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Stochastic Optimal Control Problems with a Bounded Memory*

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Abstract This paper treats a finite time horizon optimal control problem in which the controlled state dynamics is governed by a general system of stochastic functional differential equations with a bounded memory. An infinite-dimensional HJB equation is derived using a Bellman-type dynamic programming principle. It is shown that the value function is the unique viscosity solution of the HJB equation. In addition, the computation issues are also studied. More particularly, a finite difference scheme is obtained to approximate the viscosity solution of the infinite dimensional HJB equation. The convergence of the scheme is proved using the Banach fixed point theorem. The computational algorithm is also provided based on the scheme obtained.

Keywords stochastic control; stochastic functional differential equations; viscosity solutions; finite difference approximation

1 Introduction

The theory of stochastic functional differential equations has been widely used to describe the stochastic systems whose evolution depend on the past history of the state. It has many applications in real world applications (see Mohammed [20], [21] and Kolmanovskii and Shaikhet [13] for basic theory and some applications). The linear-quadratic regulatory problem involving stochastic delay equations was first studied in Kolmanovskii and Maizenberg [12], and optimal control problems for a class of nonlinear stochastic equations that involve a continuous delay of the following type

$$dX(s) = \alpha(s,X(s),Y(s),u(s))ds + \beta(s,X(s),Y(s),u(s))dW(s), \quad s \in [t,T], \tag{1}$$

have been studied in recent literature (see e.g. Elsanousi [8], Elsanousi et al [9], and Larssen [16], Oksendal and Sulem [22]), in which $Y(s) = \int_{-r}^{0} e^{-\delta\theta} X(s+\theta)d\theta$.

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In this paper, we will consider more general forms of the system of stochastic equations in $\mathbf{R}^n$

$$dX(s) = f(s, X_s, u(s))ds + g(s, X_s, u(s))dW(s), \quad \forall s \in [t, T] \tag{2}$$

in which $W(s)$ is a standard m-dimensional Brownian motion. In addition, the drift $f(s, X_s, u(s))$ and the diffusion coefficients $g(s, X_s, u(s))$ depend explicitly on the window of the state process X_s over the time interval $[s-r, s]$, where $X_s : [-r, 0] \to \mathbf{R}^n$ is defined by $X_s(\theta) = X(s+\theta), \theta \in [-r, 0]$. The consideration of such a system enable us to model many real world problems that have aftereffects (see [13]). Apparently, equation (1) is only a special case of (2).

In our recent work [4], we studied the system (2) using the viscosity solution concept introduced by Crandall and Lions [2], [18], [19] in order to characterize the value function as the unique viscosity solution of the associated HJB equation. In [5], we considered the finite difference method to solve the associated Hamilton-Jacobi-Bellman equation numerically. One thing we would like to point out is that the Markov Chain approximation method (see [15] for basic theory) can also be used to obtain the numerical solution for stochastic systems with delay (see Kusher [14]). In addition, in [3, 6], we considered optimal stopping time for stochastic systems with a bounded memory. In [7], we studied the application in Black-Scholes formula when the stock price is described with a stochastic delayed differential equations.

This paper is organized as follows. Notation and the statement of the problem are contained in Section 2. In Section 3, the infinite dimensional Hamilton-Jacobi-Bellman (HJB) equation for the value function is given. In Section 4, we consider the viscosity solution of the HJB equation. It is shown in Section 4 that the value function is a viscosity solution of the HJB equation. The uniqueness result for viscosity solution of the HJB equation is also given there. In Section 5, we present a finite difference method to approximate the viscosity solution of the HJB equation. The convergence results are given. An computational algorithm is given in Section 6.

2 Problem Formulation

Let $T > 0$ denote a fixed terminal time, and let $t \in [0, T]$ denote an initial time. We study the finite time horizon optimal control problem for a general system of stochastic functional differential equations on the interval $[t, T]$. Let $r > 0$ be a fixed constant, and let $\mathbb{J} = [-r, 0]$ denote the duration of the bounded memory of the equations considered in this paper. For the sake of simplicity, denote $C(\mathbb{J}; \Re^n)$, the space of continuous functions $\phi : \mathbb{J} \to \Re^n$, by $\mathbf{C}$. Note that $\mathbf{C}$ is a real separable Banach space under the sup-norm defined by

$$\|\phi\| = \sup_{t \in \mathbb{J}} |\phi(t)|, \quad \phi \in \mathbf{C},$$

where $|\cdot|$ is the Euclidean norm in $\Re^n$.

