshold levels υ_1, υ_2 and 4 coefficients a, b, c_1 and c_2. The value matching conditions at $u = \upsilon_1, \upsilon_2$ respectively are given as

$$F_{1a}(\upsilon_2) = F_{1b}(\upsilon_2), \qquad F_{1b}(\upsilon_1) = r_1 \upsilon_1 - I \tag{26}$$

$$F_{2a}(\upsilon_2) = r_2 \upsilon_2 - I. \tag{27}$$

Also, the smooth pasting conditions are

$$F'_{1a}(\upsilon_2) = F'_{1b}(\upsilon_2), \qquad F'_{1b}(\upsilon_1) = r_1 \tag{28}$$

$$F'_{2a}(\upsilon_2) = r_2. \tag{29}$$

We use these 6 equations to determine 6 unknown parameters.

Case B: As we already saw, $\upsilon_1 = \infty$ and $c_1 = 0$ in case B. Therefore, we need to calculate υ_2, a, b and c_2. The value matching and smooth pasting conditions at $u = \upsilon_2$ are

$$F_{1a}(\upsilon_2) = F_{1c}(\upsilon_2), \qquad F_{2a}(\upsilon_2) = r_2 \upsilon_2 - I, \tag{30}$$

$$F'_{1a}(\upsilon_2) = F'_{1c}(\upsilon_2), \qquad F'_{2a}(\upsilon_2) = r_2. \tag{31}$$

Unlike the single variable case in (5), it is not straightforward to solve these nonlinear equations. In the next section, we will explain how to solve them and give some numerical examples.

4 Computation of value functions and numerical results

To develop numerical procedures for computing unknown coefficients, we will consider 2 cases A and B separately.

Case A: From (27) and (29), we can derive

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(\zeta_1-\zeta_2)\upsilon_1^{\zeta_1+\zeta_2-1}} \begin{bmatrix} d_{11}\upsilon_1^{\zeta_2}+d_{12}\upsilon_1^{\zeta_2-1} \\ d_{21}\upsilon_1^{\zeta_1}+d_{22}\upsilon_1^{\zeta_1-1} \end{bmatrix} \tag{32}$$

where

$$d_{11} = (\zeta_2-1)(\frac{\alpha r_2}{\alpha+\rho-\mu}-r_1), \qquad d_{12} = \zeta_2\frac{\rho I}{\alpha+\rho},$$
$$d_{21} = (\zeta_1-1)(r_1-\frac{\alpha r_2}{\alpha+\rho-\mu}), \qquad d_{22} = -\zeta_1\frac{\rho I}{\alpha+\rho}.$$

We also derive from (26) and (28)

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\beta(\theta-\eta)\upsilon_2^{\eta+\theta-1}} \begin{bmatrix} \beta(\theta-1)\upsilon_2^{\theta}r_2-\beta\theta\upsilon_2^{\theta-1}I \\ -(\eta-1)\upsilon_2^{\eta}r_2+\eta\upsilon_2^{\eta-1}I \end{bmatrix}. \tag{33}$$

Combining (33) with (26) and (28), we can further show

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(\zeta_1-\zeta_2)\upsilon_2^{\zeta_1+\zeta_2-1}} \begin{bmatrix} (e_{21}-\zeta_2 e_{11})\upsilon_2^{\zeta_2}+(e_{22}-\zeta_2 e_{12})\upsilon_2^{\zeta_2-1} \\ (\zeta_1 e_{11}-e_{21})\upsilon_2^{\zeta_1}+(\zeta_1 e_{12}-e_{22})\upsilon_2^{\zeta_1-1} \end{bmatrix} \tag{34}$$

where

$$\begin{aligned} e_{11} &= \left\{\frac{\beta(\theta-1)+\alpha(\eta-1)}{\beta(\theta-\eta)}-\frac{\alpha}{\alpha+\rho-\mu}\right\}r_2, \\ e_{12} &= \left\{\frac{\alpha}{\alpha+\rho}-\frac{\beta\theta+\alpha\eta}{\beta(\theta-\eta)}\right\}I, \\ e_{21} &= \left\{\frac{\beta\eta(\theta-1)+\alpha\theta(\eta-1)}{\beta(\theta-\eta)}-\frac{\alpha}{\alpha+\rho-\mu}\right\}r_2, \\ e_{22} &= -\frac{(\alpha+\beta)\eta\theta}{\beta(\theta-\eta)}I. \end{aligned}$$

Now let $k=\upsilon_1/\upsilon_2$, then $k>1$ from Lemma 2. We then substitute $\upsilon_1=k\upsilon_2$ into (32) and equate it to (34). With some algebras, we can show

$$\upsilon_2 = \frac{d_{12}-(e_{22}-\zeta_2 e_{12})k^{\zeta_1}}{(e_{21}-\zeta_2 e_{11})k^{\zeta_1}-d_{11}k} = \frac{d_{22}+(e_{22}-\zeta_1 e_{12})k^{\zeta_2}}{(\zeta_1 e_{11}-e_{21})k^{\zeta_2}-d_{21}k}. \tag{35}$$

Since d_{ij}'s and e_{ij}'s are known coefficients, (35) is a nonlinear equation of single variable k which can easily be solved numerically. Once k is at hand, we can calculate υ_2 from (35), $\upsilon_1=k\upsilon_2$, a and b from (33), c_1 and c_2 from (32).

