The Sixth International Symposium on Operations Research and Its Applications (ISORA'06)
Xinjiang, China, August 8–12, 2006


# A New Arbitrary Starting Simplicial Algorithm for Computing an Integer Point of an $n$-Dimensional Simplex*

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**Abstract** Determining whether there is an integer point in an $n$-dimensional simplex is an NP-complete problem. In this paper, a new arbitrary starting variable dimension algorithm is developed for computing an integer point of an $n$-dimensional simplex. The algorithm is derived from an introduction of an integer labeling rule and an application of a triangulation of the space and is composed of two phases, one of which forms a variable dimension algorithm and the other a full-dimension pivoting procedure. Starting at an arbitrary integer point, the algorithm interchanges from one phase to the other if necessary and follows a finite simplicial path that either leads to an integer point of the simplex or proves that no such points exist. An advantage of the algorithm is that all the existing superior triangulations can be its underlying triangulations without any modification.

**Keywords** integer point; simplex; Diophantine equation; nonnegative integer solution; unimodular transformation; integer programming; integer labeling; triangulation; variable dimension algorithm; pivoting procedure

## 1 Introduction

The problem we consider in this paper is as follows: Determine whether there is an integer point in $P$ given by

$$P = \{x \in R^n \mid Ax \leq b\},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{pmatrix}$$

with every entry being an integer and $b = (b_1, b_2, \cdots, b_{n+1})^\top$. The problem is an NP-complete problem (e.g., Nemhauser and Wolsey, 1988; Schrijver, 1998). We

*The work of this paper was partially supported by SRG: 7001428 of City University of Hong Kong.
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assume throughout this paper that $P$ is bounded and has an interior point. Then, $P$ is an $n$-dimensional simplex, there is a vector $\rho = (\rho_1, \rho_2, \cdots, \rho_{n+1})^\top > 0$ such that $\rho^\top A = 0$, and any $n \times n$ submatrix of $A$ is nonsingular. The problem is very general though it looks special. In fact, finding an integer point of a polytope can be reduced in polynomial-time to finding an integer point of a simplex by applying aggregation techniques (e.g., Glover and Babayev, 1995; Kannan, 1983; Zhu and Broughan, 1998). Such an aggregation procedure is given in the appendix of this paper.

Recently, stimulated by the work in Scarf (1981), a simplicial approach was developed for computing an integer point of an $n$-dimensional simplex in Dang and Maaren (1998, 1999, 2001). The idea of the approach comes from simplicial methods, which were originated in Scarf (1967) for computing fixed points of continuous or upper-semi continuous mappings. Scarf's work was built on his elegant primitive sets, but subsequential substantial developments of simplicial methods were based on triangulations (e.g., Allgower and Georg, 2000; Dang, 1991, 1995; Eaves, 1972, 1984; Eaves and Saigal, 1972; Forster, 1995; Garcia and Zangwill, 1981; Kojima and Yamamoto, 1984; Kuhn, 1968; van der Laan and Talman, 1979, 1981; Merrill, 1972; Scarf, 1973; Todd, 1976; Wright, 1981; Yamamoto, 1984).

In this paper, a new arbitrary starting variable dimension algorithm is developed for computing an integer point of an $n$-dimensional simplex. The algorithm is derived from an introduction of an integer labeling rule and an application of a triangulation of the space and is composed of two phases, one of which forms a variable dimension algorithm, which is derived from a modification of the 2-ray algorithm for computing fixed points in Yamamoto (1984), and the other a full-dimension pivoting procedure, which comes from Dang and Maaren (1999). Starting at an arbitrary integer point, the algorithm interchanges from one phase to the other if necessary and follows a finite simplicial path that either leads to an integer point of the simplex or proves that no such points exist. An advantage of the algorithm is that all the existing superior triangulations can be its underlying triangulations without any modification.

The rest of this paper is organized as follows. An integer labeling rule is introduced and its properties are discussed in Section 2. Based on the integer labeling rule and a triangulation of the space, a new arbitrary starting variable dimension algorithm is developed in Section 3.

## 2 An Integer Labeling Rule and Its Properties

The definition of an $(n+1) \times n$ matrix in canonical form was given in Dang and Maaren (1998), which is as follows.

**Definition 1.** An $(n+1) \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{pmatrix}$$

is in canonical form if $a_{ii} > 0$, $i = 1,2,\cdots,n$, and $a_{ij} \leq 0$ for any $i$ and $j$ with $i \neq j$.

We assume throughout this paper that $A$ is in the canonical form and $(a_{1n}, a_{2n}, \cdots, a_{n-1,n})^\top < 0$. If $A$ is an arbitrary integer $(n+1)\times n$ matrix satisfying that there is a positive vector $\rho$ with $\rho^\top A = 0$ and that any $n \times n$ submatrix of $A$ is nonsingular, a procedure given in Pnueli (1968) shows that, applying the following three elementary column operations to $A$,

1. interchange two columns,
2. multiply a column by $-1$,
3. add any integer times a column to another column,

one can transform $A$ in polynomial-time into a matrix in the canonical form and $(a_{1n}, a_{2n}, \cdots, a_{n-1,n})^\top < 0$. Such a unimodular transformation procedure is given in the appendix of this paper.

Let $a_i^\top$ denote the $i$th row of $A$ for $i = 1,2,\cdots,n+1$. Let $N = \{1,2,\cdots,n\}$ and $N_0 = \{1,2,\cdots,n+1\}$. For $k = 1,2,\cdots,n$, let

$$A_{kk} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}.$$

The following theorem was proved in Pnueli (1968) and can also be found in the unimodular transformation procedure in the appendix of this paper.

**Theorem 1.** *For $k = 1,2,\cdots,n$, $A_{kk}$ is invertible, the determinant of $A_{kk}$ is positive, and $A_{kk}^{-1} \geq 0$. For $k = 2,3,\cdots,n$,*

$$a_{kk} - (a_{k1}, a_{k2}, \cdots, a_{k,k-1}) A_{k-1,k-1}^{-1} (a_{1k}, a_{2k}, \cdots, a_{k-1,k})^\top > 0.$$

Let $\eta$ be an arbitrary integer point of $R^n$, which will be the starting point of the algorithm. Let $I(0) = \emptyset$, and, for any $h \in N$, let $I(h) = \{1,2,\cdots,h\}$. For $\alpha \in \{-1,1\}$ and $h \in N$, let

$$X(\eta,h,\alpha) = \{x \in R^n \mid \alpha(x_h - \eta_h) \geq 0 \text{ and } x_k = \eta_k,\ k \in N \backslash I(h)\}.$$

Let $T$ be a triangulation of $R^n$ such that every integer unit cube is subdivided into integer simplices, where an integer unit cube is a unit cube with only integer vertices and an integer simplex is a simplex with only integer vertices. Such triangulations include the $K_1$-triangulation in Freudenthal (1942), the $J_1$-triangulation in Todd (1976), the $D_1$-triangulation in Dang (1991), the $D_1'$-triangulation in Todd and Tunçel (1993), the $D_1(m)$-triangulation in Dang (2005), etc. A $q$-dimensional simplex of $T$ with vertices $y^0, y^1, \cdots, y^q$ is denoted by $< y^0, y^1, \cdots, y^q >$. Let $\mathscr{T}$ be the set of faces of simplices of $T$.

For $\sigma \in \mathscr{T}$, let $\text{grid}(\sigma) = \max\{\|x-y\| \mid x \in \sigma \text{ and } y \in \sigma\}$, where $\|\cdot\|$ stands for the infinity norm. We define $\text{mesh}(T) = \max_{\sigma\in\mathscr{T}} \text{grid}(\sigma)$. Then, $\text{mesh}(T) = 1$.

**Definition 2.** For $x \in R^n$, we assign to $x$ an integer label $l(x)$ given by

$$l(x) = \begin{cases} \min\{k \mid a_k^\top x > b_k\} & \text{if } x \notin P, \\ 0 & \text{otherwise.} \end{cases}$$

Given this integer labeling rule, we can define $h$-completeness, almost $h$-completeness, completeness, and almost completeness, which are as follows.