We denote by $(\,\cdot\,|\,\cdot\,)$ the inner product in $L^2(\mathbb{J},\mathfrak{R}^n)$, and $\langle\cdot,\cdot\rangle$ the inner product in $\mathfrak{R}^n$. Given ϕ and ψ in $\mathbf{C}$, we have defined as follows,

$$(\phi|\psi)=\int_{-r}^{0}\langle\phi(s),\psi(s)\rangle ds, \qquad \text{and} \qquad \|\phi\|_2=(\phi|\phi)^{\frac{1}{2}}.$$

Note that the space $\mathbf{C}$ can be continuously embedded into $L^2(\mathbb{J};\mathfrak{R}^n)$.

Convention 1.
Throughout the end, we use the following conventional notation for functional differential equations (see Hale [11]): If $\psi\in C([-r,\infty);\mathfrak{R}^n)$ and $t\in\mathfrak{R}_+$, let $\psi_t\in\mathbf{C}$ be defined by $\psi_t(\theta)=\psi(t+\theta)$, $\theta\in\mathbb{J}$.

Throughout the end, let $\{W(t),t\geq 0\}$ be a certain m-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega,\mathscr{F},P;\mathbf{F})$, where $\mathbf{F}=\{\mathscr{F}(t),t\geq 0\}$ is the P-augmentation of the natural filtration $\{\mathscr{F}^W(t),t\geq 0\}$ generated by the Brownian motion $\{W(t),t\geq 0\}$, *i.e.,* if $t\geq 0$,

$$\mathscr{F}^W(t)=\sigma\{W(s),0\leq s\leq t\}$$

and

$$\mathscr{F}(t)=\mathscr{F}^W(t)\vee\{A\subset\Omega|\exists B\in\mathscr{F}\text{ such that }A\subset B\text{ and }P(B)=0\}$$

where the operator $\vee$ denotes that $\mathscr{F}(t)$ is the smallest σ-algebra such that $\mathscr{F}^W(t)\subset\mathscr{F}(t)$ and

$$\{A\subset\Omega|\exists B\in\mathscr{F}\text{ such that }A\subset B\text{ and }P(B)=0\}\subset\mathscr{F}(t).$$

Let $L^2(\Omega,\mathbf{C})$ be the space of $\mathbf{C}$-valued random variables $\Xi:\Omega\to\mathbf{C}$ such that

$$\|\Xi\|_{L^2}=\left\{\int_\Omega\|\Xi(\omega)\|^2dP(\omega)\right\}^{\frac{1}{2}}<\infty.$$

In addition, let $L^2(\Omega,\mathbf{C};\mathscr{F}(t))$ be those $\Xi\in L^2(\Omega,\mathbf{C})$ which are $\mathscr{F}(t)$-measurable.

We consider the following system of controlled stochastic functional differential equations with a bounded memory:

$$dX(s)=f(s,X_s,u(s))ds+g(s,X_s,u(s))dW(s), \quad s\in[t,T], \tag{3}$$

with the initial function $X_t=\psi_t$, where $\psi_t\in L^2(\Omega,\mathbf{C};\mathscr{F}(t))$ and $u(\cdot)=\{u(s),s\in[t,T]\}$ is a control process taking values in a compact set U (of an Euclidean space). The functions, $f:[0,T]\times\mathbf{C}\times U\to\mathfrak{R}^n$ and $g:[0,T]\times\mathbf{C}\times U\to\mathfrak{R}^{n\times m}$ are given deterministic functions.

Definition 1. Given the m-dimensional standard Brownian motion $\{W(s),s\in[0,T]\}$ and the control process $\{u(s),\ s\in[t,T]\}$, a process $\{X(s;t,\psi_t,u(\cdot)),s\in[t-r,T]\}$ is said to be a (strong) solution of the controlled equation (3) on the interval $[t-r,T]$ and through the initial datum $(t,\psi_t)\in[0,T]\times L^2(\Omega,\mathbf{C};\mathscr{F}(t))$ if it satisfies the following conditions:

1. $X_t(\cdot;t,\psi_t,u(\cdot)) = \psi_t$;
2. $X(s;t,\psi_t,u(\cdot))$ is $\mathscr{F}(s)$-measurable for each $s \in [t,T]$;
3. The process $\{X(s;t,\psi_t,u(\cdot)), s \in [t,T]\}$ is continuous and satisfies the following stochastic integral equation P-a.s.

$$X(s) = \psi_t(0) + \int_t^s f(\lambda, X_\lambda, u(\lambda))d\lambda + \int_t^s g(\lambda, X_\lambda, u(\lambda))dW(\lambda). \quad (4)$$

In addition, the solution process $\{X(s;t,\psi_t,u(\cdot)), s \in [t-r,T]\}$ is said to be (strongly) unique if $\{\tilde{X}(s;t,\psi_t,u(\cdot)), s \in [t-r,T]\}$ is also a solution of (3) on $[t-r,T]$ with the control process $u(\cdot)$ and through the same initial datum (t,ψ_t), then

$$P\{X(s;t,\psi_t,u(\cdot)) = \tilde{X}(s;t,\psi_t,u(\cdot)), \forall s \in [t,T]\} = 1.$$

