Polynomial-time Perfect Sampler for Closed Jackson Networks with Single Servers

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Abstract *In this paper, we propose a sampler for the product-form solution of basic queueing networks, closed Jackson networks with single servers. Our approach is sampling via Markov chain, but it is NOT a simulation of behavior of customers in queueing networks. We propose a new ergodic Markov chain whose unique stationary distribution is the product form solution of a closed Jackson Network, thus we can sample from the product form solution without knowing the value of its normalizing constant. The sampler is based on monotone Coupling from the Past (monotone CFTP) and realizes the sampling from the target distribution exactly. We show that the chain is monotone and rapidly mixing, and that the expected running time of the sampling algorithm is* $\mathrm{O}(n^3 \ln(Kn))$ *where* n *is the number of nodes in the network and* K *is the number of customers.*

1 Introduction

The Jackson network is proposed by Jackson in 1957 [14], and is one of the basic and significant models in queueing network theory. The Jackson network consists of some nodes, each of which has one or more servers. In the network, a customer receives a service by a server on a node according to the exponentially distributed service time, moves stochastically to a next node after the service, and waits one's turn in a line on first-come-first-served (FCFS) basis. It is well known that the steady-state distribution of customers in a Jackson network is a product form [14, 12, 11].

We say a network is closed if no customers leave or enter the network. By computing the normalizing constant of the product form solution of a given closed queueing network, we can obtain significant evaluated value like as throughput,

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rates of utilization of stations, and so on [11]. There is well-known Buzen's algorithm [5], which computes the normalizing constant of closed queueing networks. However the running time of Buzen's algorithm is pseudo-polynomial time that depends on the number of customers in a closed network.

One of hopeful efficient approximations is randomized algorithm. In particular, MCMC (Markov chain Monte Carlo) is useful and practically used for computing a normalizing constant of a distribution. It is important to discuss the convergence speed of Markov chain for an efficient algorithm based on MCMC. Chen and O'Cinneide [7] proposed a randomized algorithm based on MCMC, but their algorithm is weakly polynomial-time in some very special cases. Ozawa [22] proposed a perfect sampler for closed Jackson networks with single servers, however his chain mixes in pseudo-polynomial-time.

In many practical situations, each node of a network has a single server. In this paper, we are concerned with a closed Jackson network with single servers. We propose a Markov chain and show that the chain is *monotone* and rapidly $\mathrm{O}(n^3 \ln(Kn))$ mixing, where n is the number of nodes and K is the number of customers in a closed Jackson network. Here we note that the chain is not a simulation of customer's move in a queueing network, but just have a unique stationary distribution which is the same as the product form solution for a given network.

An ordinary sampling via Markov chain is an approximate sampler, whereas Propp and Wilson devised monotone CFTP (Coupling from the Past) algorithm which realizes a perfect (exact) sampling from stationary distribution in probabilistically finite time by ingeniously simulating the chain [23]. Thus our chain provides an efficient perfect sampler based on monotone CFTP. One of the great advantages of perfect sampling is that we never need to determine the error rate ε when to use. Another is that a perfect sampler becomes faster than any approximate sampler based on a Markov chain when we need a sample according to highly accurate distribution.

There are two benefits at least, if we have a fast sampler. One is that we may design a fast randomized algorithms for computing normalizing constant, and so for throughput. Actually, we can design a polynomial-time randomized approximation scheme, though we will not deal with it in this paper. The other is that a fast sampler finds a state with respect to the steady-state distribution of networks, thus we can use it as an initial state of a simulation of behavior of customers.

2 Preliminaries

2.1 Product-form Solution

We denote the set of real numbers (non-negative, positive real numbers) by $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$), and the set of integers (non-negative, positive integers) by $\mathbb{Z}$ ($\mathbb{Z}_+$, $\mathbb{Z}_{++}$), respectively. A closed Jackson network is a queueing network model satisfying the followings;

(i) The network has $n \in \mathbb{Z}_{++}$ nodes. Each node contains exactly one server, thus

at most one customer can receive a service on a node at a time.
(ii) In each node, customers are served one by one on first-come-first-served (FCFS) basis. The servicing time on node $i \in \{1, \dots, n\}$ is exponentially distributed with mean $1/\mu_i$.
(iii) Once served in node $i \in \{1, \dots, n\}$, a customer goes to node $j \in \{1, \dots, n\}$ with probability $W_{ij} \in \mathrm{R}_+$. We assume that the matrix $W = (W_{ij})$ of transition probability of customers is irreducible and aperiodic, so ergodic.
(iv) No customers leave or enter the network. Thus, there are always $K \in \mathbb{Z}_{++}$ customers in the network.