**Definition 3.**

1. For $h = 0, 1, \cdots, n$, a simplex $\sigma = < y^0, y^1, \cdots, y^h >$ is $h$-complete if $\sigma$ carries $h+1$ different nonzero integer labels and $h$ of these labels are in $I(h)$.
2. For $h = 1, 2, \cdots, n$, a simplex $\sigma = < y^0, y^1, \cdots, y^h >$ is almost $h$-complete if $\sigma$ carries either only all the integer labels in $I(h)$, or all the integer labels in $I(h-1)$, no integer labels 0 and $h$, and at least one integer label in $N \backslash I(h)$.
3. A simplex $\sigma = < y^0, y^1, \cdots, y^q >$ is complete if $\sigma$ carries $q+1$ different nonzero integer labels.
4. A simplex $\sigma = < y^0, y^1, \cdots, y^q >$ is almost complete if $\sigma$ carries exactly $q$ different nonzero integer labels.

We remark that an $n$-complete simplex is the same as a complete $n$-dimensional simplex and viceversa.

**Lemma 2.** *There are finitely many n-complete simplices.*

**Proof.** Let $\sigma = < y^0, y^1, \cdots, y^n >$ be an arbitrary $n$-complete simplex. Without loss of generality, we assume that $l(y^0) = n+1$ and $l(y^i) = i$, $i = 1, 2, \cdots, n$. For $i = 1, 2, \cdots, n$, we define

$$f_i(x) = a_i^\top x - b_i.$$

Let $x$ be an arbitrary point of $\sigma$. Then, for $i = 1, 2, \cdots, n$, since $f_i(y^0) \le 0$ and $f_i(y^i) > 0$,

$$f_i(x) = f_i(x) - f_i(y^0) + f_i(y^0) \le f_i(x) - f_i(y^0) = a_i^\top (x - y^0)$$

and

$$f_i(x) = f_i(x) - f_i(y^i) + f_i(y^i) > f_i(x) - f_i(y) = a_i^\top (x - y^i).$$

From mesh$(T) = 1$, we obtain that $-e \le x - y^i \le e$, $i = 0, 1, \cdots, n$, where $e = (1,\ 1,\ \cdots,\ 1)^\top \in R^n$. Thus, for $i = 1, 2, \cdots, n$,

$$\min_{-e \le y \le e} a_i^\top y \le f_i(x) \le \max_{-e \le y \le e} a_i^\top y.$$

Let

$$\Delta = \{x \in R^n \mid \min_{-e \le y \le e} a_i^\top y \le f_i(x) \le \max_{-e \le y \le e} a_i^\top y,\ i = 1, 2, \cdots, n\}.$$

Then, all the $n$-complete simplices are contained in $\Delta$. Clearly, $f(x) = (f_1(x),\ f_2(x),\ \cdots,\ f_n(x))^\top$ is bounded on $\Delta$. Let $b_{-(n+1)} = (b_1, b_2, \cdots, b_n)^\top$. Since $A_{nn}$ is invertible, hence, for any $x \in \Delta$,

$$x = A_{nn}^{-1}(f(x) + b_{-(n+1)}).$$

Thus, for any $x \in \Delta$,

$$\|x\| = \|A_{nn}^{-1}(f(x) + b_{-(n+1)})\| \leq \|A_{nn}^{-1}\|(\|f(x)\| + \|b_{-(n+1)}\|).$$

Therefore, $\Delta$ is bounded. The lemma follows. □

**Lemma 3.** *There are finitely many almost h-complete simplices carrying only all the integer labels in* $I(h)$ *and contained in* $X(\eta, h, -1)$

**Proof.** Let $\sigma = < y^0, y^1, \cdots, y^h >$ be an arbitrary almost $h$-complete simplex carrying only all the integer labels in $I(h)$ and contained in $X(\eta, h, -1)$. Without loss of generality, we assume that $l(y^i) = i$, $i = 1, 2, \cdots, h$. For $i = 1, 2, \cdots, h$, we define

$$f_i(x) = a_i^\top x - b_i.$$

Let $x$ be an arbitrary point of $\sigma$. Then, for $i = 1, 2, \cdots, h-1$, since $f_i(y^i) > 0$ and $f_i(y^h) \leq 0$,

$$f_i(x) = f_i(x) - f_i(y^h) + f_i(y^h) \leq f_i(x) - f_i(y^h) = a_i^\top (x - y^h)$$

and

$$f_i(x) = f_i(x) - f_i(y^i) + f_i(y^i) > f_i(x) - f_i(y^i) = a_i^\top (x - y^i),$$

and, since $f_h(y^h) > 0$,

$$f_h(x) = f_h(x) - f_h(y^h) + f_h(y^h) > f_h(x) - f_h(y^h) = a_h^\top (x - y^h).$$

From mesh$(T) = 1$, we obtain that $-e \leq x - y^i \leq e$, $i = 1, 2, \cdots, h$, where $e = (1, 1, \cdots, 1)^\top \in R^n$. Thus, for $i = 1, 2, \cdots, h-1$,

$$\min_{-e \leq y \leq e} a_i^\top y \leq f_i(x) \leq \max_{-e \leq y \leq e} a_i^\top y,$$

and

$$\min_{-e \leq y \leq e} a_h^\top y \leq f_h(x).$$

Therefore, $\sigma$ is contained in

$$\Lambda = \left\{ x \in X(\eta, h, -1) \;\middle|\; \begin{array}{l} \min_{-e \leq y \leq e} a_i^\top y \leq f_i(x) \leq \max_{-e \leq y \leq e} a_i^\top y, \\ i = 1, 2, \cdots, h-1, \text{ and } \min_{-e \leq y \leq e} a_h^\top y \leq f_h(x) \end{array} \right\}.$$

Let $\hat{f}(x) = (f_1(x), f_2(x), \cdots, f_{h-1}(x))^\top$. Then, $\hat{f}(x)$ is bounded on $\Lambda$. Let $\hat{x} = (x_1, x_2, \cdots, x_{h-1})^\top$, $\hat{a} = (a_{1h}, a_{2h}, \cdots, a_{h-1,h})^\top$, $\hat{b} = (b_1, b_2, \cdots, b_{h-1})^\top$, $\bar{a} = (a_{h1}, a_{h2}, \cdots, a_{h,h-1})$, $\hat{d} = (a_{h,h+1}, a_{h,h+2}, \cdots, a_{hn})^\top$, $\hat{e} = (1, 1, \cdots, 1)^\top \in R^{n-h}$, and

$$B_{h-1,n-h} = \begin{pmatrix} a_{1,h+1} & a_{1,h+2} & \cdots & a_{1n} \\ a_{2,h+1} & a_{2,h+2} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{h-1,h+1} & a_{h-1,h+2} & \cdots & a_{h-1,n} \end{pmatrix}.$$

Since $A_{h-1,h-1}$ is invertible, hence, for any $x = (\hat{x}^\top, x_h, \eta\hat{e}^\top)^\top \in \Lambda$,

$$\hat{x} = A_{h-1,h-1}^{-1}(\hat{f}(x) + \hat{b} - \eta B_{h-1,n-h}\hat{e}) - A_{h-1,h-1}^{-1}\hat{a}x_h.$$

Substituting $x = (\hat{x}^\top, x_h, \eta\hat{e}^\top)^\top$ into $\min_{-e\le y\le e} a_h^\top y \le f_h(x)$, we obtain that

$$\min_{-e\le y\le e} a_h^\top y \le \bar{a}A_{h-1,h-1}^{-1}(\hat{f}(x) + \hat{b} - \eta B_{h-1,n-h}\hat{e}) + \eta\hat{d}^\top\hat{e} + (a_{hh} - \bar{a}A_{h-1,h-1}^{-1}\hat{a})x_h.$$

From $a_{hh} - \bar{a}A_{h-1,h-1}^{-1}\hat{a} > 0$ and $x \in X(\eta, h, -1)$, we obtain that

$$\frac{\min_{-e\le y\le e} a_n^\top y - \bar{a}A_{h-1,h-1}^{-1}(\hat{f}(x) + \hat{b} - \eta B_{h-1,n-h}\hat{e}) - \eta\hat{d}^\top\hat{e}}{a_{hh} - \bar{a}A_{h-1,h-1}^{-1}\hat{a}} \le x_h \le \eta_h.$$