Definition 2. For each $t \in [0,T]$, a 5-tuples $\alpha = (\Omega, \mathscr{F}, P, W(\cdot), u(\cdot))$ is said to be an admissible control if it satisfies the following conditions:

1. $(\Omega, \mathscr{F}, P)$ is a complete probability space.
2. $W(\cdot) = \{W(s), s \in [0,T]\}$ is an m-dimensional standard Brownian motion on $(\Omega, \mathscr{F}, P)$ over $[t,T]$ with $W(t) = 0$ a.s., and $\mathscr{F}(t,s) = \sigma\{W(\tau), t \le \tau \le s\}$ augmented by the P-null sets in $\mathscr{F}$.
3. $u : [t,T] \times \Omega \to U$ is an $\{\mathscr{F}(t,s), s \in [t,T]\}$-adapted process on $(\Omega, \mathscr{F}, P)$ that is right-continuous at the initial time t.
4. Under the control process $u(\cdot) = \{u(s), s \in [t,T]\}$, equation (3) admits a unique strong solution $X^{t,\psi,u(\cdot)}(\cdot) = \{X(s;t,\psi,u(\cdot)), s \in [t,T]\}$ on $(\Omega$, $\mathscr{F}$, P; $\{\mathscr{F}(t,s)$, $s \in [t,T]\})$ through each initial datum $(t,\psi) \in [0,T] \times \mathbf{C}$.
5. The control process $u(\cdot)$ is such that

$$\mathbb{E}\Big[\int_t^T |L(s, X_s(t,\psi,u(\cdot)), u(s))|ds + |\Psi(X_T(t,\psi,u(\cdot)))|\Big] < \infty,$$

   where $L : [0,T] \times \mathbf{C} \times U \to \Re$ and $\Psi : \mathbf{C} \to \Re$ represent the running and terminal cost functions, respectively.

The collection of *admissible controls* $\alpha = (\Omega, \mathscr{F}, P, W(\cdot), u(\cdot))$ over the interval $[t,T]$ shall be denoted by $\mathscr{U}[t,T]$.

We shall write $u(\cdot) \in \mathscr{U}[t,T]$ or $\alpha = (\Omega, \mathscr{F}, P, W(\cdot), u(\cdot)) \in \mathscr{U}[t,T]$ interchangeably, whenever there is no danger of ambiguity.

Throughout the end, we assume that the functions $f : [0,T] \times \mathbf{C} \times U \to \Re^n$, and $g : [0,T] \times \mathbf{C} \times U \to \Re^{n \times m}$ satisfy the following conditions. (See Mohammed [20, 21].) The functions f and g are continuous and they satisfy the following linear growth and Lipschitz conditions.

Assumption 2.
There exists a constant $\Lambda > 0$ *such that*

$$|f(t,\varphi,u) - f(t,\phi,u)| + |g(t,\varphi,u) - g(t,\phi,u)| \le \Lambda \|\varphi - \phi\|,$$
$$\forall (t,\varphi,u), (t,\phi,u) \in [0,T] \times \mathbf{C} \times U.$$

Assumption 3.
There exists a constant $K > 0$ such that

$$|f(t,\phi,u)| + |g(t,\phi,u)| \le K(1+\|\phi\|), \qquad \forall (t,\phi,u) \in [0,T]\times \mathbf{C} \times U.$$

Given an admissible control $u(\cdot) \in \mathscr{U}[t,T]$, let $X^{t,\psi,u(\cdot)}(\cdot) = \{X(s;t,\psi,u(\cdot)), s \in [t,T]\}$ be the solution of (3) through the initial datum $(t,\psi) \in [0,T]\times\mathbf{C}$. We again consider the corresponding $\mathbf{C}$-valued process $\{X_s(t,\psi,u(\cdot)), s \in [t,T]\}$ defined by

$$X_s(\theta;t,\psi,u(\cdot)) = X(s+\theta;t,\psi,u(\cdot)), \quad \theta \in \mathbb{J}. \tag{5}$$

For notational simplicity, we often write $X(s) = X(s;t,\psi,u(\cdot))$ and $X_s = X_s(t,\psi,u(\cdot))$ for $s \in [t,T]$ whenever there is no danger of ambiguity.

It can be shown under Assumptions 2–3 that the $\mathbf{C}$-valued process $\{X_s(t,\ \psi,\ u(\cdot)),\ s \in [t,T]\}$ is a Markov process (see Mohammed [20], [21]).