Case B: In this case, equations (30) and (31) can be solved explicitly. Specifically, we obtain after some algebras

$$u_2 = \frac{(\alpha+\rho-\mu)((\zeta_2-\eta)\beta\rho\theta+(\zeta_2-\theta)(\alpha+\rho+\beta)\alpha\eta)I}{(\alpha+\rho)((\zeta_2-\theta)(\eta-1)(\alpha+\rho-\mu+\beta)\alpha+(\zeta_2-\eta)(\rho-\mu)(\theta-1)\beta)r_2}.$$

Other parameters can be also calculated as

$$\begin{aligned} a &= \frac{(\theta-1)\beta u_2 r_2 - \theta\beta I}{(\theta-\eta)\beta u_2^{\eta}}, \\ b &= \frac{(1-\eta)u_2 r_2 + \eta I}{(\theta-\eta)\beta u_2^{\theta}}, \\ c_2 &= \frac{(\eta-1)(1+\frac{\beta}{\alpha+\rho-\mu})\alpha r_2 u_2 - (1+\frac{\beta}{\alpha+\rho})\alpha\eta I}{(\zeta_2-\eta)\beta u_2^{\zeta_2}}. \end{aligned}$$

Note that r_1 does not appear in the solution since we never invest when $R_t = r_1$ in case B.

In what follows, we will show some numerical results to demonstrate the above procedures work properly. We use the parameters in Table 1 as a base case of the numerical results. Roughly speaking, annual growth rate of the project value is 5%

Table 1: Parameters for the base case of numerical examples.

μ	σ	ρ	α	β	r_1	r_2	I
0.05	0.1	0.1	0.1	0.1	1	1.4	1

with volatility 10%, and future payoff will be discounted 10% per year. The regime process changes its state once a 10 year in average.

Table 2 shows how the ratio r_2/r_1 affects the threshold levels (we fix $r_1 = 1$). For the above parameters, case B in (11) is equivalent to $r_2 \geq 1.5$. In this case, $\upsilon_1 = \infty$ and investment will be executed only when $R_t = r_2$. υ_2 decreases as r_2 increases due to stronger incentive to invest before the regime process drops into bad state. In fact, the payoff obtained by investing at υ_2 is around 1.95 for all values of r_2 listed in Table 2. Figure 1 shows the value functions $F_1(u)$ and $F_2(u)$ for $r_2 = 1.2$ and $r_2 = 2.0$,

Table 2: r_2 and threshold levels υ_1 and υ_2 ($r_1 = 1$).

r_2	1.2	1.4	1.6	1.8	2.0
υ_1	3.61	10.8	∞	∞	∞
υ_2	1.64	1.40	1.22	1.08	0.97

respectively. As we explained in Section 3, each curve is composed of smoothly pasted 2 or 3 different functions.

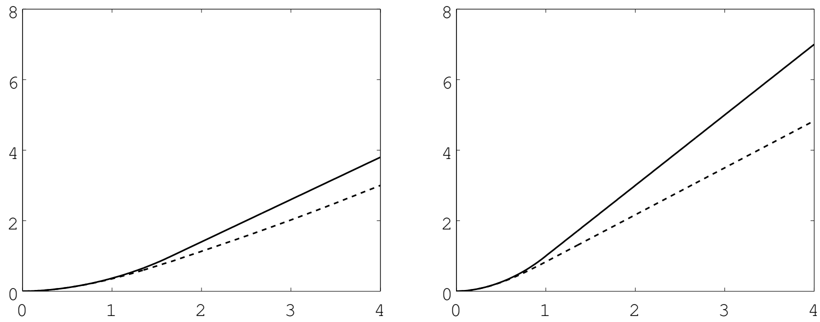

Figure 1: Value functions $F_1(u)$ (dashed line) and $F_2(u)$ (solid line) for $r_2 = 1.2$ (left panel) and $r_2 = 2.0$ (right panel). Other parameters are listed in Table 1.

Table 3 shows threshold levels when transition rates α and β of regime switch range from 0.1 to 1.0 (from 10 years to 1 year in average). υ_2 increases as $\alpha = \beta$ increases since the possibility of risk to stay at a bad state for longer time decreases. However, the difference is rather small in the current setting since it is almost unlikely to occur to invest in a bad state.

Table 3: Rate of regime transitions and threshold levels.

$\alpha = \beta$	0.1	0.2	0.5	1.0
υ_1	10.8	∞	∞	∞
υ_2	1.40	1.43	1.48	1.51

5 Concluding remarks

In this paper, we consider an optimal investment problem with regime switching. In contrast to existing studies of real option analysis, project value is affected by two uncertainties, i.e., its fundamental value and a Markovian regime process. By investigating the optimal policy, we identify the form of the value functions and develop numerical procedures to compute unknown coefficients of value functions. Some numerical examples are also provided to see how optimal policy changes by model parameters.

There are some possibilities to extend the results obtained so far. From practical viewpoint, it would be helpful if we could solve similar problems having more general payoff functions. It is also interesting and challenging to solve a similar problem with more than three regime states. Extensions along these directions are under preparation and we expect them to widen practical applicability of real option analysis since it is quite common in reality that a project value is affected by several factors.

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