Since $\hat{f}(x)$ is bounded on $\Lambda$, hence, $x_h$ is bounded on $\Lambda$. Therefore, $\Lambda$ is bounded. The lemma follows. □

**Lemma 4.** *There are finitely many almost h-complete simplices contained in $X(\eta$, $h$, $1)$ and carrying all the integer labels in $I(h-1)$, no integer labels $0$ and h, and at least one integer label in $N\backslash\{I(h)\}$.*

**Proof.** Let $\sigma =< y^0, y^1, \cdots, y^h >$ be an arbitrary almost $h$-complete simplices contained in $X(\eta, h, 1)$ and carrying all the integer labels in $I(h-1)$, no integer labels 0 and $h$, and at least one integer label in $N\backslash\{I(h)\}$. Without loss of generality, we assume that $l(y^i) = i$, $i = 1, 2, \cdots, h-1$, and $l(y^h) = q > h$. For $i = 1, 2, \cdots, h$, we define

$$f_i(x) = a_i^\top x - b_i.$$

Then, for $i = 1, 2, \cdots, h-1$, since $f_i(y^i) > 0$ and $f_i(y^h) \le 0$,

$$f_i(x) = f_i(x) - f_i(y^h) + f_i(y^h) \le f_i(x) - f_i(y^h) = a_i^\top(x - y^h)$$

and

$$f_i(x) = f_i(x) - f_i(y^i) + f_i(y^i) > f_i(x) - f_i(y^i) = a_i^\top(x - y^i),$$

and, since $f_h(y^h) \le 0$,

$$f_h(x) = f_h(x) - f_h(y^h) + f_h(y^h) \le f_h(x) - f_h(y^h) = a_h^\top(x - y^h).$$

From mesh$(T) = 1$, we obtain that $-e \le x - y^i \le e$, $i = 1, 2, \cdots, h$, where $e = (1, 1, \cdots, 1)^\top \in R^n$. Thus, for $i = 1, 2, \cdots, h-1$,

$$\min_{-e\le y\le e} a_i^\top y \le f_i(x) \le \max_{-e\le y\le e} a_i^\top y,$$

and

$$f_h(x) \le \max_{-e\le y\le e} a_h^\top y.$$

Therefore, $\sigma$ is contained in

$$\Lambda = \left\{ x \in X(\eta, h, 1) \;\middle|\; \begin{array}{l} \min_{-e \le y \le e} a_i^\top y \le f_i(x) \le \max_{-e \le y \le e} a_i^\top y, \\ i = 1, 2, \cdots, h-1, \text{ and } f_h(x) \le \max_{-e \le y \le e} a_h^\top y \end{array} \right\}.$$

Let $\hat{f}(x) = (f_1(x), f_2(x), \cdots, f_{h-1}(x))^\top$. Then, $\hat{f}(x)$ is bounded on $\Lambda$. Let $\hat{x} = (x_1, x_2, \cdots, x_{h-1})^\top$, $\hat{a} = (a_{1h}, a_{2h}, \cdots, a_{h-1,h})^\top$, $\hat{b} = (b_1, b_2, \cdots, b_{h-1})^\top$, $\bar{a} = (a_{h1}, a_{h2}, \cdots, a_{h,h-1})$, $\hat{d} = (a_{h,h+1}, a_{h,h+2}, \cdots, a_{hn})^\top$, $\hat{e} = (1, 1, \cdots, 1)^\top \in R^{n-h}$, and

$$B_{h-1,n-h} = \begin{pmatrix} a_{1,h+1} & a_{1,h+2} & \cdots & a_{1n} \\ a_{2,h+1} & a_{2,h+2} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{h-1,h+1} & a_{h-1,h+2} & \cdots & a_{h-1,n} \end{pmatrix}.$$

Since $A_{h-1,h-1}$ is invertible, hence, for any $x = (\hat{x}^\top, x_h, \eta \hat{e}^\top)^\top \in \Lambda$,

$$\hat{x} = A_{h-1,h-1}^{-1}(\hat{f}(x) + \hat{b} - \eta B_{h-1,n-h}\hat{e}) - A_{h-1,h-1}^{-1}\hat{a}x_h.$$

Substituting $x = (\hat{x}^\top, x_h, \eta \hat{e}^\top)^\top \in \Lambda$ into $\max_{-e \le y \le e} a_h^\top y \ge f_h(x)$, we obtain that

$$\max_{-e \le y \le e} a_h^\top y \ge \bar{a}A_{h-1,h-1}^{-1}(\hat{f}(x) + \hat{b} - \eta B_{h-1,n-h}\hat{e}) + \eta \hat{d}^\top \hat{e} + (a_{hh} - \bar{a}A_{h-1,h-1}^{-1}\hat{a})x_h.$$

From $a_{hh} - \bar{a}A_{h-1,h-1}^{-1}\hat{a} > 0$ and $x \in X(\eta, h, 1)$, we obtain that

$$\frac{\max_{-e \le y \le e} a_n^\top y - \bar{a}A_{h-1,h-1}^{-1}(\hat{f}(x) + \hat{b} - \eta B_{h-1,n-h}\hat{e}) - \eta \hat{d}^\top \hat{e}}{a_{hh} - \bar{a}A_{h-1,h-1}^{-1}\hat{a}} \ge x_h \ge \eta_h.$$

Since $\hat{f}(x)$ is bounded on $\Lambda$, hence, $x_h$ is bounded on $\Lambda$. Therefore, $\Lambda$ is bounded. The lemma follows. □

For any $\xi \in R^n$ and $K \subseteq N$, we define

$$H(\xi, K) = \{\xi + x \mid 0 \le x_i,\ i \in K, \text{ and } x_i = 0,\ i \notin K\}.$$

The following lemma can be found in Dang and Maaren (1998).

**Lemma 5.** *If $z^0$ is an integer point in $P$, then, for any $K \subseteq N$, each point of $\in H(z^0, K)$ carries either integer label* 0 *or an integer label in $K$.*

As a corollary of Lemma 5, we obtain

**Corollary 6.** *If $z^0$ is an integer point in $P$, there is no $n$-complete simplex in $H(z^0, N)$, and, for any $j \in N$, there is no complete $(n-1)$-dimensional simplex in $H(z^0, N \backslash \{j\})$ carrying only integer labels in $N$.*

**Lemma 7.** *If $z^0$ is an integer point in $P$, there are finitely many almost complete $n$-dimensional simplices carrying all the integer labels in $N$ and contained in $R^n \setminus H(z^0, N)$.*

**Proof.** Let $\sigma = < y^0, y^1, \cdots, y^n >$ be an arbitrary almost complete $n$-dimensional simplex carrying all the integer labels in $N$. Without loss of generality, we assume that $l(y^i) = i$, $i = 1,2,\cdots,n$. For $i = 1,2,\cdots,n$, we define

$$f_i(x) = a_i^\top x - b_i.$$

Let $x$ be an arbitrary point of $\sigma$. Then, for $i = 1,2,\cdots,n-1$, since $f_i(y^i) > 0$ and $f_i(y^n) \leq 0$,

$$f_i(x) = f_i(x) - f_i(y^n) + f_i(y^n) \leq f_i(x) - f_i(y^n) = a_i(x - y^n)$$

and

$$f_i(x) = f_i(x) - f_i(y^i) + f_i(y^i) > f_i(x) - f_i(y^i) = a_i(x - y^i),$$

and, since $f_n(y^n) > 0$,

$$f_n(x) = f_n(x) - f_n(y^n) + f_n(y^n) > f_n(x) - f_n(y^n) = a_n^\top(x - y^n).$$

From mesh$(T) = 1$, we obtain that $-e \leq x - y^i \leq e$, $i = 1,2,\cdots,n$, where $e = (1, 1, \cdots, 1)^\top \in R^n$. Thus, for $i = 1,2,\cdots,n-1$,

$$\min_{-e\leq y\leq e} a_i^\top y \leq f_i(x) \leq \max_{-e\leq y\leq e} a_i^\top y,$$

and

$$\min_{-e\leq y\leq e} a_n^\top y \leq f_n(x).$$

Let

$$\Lambda = \left\{ x \in R^n \;\middle|\; \begin{array}{l} \min_{-e\leq y\leq e} a_i^\top y \leq f_i(x) \leq \max_{-e\leq y\leq e} a_i^\top y, \\ i = 1,2,\cdots,n-1, \text{ and } \min_{-e\leq y\leq e} a_n^\top y \leq f_n(x) \end{array} \right\}.$$

Then, all the almost complete $n$-dimensional simplices carrying all the integer labels in $N$ are contained in $\Lambda$.