Let L and Ψ be two continuous real-valued functions on $[0,T]\times\mathbf{C}\times U$ and $[0,T]\times\mathbf{C}$, respectively. Moreover, we assume that they both have at most polynomial growth in $L^2(\mathbb{J};\Re)$. In other words, there exist constants Λ, k such that

$$|L(t,\phi,u)| \le \Lambda(1+\|\phi\|_2)^k, \qquad \text{and} \qquad |\Psi(t,\phi)| \le \Lambda(1+\|\phi\|_2)^k,$$

for all $(t,\phi,u) \in [0,T]\times\mathbf{C}\times U$, for some positive integer k .

Given any initial data $(t,\psi) \in [0,T]\times\mathbf{C}$ and any admissible control $u(\cdot) \in \mathscr{U}[t,T]$, we define the objective function

$$J(t,\psi;u(\cdot)) \equiv \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} L(s,X_s(t,\psi,u(\cdot)),u(s))ds + e^{-\rho(T-t)}\Psi(X_T(t,\psi,u(\cdot)))\right], \tag{6}$$

where $\rho > 0$ denotes a discount factor. For each initial datum $(t,\psi) \in [0,T]\times\mathbf{C}$, the optimal control problem is to find $u(\cdot) \in \mathscr{U}[t,T]$ so as to maximize the objective function J. In this case, the value function $V : [0,T]\times\mathbf{C} \to \Re$ is defined to be

$$V(t,\psi) = \sup_{u(\cdot)\in\mathscr{U}[t,T]} J(t,\psi;u(\cdot)). \tag{7}$$

3 The HJB Equation

Let $\mathbf{C}^*$ and $\mathbf{C}^\dagger$ be the space of bounded linear functionals $\Phi : \mathbf{C} \to \Re$ and bounded bilinear functionals $\tilde{\Phi} : \mathbf{C}\times\mathbf{C} \to \Re$, of the space $\mathbf{C}$, respectively. They are equipped with the operator norms which will be, respectively, denoted by $\|\cdot\|^*$ and $\|\cdot\|^\dagger$.

Let $\mathbf{B} = \{v\mathbf{1}_{\{0\}}, v \in \Re^n\}$, where $\mathbf{1}_{\{0\}} : [-r,0] \to \Re$ is defined by

$$\mathbf{1}_{\{0\}}(\theta) = \begin{cases} 0 & \text{for } \theta \in [-r,0), \\ 1 & \text{for } \theta = 0. \end{cases}$$

We form the direct sum

$$\mathbf{C}\oplus\mathbf{B} = \{\phi + v\mathbf{1}_{\{0\}} \mid \phi\in\mathbf{C},\ v\in\Re^n\}$$

and equip it with the norm $\|\cdot\|$ defined by

$$\|\phi + v\mathbf{1}_{\{0\}}\| = \sup_{\theta\in[-r,0]} |\phi(\theta)| + |v|, \quad \phi\in\mathbf{C},\ v\in\Re^n.$$

Note that for each sufficiently smooth function $\Phi:\mathbf{C}\to\Re$, its first order Fréchet derivative (with respect to $\phi\in\mathbf{C}$), $D\Phi(\varphi)\in\mathbf{C}^*$, has a unique and continuous linear extension $\overline{D\Phi(\varphi)}\in(\mathbf{C}\oplus\mathbf{B})^*$. Similarly, its second order Fréchet derivative, $D^2\Phi(\varphi)\in\mathbf{C}^\dagger$, has a unique and continuous linear extension $\overline{D^2\Phi(\varphi)}\in(\mathbf{C}\oplus\mathbf{B})^\dagger$. In above, $(\mathbf{C}\oplus\mathbf{B})^*$ and $(\mathbf{C}\oplus\mathbf{B})^\dagger$ are spaces of bounded linear and bilinear functionals of $\mathbf{C}\oplus\mathbf{B}$, respectively. (See Lemma (3.1) and Lemma (3.2) on pp 79-83 of Mohammed [20] for details).

For a Borel measurable function $\Phi:\mathbf{C}\to\Re$, we also define

$$\mathscr{S}(\Phi)(\phi) = \lim_{h\to 0+}\frac{1}{h}\left[\Phi(\tilde{\phi}_h) - \Phi(\phi)\right] \tag{8}$$

for all $\phi\in\mathbf{C}$, where $\tilde{\phi}:[-r,T]\to\Re^n$ is an extension of ϕ defined by

$$\tilde{\phi}(t) = \begin{cases}\phi(t) & \text{if } t\in[-r,0)\\ \phi(0) & \text{if } t\geq 0,\end{cases}$$

and again $\tilde{\phi}_t\in\mathbf{C}$ is defined by

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t+\theta), \quad \theta\in[-r,0].$$

Let $\mathscr{D}(\mathscr{S})$, the domain of the operator $\mathscr{S}$, be the set of $\Phi:\mathbf{C}\to\Re$ such that the above limit exists for each $\phi\in\mathbf{C}$.