Let $\hat{f}(x) = (f_1(x), f_2(x), \cdots, f_{n-1}(x))^\top$. Clearly, $\hat{f}(x)$ is bounded on $\Lambda$. Let $\hat{x} = (x_1, x_2, \cdots, x_{n-1})^\top$, $\hat{a} = (a_{1n}, a_{2n}, \cdots, a_{n-1,n})^\top$, $\hat{b} = (b_1, b_2, \cdots, b_{n-1})^\top$, and $\bar{a} = (a_{n1}, a_{n2}, \cdots, a_{n,n-1})$. Since $A_{n-1,n-1}$ is invertible, hence, for any $x = (\hat{x}^\top, x_n)^\top \in \Lambda$,

$$\hat{x} = A_{n-1,n-1}^{-1}(\hat{f}(x) + \hat{b}) - A_{n-1,n-1}^{-1}\hat{a}x_n.$$

Substituting $x = (\hat{x}^\top, x_n)^\top \in \Lambda$ into $\min_{-e\leq y\leq e} a_n^\top y \leq f_n(x)$, we obtain

$$\min_{-e\leq y\leq e} a_n^\top y \leq \bar{a}A_{n-1,n-1}^{-1}(\hat{f}(x) + \hat{b}) + (a_{nn} - \bar{a}A_{n-1,n-1}^{-1}\hat{a})x_n.$$

From $a_{nn} - \bar{a}A_{n-1,n-1}^{-1}\hat{a} > 0$, we get

$$\frac{\min_{-e\leq y\leq e} a_n^\top y - \bar{a}A_{n-1,n-1}^{-1}(\hat{f}(x) + \hat{b})}{a_{nn} - \bar{a}A_{n-1,n-1}^{-1}\hat{a}} \leq x_n.$$

Since $\hat{f}(x)$ is bounded on $\Lambda$ and $-A_{n-1,n-1}^{-1}\hat{a} > 0$, hence, $\Lambda \backslash H(z^0, N)$ is bounded, where $-A_{n-1,n-1}^{-1}\hat{a} > 0$ comes from $\hat{a} < 0$ and $A_{n-1,n-1}^{-1} \leq 0$. The lemma follows. □

# 3 An Arbitrary Starting Variable Dimension Algorithm

In this section, based on the results in Section 2, we develop an algorithm for computing an integer point in $P$. As a result of the assumption on $A$, we obtain

**Lemma 8.** *If* $x^1 = (x_1^1, x_2^1, \cdots, x_n^1)^\top \in P$ *and* $x^2 = (x_1^2, x_2^2, \cdots, x_n^2)^\top \in P$,

$$\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)^\top = (\max\{x_1^1, x_1^2\}, \max\{x_2^1, x_2^2\}, \cdots, \max\{x_n^1, x_n^2\})^\top \in P.$$

**Proof.** Since $a_{n+1} \leq 0$,
$$a_{n+1}^\top \bar{x} \leq a_{n+1}^\top x^1 \leq b_{n+1}.$$
For any given $i \in N$, without loss of generality, we assume that $\bar{x}_i = x_i^1$. Then, from $a_{ii} > 0$ and $a_{ik} \leq 0$ for any $k \neq i$, we obtain that

$$a_i^\top \bar{x} = a_{ii}\bar{x}_i + \sum_{k \neq i} a_{ik}\bar{x}_k = a_{ii}x_i^1 + \sum_{k \neq i} a_{ik}\bar{x}_k \leq a_{ii}x_i^1 + \sum_{k \neq i} a_{ik}x_k^1 = a_i^\top x^1 \leq b_i.$$

Thus, $\bar{x} \in P$. □

Let $x^{\max}$ denote the unique solution of

$$\max_{x \in P} e^\top x,$$

where $e = (1, 1, \cdots, 1)^\top \in R^n$.

**Lemma 9.** *For any point* $x \in P$, $x \leq x^{\max}$.

**Proof.** Suppose that there is a point $\hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)^\top \in P$ satisfying $\hat{x}_q > x_q^{\max}$ for some $q \in N$. Then, from Lemma 8, we obtain that

$$\bar{x} = (\max\{\hat{x}_1, x_1^{\max}\}, \max\{\hat{x}_2, x_2^{\max}\}, \cdots, \max\{\hat{x}_n, x_n^{\max}\})^\top \in P.$$

Clearly, $e^\top \bar{x} > e^\top x^{\max}$, which contradicts that $e^\top x^{\max} = \max_{x \in P} e^\top x$. The lemma follows. □

Since $P$ has an interior point, hence, $x^{\max}$ is the unique solution of

$$A_{nn}x = (b_1, b_2, \cdots, b_n)^\top.$$

For any number $\alpha$, let $\lfloor \alpha \rfloor$ denote the greatest integer less than or equal to $\alpha$. We define $x^u = (x_1^u, x_2^u, \ldots, x_n^u)^\top$ with $x_i^u = \lfloor x_i^{\max} \rfloor$ for $i = 1, 2, \ldots, n$. Then, $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \cdots, \lfloor x_n \rfloor)^\top \leq x^u$ for any $x \in P$. If $x^u \in P$, an integer point of $P$ has been found. We assume that $x^u \notin P$.

Given the integer labeling rule in Section 2 and a triangulation $T$ of the space, an arbitrary starting variable dimension algorithm is developed for computing an integer point of an $n$-dimensional simplex, which is as follows.

**Initialization:** Compute $l(\eta)$. If $l(\eta) = 0$, the algorithm terminates, and an integer point of $P$ has been found. Otherwise, let $maxK = l(\eta)$, $h = 1$,

$$\alpha = \begin{cases} -1 & \text{if } h = maxK, \\ 1 & \text{otherwise,} \end{cases}$$

$\tau_0 =< \eta >$, $\sigma_0$ be the unique $h$-dimensional simplex in $X(\eta, h, \alpha)$ having $\tau_0$ as a facet, $y^+$ be the vertex of $\sigma_0$ opposite to $\tau_0$, $p = 1$, and $k = 0$. Go to Step 1.

**Step 1:** Compute $l(y^+)$. If $l(y^+) = 0$, the algorithm terminates, and an integer point of $P$ has been found. If either $l(y^+) = h$ and $maxK > h$ or $l(y^+) > h$ and $maxK = h$, then $\sigma_k$ is $h$-complete and go to Step 3. Otherwise, do as follows: Let $y^-$ be the unique vertex of $\tau_k$ such that

$$l(y^-) = \begin{cases} l(y^+) & \text{if } l(y^+) \leq h, \\ maxK & \text{otherwise,} \end{cases}$$

and $\tau_{k+1}$ the facet of $\sigma_k$ opposite to $y^-$. Let $maxK = l(y^+)$ if $l(y^+) > h$, and go to Step 2.

**Step 2:** If $\tau_{k+1} \subset X(\eta, h-1, \alpha)$ for some $\alpha \in \{-1, 1\}$, go to Step 4. Otherwise, do as follows: Let $\sigma_{k+1}$ be the unique simplex that is adjacent to $\sigma_k$ and has $\tau_{k+1}$ as a facet. Let $y^+$ be the vertex of $\sigma_{k+1}$ opposite to $\tau_{k+1}$ and $k = k+1$. Go to Step 1.

**Step 3:** If $h = n$ then go to Step 5. Otherwise, do as follows: Let $maxK = l(y^+)$ if $l(y^+) > h$. Let $h = h+1$, $\tau_{k+1} = \sigma_k$, and

$$\alpha = \begin{cases} -1 & \text{if } h = maxK, \\ 1 & \text{otherwise.} \end{cases}$$

Let $\sigma_{k+1}$ be the unique $h$-dimensional simplex in $X(\eta, h, \alpha)$ having $\tau_{k+1}$ as a facet, and $y^+$ be the vertex of $\sigma_{k+1}$ opposite to $\tau_{k+1}$. Let $k = k+1$, and go to Step 1.

**Step 4:** Let $\sigma_{k+1} = \tau_{k+1}$, $y^-$ be the unique vertex of $\sigma_{k+1}$ such that

$$l(y^-) = \begin{cases} h-1 & \text{if } \alpha = 1, \\ maxK & \text{otherwise,} \end{cases}$$

and $\tau_{k+2}$ the facet of $\sigma_{k+1}$ opposite to $y^-$. Let $maxK = h-1$ if $\alpha = -1$. Let $h = h-1$ and $k = k+1$, and go to Step 2.

**Step 5:** If $p$ is even, let $p = p+1$, $\alpha \in \{-1, 1\}$ satisfying $\sigma_k \subset X(\eta, n, \alpha)$, $y^-$ be the unique vertex of $\sigma_k$ such that

$$l(y^-) = \begin{cases} n & \text{if } \alpha = 1, \\ n+1 & \text{otherwise,} \end{cases}$$

$\tau_{k+1}$ the facet of $\sigma_k$ opposite to $y^-$,

$$maxK = \begin{cases} n+1 & \text{if } \alpha = 1, \\ n & \text{otherwise,} \end{cases}$$

$h = n$, and go to Step 2. If $p$ is odd, do as follows: Let $p = p+1$, $y^-$ be the vertex of $\sigma_k$ carrying integer label $n+1$ and $\tau_{k+1}$ the facet of $\sigma_k$ opposite to $y^-$. Go to Step 6.

**Step 6:** Let $\sigma_{k+1}$ be the unique simplex that is adjacent to $\sigma_k$ and has $\tau_{k+1}$ as a facet, and $y^+$ the vertex of $\sigma_{k+1}$ opposite to $\tau_{k+1}$. Let $k = k+1$ and go to Step 7.

**Step 7:** Compute $l(y^+)$. If $l(y^+) = 0$, the algorithm terminates, and an integer point of $P$ has been found. If $x^u \leq y^+$, the algorithm terminates, and there is no integer point in $P$. If $l(y^+) = n+1$, go to Step 5. If $l(y^+) \neq n+1$, let $y^-$ be the vertex of $\sigma_k$ other than $y^+$ and carrying integer label $l(y^+)$, and $\tau_{k+1}$ the facet of $\sigma_k$ opposite to $y^-$. Go to Step 6.

We remark that the algorithm is composed of two phases, one of which consists of Steps 1-4 of the algorithm and the other Steps 6-7 of the algorithm. Step 5 of the algorithm is a bridge for interchanging between these two phases. Steps 1-4 of the algorithm forms a variable dimension algorithm, which comes from a modification of the 2-ray algorithm in Yamamoto (1984), and Steps 6-7 of the algorithm a full-dimension pivoting procedure, which comes from Dang and Maaren (1999).

**Theorem 10.** *Within a finite number of iterations, the algorithm either yields an integer point in P or proves that no such points exist.*

**Proof.** As that in Dang and Maaren (1999), by constructing an undirected graph and following a standard argument, one can show that the algorithm will never cycle. When Steps 1-4 of the algorithm are implemented, Lemma 3 and Lemma 4 imply that, within a finite number of iterations, either an integer point in $P$ is found or an n-complete simplex is generated. If an $n$-complete simplex is generated in Steps 1-4 of the algorithm, Steps 6-7 of the algorithm will be implemented. When Steps 6-7 of the algorithm are implemented, Lemma 7 implies that, within a finite number of iterations, one of the following events will occur: an $n$-complete simplex is generated, an integer point in $P$ is found, or an integer point greater than or equal to $x^u$ is met. If an $n$-complete simplex is generated in Steps 6-7 of the algorithm, Steps 1-4 of the algorithm will be implemented. The algorithm interchanges between two phases. From Lemma 2, we know that there are finitely many $n$-complete simplices. Therefore, the algorithm interchanges between two phases at most finitely many times.

From Corollary 6, we know that, if $z^0$ is an integer point in $P$, there is no $n$-complete simplex in $H(z^0,N)$ and no $(n-1)$-dimensional complete simplex carrying only integer labels in $N$ and contained in the boundary of $H(z^0,N)$, $\cup_{j\in N} H(z^0, N\setminus\{j\})$. Thus, if $P$ has an integer point, when Steps 6-7 are implemented in the algorithm, within a finite number of iterations, the algorithm will either generate an $n$-complete simplex or find an integer point in $P$ since any integer point in $P$ is less than or equal to $x^u$. Therefore, when Steps 6-7 are implemented in the algorithm, the algorithm meets an integer point greater than or equal to $x^u$, which implies that $P$ has no integer point. The theorem follows. □

The following example shows how the algorithm works when $n = 2$.

**Example 1.** Consider

$$P = \{x = (x_1, x_2)^\top \mid Ax \leq b\},$$

where

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 3 \\ -1 & -1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

Let $\eta = (-2, 2)^\top$. Computing $l(\eta)$, we obtain $l(\eta) = 2$. Let $maxK = 2$, $h = 1$, $\alpha = 1$, $\tau_0 =< \eta >$, $y^0 = \eta$, $y^1 = (-1, 2)^\top$, $\sigma =< y^0, y^1 >$, and $y^+ = y^1$.

**Iteration 1:** Computing $l(y^+)$, we obtain $l(y^+) = 2$. Let $y^- = y^0$, $\tau_1 =< y^1 >$, $y^0 = (0, 2)^\top$, $\sigma_1 =< y^0, y^1 >$, and $y^+ = y^0$.
**Iteration 2:** Computing $l(y^+)$, we obtain $l(y^+) = 2$. Let $y^- = y^1$, $\tau_2 =< y^0 >$, $y^1 = (1, 2)^\top$, $\sigma_2 =< y^0, y^1 >$, and $y^+ = y^1$.
**Iteration 3:** Computing $l(y^+)$, we obtain $l(y^+) = 2$. Let $y^- = y^0$, $\tau_3 =< y^1 >$, $y^0 = (2, 2)^\top$, $\sigma_3 =< y^0, y^1 >$, and $y^+ = y^0$.
**Iteration 4:** Computing $l(y^+)$, we obtain $l(y^+) = 1$. Let $h = h + 1 = 2$, $\tau_4 =< y^0, y^1 >$, $\alpha = -1$, $y^2 = (2, 1)$, $\sigma_4 =< y^0, y^1, y^2 >$, and $y^+ = y^2$.
**Iteration 5:** Computing $l(y^+)$, we obtain $l(y^+) = 1$. Let $y^- = y^0$, $\tau_5 =< y^1, y^2 >$, $y^0 = (1, 1)$, $\sigma_5 =< y^0, y^1, y^2 >$, and $y^+ = y^0$.
**Iteration 6:** Computing $l(y^+)$, we obtain $l(y^+) = 1$. Let $y^- = y^2$, $\tau_6 =< y^0, y^1 >$, $y^2 = (0, 1)$, $\sigma_6 =< y^0, y^1, y^2 >$, and $y^+ = y^2$.
**Iteration 7:** Computing $l(y^+)$, we obtain $l(y^+) = 2$. Let $y^- = y^1$, $\tau_7 =< y^0, y^2 >$, $y^1 = (1, 0)$, $\sigma_7 =< y^0, y^1, y^2 >$, and $y^+ = y^1$.
**Iteration 8:** Computing $l(y^+)$, we obtain $l(y^+) = 1$. Let $y^- = y^0$, $\tau_8 =< y^1, y^2 >$, $y^0 = (0, 0)$, $\sigma_8 =< y^0, y^1, y^2 >$, and $y^+ = y^0$.
**Iteration 9:** Computing $l(y^+)$, we obtain $l(y^+) = 0$. An integer point of $P$ has been obtained.