Throughout the end, let $C^{1,2}_{lip}([0,T]\times\mathbf{C})$ be the space of functions $\Phi:[0,T]\times\mathbf{C}\to\Re$ such that $\frac{\partial\Phi}{\partial t}:[0,T]\times\mathbf{C}\to\Re$ and $D^2\Phi:[0,T]\times\mathbf{C}\to\mathbf{C}^\dagger$ exist and are continuous and satisfy the following Lipschitz condition:

$$\|D^2\Phi(t,\phi) - D^2\Phi(t,\varphi)\|^\dagger \leq K\|\phi-\varphi\| \quad \forall t\in[0,T],\ \phi,\varphi\in\mathbf{C}.$$

We can derive the HJB equation for V, which is given in the following theorem:

Theorem 4. *Suppose V is the value function defined by (7) and satisfies $V\in C^{1,2}_{lip}([0,T]\times\mathbf{C})\cap\mathscr{D}(\mathscr{S})$. Then the value function V satisfies the following HJB equation:*

$$\rho V(t,\psi) - \frac{\partial V}{\partial t}(t,\psi) - \max_{v\in U}[\mathscr{A}^v V(t,\psi) + L(t,\psi,v)] = 0 \tag{9}$$

on $[0,T]\times\mathbf{C}$, and $V(T,\psi)=\Psi(\psi)$, $\forall\psi\in\mathbf{C}$, where $\mathscr{A}$ is defined by

$$\begin{aligned}\mathscr{A}^v V(t,\psi) \equiv{}& \mathscr{S}(V)(t,\psi) + \overline{DV(t,\psi)}(f(t,\psi,v)\mathbf{1}_{\{0\}})\\ &+\frac{1}{2}\sum_{i=1}^m \overline{D^2V(t,\psi)}(g(t,\psi,v)\mathbf{e}_i\mathbf{1}_{\{0\}}, g(t,\psi,v)\mathbf{e}_i\mathbf{1}_{\{0\}}).\end{aligned} \tag{10}$$

For details about how to derive the HJB equation, please refer to [4].

Note that it is not known that the value function V satisfies the necessary smoothness condition $V \in C^{1,2}_{lip}([0,T]\times\mathbf{C})\cap\mathscr{D}(\mathscr{S})$. Therefore, in general we need to consider viscosity solution instead of a classical solution for HJB equation (9). In fact, it will be shown that the value function is a unique viscosity solution of the HJB equation (9). These results shall be given in the next section.

4 Viscosity Solution of the HJB Equation

In this section, we shall show that the value function V defined by (7) is actually a viscosity solution of the HJB equation (9). First, let us define the viscosity solution of (9) as follows.

Definition 3. Let $w \in C([0,T]\times\mathbf{C})$. We say that w is a viscosity subsolution of (9) if, for every $\Gamma \in C^{1,2}_{lip}([0,T]\times\mathbf{C})\cap\mathscr{D}(\mathscr{S})$, for $(t,\psi)\in[0,T]\times\mathbf{C}$ satisfying $\Gamma \geq w$ on $[0,T]\times\mathbf{C}$ and $\Gamma(t,\psi)=w(t,\psi)$, we have

$$\rho\Gamma(t,\psi)-\frac{\partial\Gamma}{\partial t}(t,\psi)-\max_{v\in U}[\mathscr{A}^v\Gamma(t,\psi)+L(t,\psi,v)]\leq 0.$$

We say that w is a viscosity super solution of (9) if, for every $\Gamma \in C^{1,2}_{lip}([0,T]\times\mathbf{C})\cap\mathscr{D}(\mathscr{S})$, and for $(t,\psi)\in[0,T]\times\mathbf{C}$ satisfying $\Gamma \leq w$ on $[0,T]\times\mathbf{C}$ and $\Gamma(t,\psi)=w(t,\psi)$, we have

$$\rho\Gamma(t,\psi)-\frac{\partial\Gamma}{\partial t}(t,\psi)-\max_{v\in U}[\mathscr{A}^v\Gamma(t,\psi)+L(t,\psi,v)]\geq 0.$$

We say that w is a viscosity solution of (9) if it is both a viscosity supersolution and a viscosity subsolution of (9).

For our value function V defined by (7), we have the following results:

Theorem 5. *The value function V is a viscosity solution of the HJB equation*

$$\rho V(t,\psi)-\frac{\partial V}{\partial t}(t,\psi)-\max_{v\in U}[\mathscr{A}^v V(t,\psi)+L(t,\psi,v)]=0 \tag{11}$$

on $[0,T]\times\mathbf{C}$*, and* $V(T,\psi)=\Psi(\psi)$, $\forall\psi\in\mathbf{C}$.