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## Appendix A Aggregations

In this section, we consider how to find two relatively prime integers $t_1$ and $t_2$ satisfying that the set of nonnegative integer solutions of

$$\sum_{j=1}^{n-1} a_{1j}x_j = b_1,\ \sum_{j=1}^{n-1} a_{2j}x_j = b_2,\ \sum_{j=1}^{n-1} x_j \le u, \tag{A.1}$$

is equal to the set of nonnegative integer solutions of

$$\sum_{j=1}^{n-1}(t_1a_{1j}+t_2a_{2j})x_j = t_1b_1+t_2b_2,\ \sum_{j=1}^{n-1} x_j \le u, \tag{A.2}$$

where $0 < u$. The results in this section follow those in Zhu (1998).

Assume that $0 \leq b_1$, $0 \leq b_2$, and $0 < b_1 + b_2$. Let $\mu_j = b_1 a_{2j} - b_2 a_{1j}$ for $j = 1, 2, \cdots, n-1$.

**Lemma A.1.** *The set of nonnegative integer solutions of*

$$\sum_{j=1}^{n-1} a_{1j} x_j = b_1, \; \sum_{j=1}^{n-1} a_{2j} x_j = b_2, \; \sum_{j=1}^{n-1} x_j \leq u,$$

*is equal to the set of nonnegative integer solutions of*

$$\sum_{j=1}^{n-1} (t_1 a_{1j} + t_2 a_{2j}) x_j = t_1 b_1 + t_2 b_2, \; \sum_{j=1}^{n-1} x_j \leq u,$$

*where $t_1$ and $t_2$ are two relatively prime integers satisfying*

$$t_1 b_1 + t_2 b_2 > u \max_{1 \leq j \leq n-1} |\mu_j|.$$

**Proof.** Assume that $x$ is a nonnegative integer solution of (A.2). Then,

$$t_1 \sum_{j=1}^{n-1} a_{1j} x_j = t_1 b_1 + t_2 b_2 - t_2 \sum_{j=1}^{n-1} a_{2j} x_j.$$

Thus,

$$\sum_{j=1}^{n-1} a_{1j} x_j = b_1 + t_2 (b_2 - \sum_{j=1}^{n-1} a_{2j} x_j)/t_1.$$

Let $q = (b_2 - \sum_{j=1}^{n-1} a_{2j} x_j)/t_1$. Then, $q$ is an integer since $t_1$ and $t_2$ are relatively prime. Therefore,

$$\sum_{j=1}^{n-1} a_{2j} x_j = b_2 + t_1 (b_1 - \sum_{j=1}^{n-1} a_{1j} x_j)/t_2 = b_2 - t_1 q.$$

Observe that

$$\begin{aligned} t_1 b_1 + t_2 b_2 &> u \max_{1 \leq j \leq n-1} |\mu_j| \\ &\geq \max_{1 \leq j \leq n-1} |\mu_j| \sum_{j=1}^{n-1} x_j \\ &\geq |\sum_{j=1}^{n-1} \mu_j x_j| \\ &= |\sum_{j=1}^{n-1} (b_1 a_{2j} - b_2 a_{1j}) x_j| \\ &= |b_1 \sum_{j=1}^{n-1} a_{2j} x_j - b_2 \sum_{j=1}^{n-1} a_{1j} x_j| \end{aligned}$$

$$=|b_1(b_2 - t_1 q) - b_2(b_1 + t_2 q)|$$
$$=(t_1 b_1 + t_2 b_2)|q|.$$

Then, $|q| < 1$. Thus, $q = 0$. The lemma follows. □

One solution of $t_1$ and $t_2$ is given by $t_1 = p$ and $t_2 = p+1$, where $p$ is the smallest integer satisfying

$$p > \frac{u \max_{1\leq j\leq n-1} |\mu_j| - b_2}{b_1 + b_2}.$$

Let $r = t_1 b_1 + t_2 b_2$, $c_j = t_1 a_{1j} + t_2 a_{2j}$ for $j = 1, 2, \cdots, n-1$, and $c_n = 0$. Consider

$$\sum_{j=1}^{n} c_j x_j = r,\ \sum_{j=1}^{n} x_j = u,\ 0 \leq x. \tag{A.3}$$

Let $k$ be an index satisfying

$$c_k = \min_{1\leq j\leq n} c_j$$

and $l$ an index satisfying

$$c_l = \max_{1\leq j\leq n} c_j.$$

Let $\rho_j = r - u c_j$ for $j = 1, 2, \cdots, n$.

**Lemma A.2.** *The set of integer solutions of*

$$\sum_{j=1}^{n} c_j x_j = r,\ \sum_{j=1}^{n} x_j = u,\ 0 \leq x,$$

*is equal to the set of integer solutions of*

$$\sum_{j=1}^{n} (s_1 c_j + s_2) x_j = s_1 r + s_2 u,\ 0 \leq x, \tag{A.4}$$

*where $s_1$ and $s_2$ are two relatively prime integers satisfying*

$$s_1 c_k + s_2 > \max\{0, \rho_k\},\ s_1 c_l + s_2 > \max\{0, -\rho_l\},\ s_1 r + s_2 u \neq 0.$$

**Proof.** Assume that $x$ is an integer solution of (A.4). Then, there is an integer $q$ satisfying

$$\sum_{j=1}^{n} c_j x_j = r + s_2 q.$$

Thus,

$$\sum_{j=1}^{n} x_j = u - s_1 q.$$

Using $s_1c_k + s_2 > \max\{0, \rho_k\}$ and $s_1c_l + s_2 > \max\{0, -\rho_l\}$, we obtain

$$s_1c_j + s_2 > 0,\ j = 1, 2, \cdots, n.$$

If $s_1r + s_2u < 0$, then both (A.3) and (A.4) are infeasible. So, we only need to consider

$$s_1r + s_2u > 0.$$

Then,

$$(s_1r + s_2u)(s_1c_k + s_2) > (s_1r + s_2u)\max\{0, \rho_k\} \geq (s_1r + s_2u)(r - uc_k)$$

and

$$(s_1r + s_2u)(s_1c_l + s_2) > (s_1r + s_2u)\max\{0, -\rho_l\} \geq (s_1r + s_2u)(uc_l - r).$$

Let

$$f(\alpha) = \frac{r - u\alpha}{s_1\alpha + s_2}.$$

Then,

$$\frac{df}{d\alpha} = \frac{-u(s_1\alpha + s_2) - s_1(r - u\alpha)}{(s_1\alpha + s_2)^2} = -\frac{s_1r + s_2u}{(s_1\alpha + s_2)^2} < 0.$$

Thus,

$$\begin{aligned}
s_1r + s_2u &> (s_1r + s_2u)\frac{r - uc_k}{s_1c_k + s_2} \\
&= (s_1r + s_2u)\max_{1 \leq p \leq n}\frac{r - uc_p}{s_1c_p + s_2} \\
&= \sum_{j=1}^{n}(s_1c_j + s_2)x_j \max_{1 \leq p \leq n}\frac{r - uc_p}{s_1c_p + s_2} \\
&\geq \sum_{j=1}^{n}(s_1c_j + s_2)(\frac{r - uc_j}{s_1c_j + s_2})x_j \\
&= \sum_{j=1}^{n}(r - uc_j)x_j
\end{aligned}$$

and

$$\begin{aligned}
s_1r + s_2u &> (s_1r + s_2u)\frac{uc_l - r}{s_1c_l + s_2} \\
&= (s_1r + s_2u)\max_{1 \leq p \leq n}\frac{uc_p - r}{s_1c_p + s_2} \\
&= \sum_{j=1}^{n}(s_1c_j + s_2)x_j \max_{1 \leq p \leq n}\frac{uc_p - r}{s_1c_p + s_2} \\
&\geq \sum_{j=1}^{n}(s_1c_j + s_2)(\frac{uc_j - r}{s_1c_j + s_2})x_j
\end{aligned}$$

$$= -\sum_{j=1}^{n}(r - uc_j)x_j.$$

Combining these two inequalities together, we obtain

$$\begin{aligned} s_1 r + s_2 u &> |\sum_{j=1}^{n}(r - uc_j)x_j| \\ &= |r\sum_{j=1}^{n}x_j - u\sum_{j=1}^{n}c_j x_j| \\ &= |r(u - s_1 q) - u(r + s_2 q)| \\ &= (s_1 r + s_2 u)|q|. \end{aligned}$$

Then, $|q| < 1$. Thus, $q = 0$. The lemma follows. □

One solution of $s_1$ and $s_2$ is given by $s_1 = 1$ and $s_2$ being equal to the smallest integer satisfying

$$s_2 > \max\{\max\{0, \rho_k\} - c_k,\ \max\{0, -\rho_l\} - c_l\} \text{ and } r + s_2 u \neq 0.$$

Let $d_j = s_1 c_j + s_2$ for $j = 1, 2, \cdots, n$, and $h = s_1 r + s_2 u$. Then, $0 < d_j$ for $j = 1, 2, \cdots, n$, and

$$\sum_{j=1}^{n} d_j x_j = h,\ 0 \leq x.$$

Repeating the above procedure, one can aggregate a polytope into a simplex.

## Appendix B Extended GCD via Lattice Basis Reduction

Let $p^\top = (p_1, p_2, \cdots, p_m)$ with every component being an integer. The extended GCD via lattice basis reduction (Havas et al., 1998) generates a unimodular matrix $U$ such that $a^\top U = (0, 0, \cdots, 0, GCD(p))^\top$, where $GCD(p)$ is the greatest common devisor of $p_1$, $i = 1, 2, \cdots, m$ and which can be stated as follows.

**Initialization:** For $i = 1, 2, \cdots, m$, let $u_i^\top$ be the $i$th row of the $m \times m$ identity matrix $I_m$. For $r = 2, 3, \cdots, m$, let $\lambda_{rs} = 0$ for $s = 1, 2, \cdots, r-1$. Let $D_i = 1$, $i = 0, 1, \cdots, m$, $m_1 = 3$, $n_1 = 4$, and $k = 2$. Go to Step 1.

**Step 1:** Implement **Reduce**$(k, k-1)$. If either $p_{k-1} \neq 0$ or it holds that $p_{k-1} = 0$, $p_k = 0$ and $n_1(D_{k-2}D_k + \lambda_{k,k-1}^2) < m_1 D_{k-1}^2$, implement **Swap**$(k)$, let $k = k-1$ when $k > 2$, and go to Step 2. Otherwise, for $i = k-2, k-3, \cdots, 1$, implement **Reduce**$(k, i)$, let $k = k+1$, and go to Step 2.

**Step 2:** If $k \leq m$, go to Step 1. Otherwise, stop. If $p_m < 0$, let $p_m = -p_m$ and $u_m = -u_m$.

In the procedure of the extended GCD via lattice basis reduction, for any $k$ and $i$, **Reduce**$(k, i)$ and **Swap**$(k)$ are given as follows:

**Reduce**$(k,i)$: If $p_i \neq 0$, let $q = \lceil p_k/p_i \rfloor$, where $\lceil \theta \rfloor$ denotes the nearest integer to $\theta$ with $\lceil \theta \rfloor = \theta - \frac{1}{2}$ for $\theta$ being a half integer. Otherwise, if $2|\lambda_{ki}| > D_i$, let $q = \lceil \lambda_{ki}/D_i \rfloor$, and if $2|\lambda_{ki}| \leq D_i$, let $q = 0$. If $q \neq 0$, let $p_k = p_k - qp_i$, $u_k = u_k - qu_i$, $\lambda_{ki} = \lambda_{ki} - qD_i$, and $\lambda_{kj} = \lambda_{kj} - q\lambda_{ij}$ for $j = 1, 2, \cdots, i-1$.

**Swap**$(k)$: Let $h = p_k$, $p_k = p_{k-1}$, $p_{k-1} = h$, $g = u_k$, $u_k = u_{k-1}$, and $u_{k-1} = g$. For $j = 1, 2, \cdots, k-2$, let $h = \lambda_{kj}$, $\lambda_{kj} = \lambda_{k-1,j}$, and $\lambda_{k-1,j} = h$. For $i = k+1, \cdots, m$, $t = \lambda_{i,k-1}D_k - \lambda_{ik}\lambda_{k,k-1}$, $\lambda_{i,k-1} = (\lambda_{i,k-1}\lambda_{k,k-1} + \lambda_{ik}D_{k-2})/D_{k-1}$, and $\lambda_{ik} = t/D_{k-1}$. Let $D_{k-1} = (D_{k-2}D_k + \lambda_{k,k-1}^2)/D_{k-1}$.

Consider

$$a^\top x = d,\ 0 \leq x, \tag{B.1}$$

where $a^\top = (a_1, a_2, \cdots, a_{n+1}) > 0$ and every component of $a$ is an integer. Apply the procedure of the extended GCD via lattice basis reduction to obtain a unimodular matrix $U$ such that $a^\top U = (0, 0, \cdots, 0, GCD(a))$. Let $x = Uy$ and $u^i$ denote the $i$th column of $U$ for $i = 1, 2, \cdots, n+1$. $U = (u^1, u^2, \cdots, u^{n+1})$. Substituting $x = Uy$ into (B.1), we obtain that $0 \leq Uy$ and $GCD(a)y_{n+1} = d$. Let $y_{n+1} = d/GCD(a)$. Then,

$$-(y_1u^1 + y_2u^2 + \cdots + y_nu^n) \leq (d/GCD(a))u^{n+1}.$$

Let $A = (-u^1, -u^2, \cdots, -u^n)$, $b = (d/r_n)u^{n+1}$, and $z = (y_1, y_2, \cdots, y_n)^\top$. Then,

$$Az \leq b,$$

where $A$ is an $(n+1) \times n$ matrix with every component being an integer. If $d/GCD(a)$ is not an integer, the equation (B.1) has no integer solutions.

# Appendix C Canonical Form

**Definition C.1.** An $(n+1) \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{pmatrix}$$

is in canonical form if $a_{ii} > 0$, $i = 1, 2, \cdots, n$, and $a_{ij} \leq 0$ for any $i$ and $j$ with $i \neq j$.

Let $A$ be an arbitrary $(n+1) \times n$ matrix satisfying that there is a positive vector $\rho = (\rho_1, \rho_2, \cdots, \rho_{n+1})^\top$ such that $\rho^\top A = 0$ and that any $n \times n$ submatrix of $A$ is nonsingular. In order to transform $A$ into a matrix in the canonical form, the following three elementary column operations can be applied:

- interchange two columns,
- multiply a column by $-1$, and
- add an integer times a column to another column.

**Theorem C.1.** *(Pnueli, 1968) Let A be an* $(n+1)\times n$ *matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{pmatrix}$$

*satisfying that there is a positive vector* $\rho > 0$ *with* $\rho^\top A = 0$ *and every* $n\times n$ *submatrix of A is nonsingular. Then, applying three elementary column operations to A, one can transform A in polynomial-time into a matrix in the canonical form.*

Let $I_n$ be the $n\times n$ identity matrix and

$$W = \begin{pmatrix} A \\ I_n \end{pmatrix}.$$

For $i = 1,2,\cdots,2n+1$ and $j = 1,2,\cdots,n$, let $w_{ij}$ denote the entry of $W$ in the $i$th row and the $j$th column, and, for $i = 1,2,\cdots,n$, let $w^i$ denote the $i$th column of $W$.

**Backward Reduction Procedure:**

**Initialization:** Let $h = n+1$, $m = n$,

$$p^\top = (p_1, p_2, \cdots, p_m) = (w_{n+1,1}, w_{n+1,2}, \cdots, w_{n+1,n}),$$

$W_h = (w^1, w^2, \cdots, w^m)$, and $W\backslash W_h = (w^{m+1}, w^{m+2}, \cdots, w^n)$. Go to Step 1.