Since a viscosity solution is both a subsolution and a supersolution, the uniqueness result will follow immediately after establishing the following comparison principle:

Theorem 6 (Comparison Principle)**.** *Assume that $V_1(t,\psi)$ and $V_2(t,\psi)$ are both continuous with respect to the argument (t,ψ) and are respectively viscosity subsolution and supersolution of (9) with at most a polynomial growth. In other terms, there exists a real number $\Lambda>0$ and a positive integer $k>0$ such that,*

$$|V_i(t,\psi)|\leq\Lambda(1+\|\psi\|_2)^k,\quad \textit{for } (t,\psi)\in[0,T]\times\mathbf{C},\ i=1,2.$$

Then

$$V_1(t,\psi)\leq V_2(t,\psi)\quad \textit{for all } (t,\psi)\in[0,T]\times\mathbf{C}. \tag{12}$$

The proofs for Theorem 5 and Theorem 6 can be found in [4], so we omit them here.

5 A Finite Difference Scheme

In this section, we consider an explicit finite difference scheme and show that it converges to the unique viscosity solution of equation (9). We will use a method introduced by Barles and Souganidis [1]. Given a positive integer M, we consider the following truncated optimal control problem with value function $V_M : [0,T]\times\mathbf{C}\to\mathbf{R}$

$$V_M(t,\psi) = \sup_{u(\cdot)\in\mathscr{U}[t,T]} \mathbb{E}\bigg[\int_t^T e^{-\rho(s-t)}(L(s,X_s,u(s))\wedge M)ds \\ + e^{-\rho(T-t)}(\Psi(X_T)\wedge M)\bigg], \quad (13)$$

where $a\wedge b$ is defined by $a\wedge b=\min\{a,b\}$ for all $a,b\in\mathbf{R}$. It is easy to see that $V_M\to V$ as $M\to\infty$. In view of these, we need only find the numerical solution for V_M.

Similar to the proof of Theorem 5 (see [4]), it can be shown that the value function V_M is the unique viscosity solution of the corresponding HJB equation

$$\rho V_M(t,\psi) - \frac{\partial V_M}{\partial t}(t,\psi) - \max_{u\in U}[\mathscr{A}^u V_M(t,\psi) + L(t,\psi,u)\wedge M] = 0 \quad (14)$$

on $[0,T]\times\mathbf{C}$, and $V(T,\psi)=\Psi(\psi)\wedge M$, $\forall\psi\in\mathbf{C}$.

Let ε with $0<\varepsilon<1$ be the stepsize for variable ψ and η with $0<\eta<1$ be the stepsize for t. We consider the finite difference operators Δ_ε, Δ_η and Δ_η^2 defined by

$$\begin{aligned}
\Delta_\eta W(t,\psi) &= \frac{W(t+\eta,\psi)-W(t,\psi)}{\eta},\\
\Delta_\varepsilon W(t,\psi)(h+v\mathbf{1}_{\{0\}}) &= \frac{W(t,\psi+\varepsilon(h+v\mathbf{1}_{\{0\}}))-W(t,\psi)}{\varepsilon},\\
\Delta_\varepsilon^2 W(t,\psi)(h+v\mathbf{1}_{\{0\}},k+w\mathbf{1}_{\{0\}}) &= \frac{W(t,\psi+\varepsilon(h+v\mathbf{1}_{\{0\}}))-W(t,\psi)}{\varepsilon^2}\\
&\quad+\frac{W(t,\psi-\varepsilon(k+w\mathbf{1}_{\{0\}}))-W(t,\psi)}{\varepsilon^2}.
\end{aligned}$$

where $h,k\in\mathbf{C}$ and $v,w\in\mathbf{R}^n$. Recall that,

$$\mathscr{S}(\Phi)(\phi) = \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\left[\Phi(\tilde{\phi}_\varepsilon)-\Phi(\phi)\right].$$

Therefore we define,

$$\mathscr{S}_\varepsilon(\Phi)(\phi) = \frac{1}{\varepsilon}\left[\Phi(\tilde{\phi}_\varepsilon)-\Phi(\phi)\right].$$

It is clear that $\mathscr{S}_\varepsilon(\Phi)$ is an approximation of $\mathscr{S}(\Phi)$.

We have the following lemma:

Lemma 7. *For any* $W : [0,T] \times \mathbf{C} \to \mathbf{R}$, $W \in \mathscr{C}^{1,2}([0,T] \times \mathbf{C})$ *such that* W *can be smoothly extended on* $[0,T] \times (\mathbf{C} \oplus \mathbf{B})$, *we have*

$$\lim_{\varepsilon \to 0} \Delta_\varepsilon W(t,\psi)(h + v\mathbf{1}_{\{0\}}) = \overline{DW(t,\psi)}(h + v\mathbf{1}_{\{0\}}), \tag{15}$$

and

$$\lim_{\varepsilon \to 0} \Delta_\varepsilon^2 W(t,\psi)(h + v\mathbf{1}_{\{0\}}) = \overline{D^2W(t,\psi)}(h + v\mathbf{1}_{\{0\}}, k + w\mathbf{1}_{\{0\}}). \tag{16}$$

The proof can be found in [5].

Let $\varepsilon, \eta > 0$. The corresponding discrete version of equation (14) is given by

$$\begin{aligned} \rho V_M(t,\psi) = {} & \frac{1}{\varepsilon}\left[V_M(t,\tilde{\psi}_\varepsilon) - V_M(t,\psi)\right] + \frac{V_M(t+\eta,\psi) - V_M(t,\psi)}{\eta} \\ & + \max_{u \in U}\left[\frac{V_M(t,\psi + \varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}})) - V_M(t,\psi)}{\varepsilon}\right. \\ & + \frac{1}{2}\sum_{i=1}^{m}\left(\frac{V_M(t,\psi + \varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}})) - V_M(t,\psi)}{\varepsilon^2}\right. \\ & \left.\left. + \frac{V_M(t,\psi - \varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}})) - V_M(t,\psi)}{\varepsilon^2}\right) + L(t,\psi,u) \wedge M\right]. \end{aligned} \tag{17}$$

Let $C_b([0,T] \times (\mathbf{C} \oplus \mathbf{B}))$ denote the space of bounded continuous functions W from $[0,T] \times (\mathbf{C} \oplus \mathbf{B})$ to $\mathbf{R}$. Define a mapping $\mathscr{H}_M : (0,1)^2 \times [0,T] \times \mathbf{C} \times \mathbf{R}^n \times C_b([0,T] \times (\mathbf{C} \oplus \mathbf{B})) \to \mathbf{R}$ as the following

$$\begin{aligned} \mathscr{H}_M(\varepsilon,\eta,t,\psi,x,W) \equiv {} & \varepsilon \max_{u \in U}\left[\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon) + \frac{W(t+\eta,\psi)}{\eta}\right. \\ & + \frac{W(t,\psi + \varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}}))}{\varepsilon} + L(t,\psi,u) \wedge M \\ & \left. + \frac{1}{2}\sum_{i=1}^{m}\frac{W(t,\psi + \varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}})) + W(t,\psi - \varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2}\right] \\ & - \varepsilon\left(\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{m}{\varepsilon^2} + \rho\right)x. \end{aligned} \tag{18}$$

Then, (17) is equivalent to $\mathscr{H}_M(\varepsilon,\eta,t,\psi,V_M(t,\psi),V_M) = 0$. Moreover, note that the coefficient of x in $\mathscr{H}_M$ is negative. This implies that $\mathscr{H}_M$ is monotone, i.e., for all $x_1, x_2 \in \mathbf{R}^n$, $\varepsilon, \eta \in (0,1)$, $t \in [0,T]$, $\psi \in \mathbf{C}$, and $W \in C_b([0,T] \times (\mathbf{C} \oplus \mathbf{B}))$

$$\mathscr{H}_M(\varepsilon,\eta,t,\psi,x_1,W) \le \mathscr{H}_M(\varepsilon,\eta,t,\psi,x_2,W) \text{ whenever } x_1 \ge x_2,$$

where $x_1 \ge x_2$ denotes componentwise inequality.

We have the following result (see [5] for the proof):

Lemma 8. *For every $t \in [0,T]$, $\psi \in \mathbf{C} \oplus \mathbf{B}$, and for every test function $W(\cdot,\cdot)$ defined on $[0,T] \times (\mathbf{C} \oplus \mathbf{B})$ such that $W \in C_b^{1,2}([0,T] \times (\mathbf{C} \oplus \mathbf{B})) \cap \mathscr{D}(\mathscr{S})$, we have*

$$\rho W(t,\psi) - \frac{\partial W}{\partial t}(t,\psi) - \max_{u \in U}[\mathscr{A}^u W(t,\psi) + L(t,\psi,u) \wedge M] = \lim_{(\tau,\phi)\to(t,\psi),\,\varepsilon,\eta\downarrow 0,\,\xi\to 0} \frac{\mathscr{H}_M(\varepsilon,\eta,\tau,\phi,W(\tau,\phi)+\xi,W+\xi)}{\varepsilon}. \quad (19)$$

In other words, the scheme $\mathscr{H}_M$ is consistent.