**Step 1:** Apply the procedure of the extended GCD via lattice basis reduction to $p$. Let $U_m$ denote the unimodular matrix generated by the procedure, $W_h = W_h U_m^\top$, and $W = (W_h, W\backslash W_h)$. Go to Step 2.

**Step 2:** Let $h = h-1$. If $h > 2$, let $m = h-1$,

$$p^\top = (p_1, p_2, \cdots, p_m) = (w_{h,1}, w_{h,2}, \cdots, w_{h,m}),$$

$W_h = (w^1, \cdots, w^m)$, and $W\backslash W_h = (w^{m+1}, w^{m+2}, \cdots, w^n)$, and go to Step 1. Otherwise, stop. Let $W = -W$.

**Forward Reduction Procedure:** Let $W_k$ denote the $(2n+1)\times k$ submatrix formed by the first $k$ columns of $W$, i.e.,

$$W_k = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1k} \\ w_{21} & w_{11} & \cdots & w_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ w_{2n+1,1} & w_{2n+1,2} & \cdots & w_{2n+1,k} \end{pmatrix}.$$

**Initialization:** Let $k = 2$ and $D_1 = W_1$. Go to Step 1.

**Step 1:** Use the diagonal positive entries, $d_{hh}$, $h = 1,2,\cdots,k-1$, of the first $k-1$ rows of $D_{k-1}$ to reduce the first $k-1$ entries of the $k$th column of $W$, $w^k$, to nonpositive entries with $|w_{hk}| < d_{hh}$: For $h = 1,2,\cdots,k-1$, if $w_{hk} > 0$, let $w^k = w^k - \lceil w_{hk}/d_{hh}\rceil d^h$, and if $-w_{hk} \geq d_{hh}$, $w^k = w^k - \lfloor -w_{hk}/d_{hh}\rfloor d^h$, where $d^h$ is the $h$th column of $D_{k-1}$. Let $D_k = (D_{k-1}, w^k)$. Go to Step 2

**Step 2:** If $k < n$, implement $\mathbf{LTM}(D_k)$, let $k = k+1$, and go to Step 2. Otherwise, stop.

$\mathbf{LTM}(D_k)$:

**Initialization:** Let $h = 1$ and go to Step 1.
**Step 1:** If $d_{hk} = 0$, go to Step 3; if $|d_{hk}| < d_{hh}$, go to Step 2; and otherwise, let $d^k = d^k - \lceil d_{hk}/d_{hh} \rfloor d^h$ and go to Step 2.
**Step 2:** If $d_{hk} = 0$, go to Step 3. Otherwise, do as follows: Let $g = d^h$, $d^h = d^k$, and $d^k = g$. If $d_{hh} < 0$, let $d^h = -d^h$. Go to Step 1.
**Step 3:** Let $h = h+1$. If $h \leq k$, go to Step 1. Otherwise, stop.

If one wants $A$ to satisfy that $|a_{ij}| < a_{ii}$ for any $i$ and $j$ with $i \neq j$, the following reduction procedure can be applied. If $|a_{kj}| \geq a_{kk}$, add $\lfloor \frac{|a_{kj}|}{a_{kk}} \rfloor$ times the $k$th column to the $j$th column. Repeat this procedure till that, for $k = 1, 2, \cdots, n$, $|a_{kj}| < a_{kk}$ for any $j \neq k$.

## Appendix D The $D_1$-Triangulation of $R^n$

A simplex of the $D_1$-triangulation of $R^n$ is the convex hull of $n+1$ vectors, $y^0$, $y^1$, $\ldots$, $y^n$, given as follows: If $p = 0$ then $y^0 = y$ and $y^k = y + s_{\pi(k)} u^{\pi(k)}$, $k = 1, 2, \cdots, n$, and, if $p \geq 1$ then $y^0 = y + s$, $y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)}$, $k = 1, 2, \cdots, p-1$, and $y^k = y + s_{\pi(k)} u^{\pi(k)}$, $k = p, p+1, \cdots, n$, where $y$ is an integer point of $R^n$ with every component being an even number, $\pi = (\pi(1), \pi(2), \cdots, \pi(n))$ a permutation of elements of $N = \{1, 2, \cdots, n\}$, $s$ a sign vector with every component being a number in $\{-1, 1\}$, and $p$ an integer with $0 \leq p \leq n-1$. Let $D_1$ be the set of all such simplices. Since a simplex of the $D_1$-triangulation is determined by $y$, $\pi$, $s$, and $p$, we use $D_1(y, \pi, s, p)$ to denote it.

We say that two simplices of $D_1$ are adjacent if they share a common facet. We show how to generate all the adjacent simplices of a simplex of the $D_1$-triangulation of $R^n$ in the following. For a given simplex $\sigma = D_1(y, \pi, s, p)$ with the vertices $y^0$, $y^1$, $\ldots$, $y^n$, its adjacent simplex opposite to a vertex, say $y^i$, is given by $D_1(\bar{y}, \bar{\pi}, \bar{s}, \bar{p})$, where $\bar{y}$, $\bar{\pi}$, $\bar{s}$, and $\bar{p}$ are generated according to the pivot rules given in the following table.

Pivot Rules of the $D_1$-Triangulation of $R^n$

| $i$ | | | $\bar{y}$ | $\bar{s}$ | $\bar{\pi}$ | $\bar{p}$ |
|---|---|---|---|---|---|---|
| 0 | $n=1$ | | $y+2s_{\pi(1)}u^{\pi(1)}$ | $s-2s_{\pi(1)}u^{\pi(1)}$ | $\pi$ | $p$ |
| | $n \geq 2$ | $p=0$ | $y$ | $s$ | $\pi$ | 1 |
| | | $p=1$ | $y$ | $s$ | $\pi$ | 0 |
| | | $2 \leq p$ | $y$ | $s-2s_{\pi(1)}u^{\pi(1)}$ | $\pi$ | $p$ |
| $1 \leq i$ | | $p=0$ | $y$ | $s-2s_{\pi(i)}u^{\pi(i)}$ | $\pi$ | $p$ |
| | | $i < p-1$ | $y$ | $s$ | $\pi^1$ | $p$ |
| | | $i = p-1$ | $y$ | $s$ | $\pi$ | $p-1$ |
| | $p-1 < i$ | $1 \leq p < n-1$ | $y$ | $s$ | $\pi^2$ | $p+1$ |
| | $i = n-1$ | $1 \leq p = n-1$ | $y+2s_{\pi(n)}u^{\pi(n)}$ | $s-2s_{\pi(n)}u^{\pi(n)}$ | $\pi$ | $p$ |
| | $i = n$ | $1 \leq p = n-1$ | $y+2s_{\pi(n-1)}u^{\pi(n-1)}$ | $s-2s_{\pi(n-1)}u^{\pi(n-1)}$ | $\pi$ | $p$ |

$\pi^1 = (\pi(1), \cdots, \pi(i+1), \pi(i), \cdots, \pi(n))$,
$\pi^2 = (\pi(1), \cdots, \pi(p-1), \pi(i), \pi(p), \cdots, \pi(i-1), \pi(i+1), \cdots, \pi(n))$.

It is clear that $D_1 + \eta$ is still a triangulation of $R^n$. Let $\mathscr{D}_1 + \eta$ be the set of faces of simplices of $D_1 + \eta$. A $q$-dimensional simplex of $\mathscr{D}_1 + \eta$ with vertices $y^0$, $y^1$, $\ldots$, $y^q$ is denoted by $< y^0, y^1, \ldots, y^q >$. The restriction of $\mathscr{D}_1 + \eta$ on $X(\eta, h, \alpha)$ for any $h \in N$ is given by

$$\mathscr{D}_1 + \eta | X(\eta, h, \alpha) = \{\sigma \in \mathscr{D}_1 + \eta \mid \sigma \subset X(\eta, h, \alpha) \text{ and } \dim(\sigma) = h\}.$$

Obviously, $\mathscr{D}_1 + \eta | X(\eta, h, \alpha)$ is a triangulation of $X(\eta, h, \alpha)$.