We define an operator $\mathscr{T}_{\varepsilon,\eta}$ on $C_b([0,T] \times (\mathbf{C} \oplus \mathbf{B}))$ as follows,

$$\mathscr{T}_{\varepsilon,\eta} W(t,\psi) \equiv \max_{u \in U} \left[\frac{1}{\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{m}{\varepsilon^2} + \rho} \left(\frac{1}{\varepsilon} W(t,\tilde{\psi}_\varepsilon) + \frac{W(t,\psi + \varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}}))}{\varepsilon} + \frac{1}{2}\sum_{i=1}^m \frac{W(t,\psi+\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}})) + W(t,\psi-\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2} + \frac{W(t+\eta,\psi)}{\eta} + L(t,\psi,u)\wedge M \right) \right]. \quad (20)$$

Then it is easy to verify that $\mathscr{H}_M(\varepsilon,\eta,t,\psi,W(t,\psi),W) = 0$ is equivalent to the equation

$$W(t,\psi) = \mathscr{T}_{\varepsilon,\eta} W(t,\psi). \quad (21)$$

It is not very hard to prove the following result (see [5] for details):

Lemma 9. *For each ε and η, $\mathscr{T}_{\varepsilon,\eta}$ is a contraction map.*

By virtue of the Banach fixed point theorem, the strict contraction $\mathscr{T}_{\varepsilon,\eta}$ has a unique fixed point that we denote by $W^M_{\varepsilon,\eta}$, which is also a unique solution of

$$\mathscr{H}_M(\varepsilon,\eta,t,\psi,W(t,\psi),W) = 0.$$

In addition, we have the following stable result (see [5])

Lemma 10. *The scheme $\mathscr{H}_M$ is* **stable**, *that is, for every $\varepsilon,\eta \in (0,1)$, there exists a bounded solution $W_{\varepsilon,\eta} \in C_b([0,T] \times (\mathbf{C} \oplus \mathbf{B}))$ to the equation*

$$\mathscr{H}_M(\varepsilon,\eta,t,\psi,W(t,\psi),W) = 0, \quad (22)$$

with the bound independent of ε, and η.

Given the above results, we can obtain the convergence result:

Theorem 11. *Let $W^M_{\varepsilon,\eta}$ denote the solution to* (22). *Then, as $(\varepsilon,\eta) \to 0$, the sequence $W^M_{\varepsilon,\eta}$ converges uniformly on $[0,T] \times \mathbf{C}$ to the unique viscosity solution V_M of* (14).

The proof can be found in [5].

6 The Computational Algorithm

Based on the results obtained in the last section, we can construct the computational algorithm to obtain a numerical solution. For example, one algorithm can be like the following:

Step 0. Choose any function $W^{(0)} \in C_b([0,T] \times \mathbf{C} \oplus \mathbf{B})$;

Step 1. Pick the starting values for $\varepsilon(1), \eta(1)$. For example, we can choose $\varepsilon(1) = 10^{-2}, \eta(1) = 10^{-3}$;

Step 2. For the given $\varepsilon, \eta > 0$, compute the function

$$W^{(1)}_{\varepsilon(1),\eta(1)} \in C_b([0,T] \times \mathbf{C} \oplus \mathbf{B})$$

by the following formula

$$W^{(1)}_{\varepsilon(1),\eta(1)} = \mathscr{T}_{\varepsilon(1),\eta(1)} W^{(0)},$$

where $\mathscr{T}_{\varepsilon(1),\eta(1)}$, which is defined on $C_b([0,T] \times \mathbf{C} \oplus \mathbf{B})$, is given by (20);

Step 3. Repeat Step 2 for $i = 2, 3, \cdots$ using

$$W^{(i)}_{\varepsilon(1),\eta(1)} = \mathscr{T}_{\varepsilon(1),\eta(1)} W^{(i-1)}_{\varepsilon(1),\eta(1)},$$

where $\mathscr{T}_{\varepsilon(1),\eta(1)}$, which is defined on $C_b([0,T] \times \mathbf{C} \oplus \mathbf{B})$, is given by (20). Stop the iteration when

$$\|W^{i+1}_{\varepsilon(1),\eta(1)}(t,\psi) - W^{i}_{\varepsilon(1),\eta(1)}(t,\psi)\| \le \delta_1,$$

where δ_1 is a preselected number which is small enough to achieve the accuracy we want. Denote the final solution by $W_{\varepsilon(1),\eta(1)}(t,\psi)$;

Step 4. Choose two sequences of $\varepsilon(k)$ and $\eta(k)$, such that

$$\lim_{k\to\infty} \varepsilon(k) = \lim_{k\to\infty} \eta(k) = 0.$$

For example, we may choose $\varepsilon(k) = \eta(k) = 10^{-(2+k)}$. Now repeat Step 2 and Step 3 for each $\varepsilon(k), \eta(k)$ until

$$\|W_{\varepsilon(k+1),\eta(k+1)}(t,\psi) - W_{\varepsilon(k),\eta(k)}(t,\psi)\| \le \delta_2,$$

where δ_2 is chosen to obtain the expected accuracy.

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