In queueing network theory, it is well known that a closed Jackson network has a *product form solution* as a steady state distribution of customers in a network. Let us consider the set of non-negative integer points

$$\Xi \stackrel{\text{def.}}{=} \left\{ \boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}_+^n \mid \textstyle\sum_{i=1}^n x_i = K \right\},$$

in an $n-1$ dimensional simplex. Clearly, a state of customers in the network with K customers is represented by $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \Xi$. Since matrix W of the transition probability of customers is ergodic, 1 is an eigenvalue and corresponding eigenvector is unique, excluding constant factor. Let $\theta \in \mathbb{R}_{++}^n$ be an eigenvector for W with corresponding to the eigenvalue 1, i.e., $\theta W = \theta$. The steady-state distribution $J : \Xi \to \mathbb{R}_{++}$ for the closed Jackson network is product form defined by

$$J(\boldsymbol{x}) = \frac{1}{G} \prod_{i=1}^{n} \alpha_i^{x_i} \left(\equiv \frac{1}{G} \prod_{i=1}^{n} \left(\frac{\theta_i}{\mu_i} \right)^{x_i} \right) \tag{1}$$

where $\alpha_i \stackrel{\text{def.}}{=} \theta_i/\mu_i$ and $G \stackrel{\text{def.}}{=} \sum_{\boldsymbol{x} \in \Xi} \prod_{i=1}^n \alpha_i^{x_i}$ is the normalizing constant [14].

2.2 Monotone Coupling from the Past

Here we review CFTP briefly. Suppose that we have an ergodic Markov chain $\mathcal{M}$ with a finite state space Ω and a transition matrix P. The transition rule of the Markov chain $X \mapsto X'$ can be described by a deterministic function $\phi : \Omega \times [0,1) \to \Omega$, called *update function*, as follows. Given a random number Λ uniformly distributed over $[0,1)$, update function ϕ satisfies that $\Pr(\phi(x, \Lambda) = y) = P(x,y)$ for any $x, y \in \Omega$. We can realize the Markov chain by setting $X' = \phi(X, \Lambda)$. Clearly, update functions corresponding to the given transition matrix P are not unique. The result of transitions of the chain from the time t_1 to t_2 ($t_1 < t_2$) with a sequence of random numbers $\boldsymbol{\lambda} = (\lambda[t_1], \lambda[t_1+1], \dots, \lambda[t_2-1]) \in [0,1)^{t_2-t_1}$ is denoted by $\Phi_{t_1}^{t_2}(x, \boldsymbol{\lambda}) : \Omega \times [0,1)^{t_2-t_1} \to \Omega$ where $\Phi_{t_1}^{t_2}(x, \boldsymbol{\lambda}) \stackrel{\text{def.}}{=} \phi(\phi(\cdots(\phi(x, \lambda[t_1]), \dots, \lambda[t_2 - 2]), \lambda[t_2-1])$. We say that a sequence $\boldsymbol{\lambda} \in [0,1)^{|T|}$ satisfies the *coalescence condition*, when $\exists y \in \Omega, \forall x \in \Omega, y = \Phi_T^0(x, \boldsymbol{\lambda})$.

Suppose that there exists a partial order "$\succeq$" on the set of states Ω. A transition rule expressed by a deterministic update function ϕ is called *monotone* (with respect to "$\succeq$") if $\forall \lambda \in [0,1), \forall x, \forall y \in \Omega, x \succeq y \Rightarrow \phi(x, \lambda) \succeq \phi(y, \lambda)$. We also say that a chain is *monotone* if the chain has a *monotone* update function. Here we suppose that there exists a unique pair of states $(x_{\max}, x_{\min})$ in partially ordered set $(\Omega, \succeq)$, satisfying $x_{\max} \succeq x \succeq x_{\min}$, $\forall x \in \Omega$.

With these preparations, a standard monotone Coupling From The Past algorithm is expressed as follows.

Algorithm 1 (Monotone CFTP Algorithm [23])

Step 1. Set the starting time period $T := -1$ to go back, and set $\boldsymbol{\lambda}$ be the empty sequence.

Step 2. Generate random real numbers $\lambda[T], \lambda[T+1], \ldots, \lambda[\lceil T/2 \rceil - 1] \in [0, 1)$, and insert them to the head of $\boldsymbol{\lambda}$ in order, i.e., put $\boldsymbol{\lambda} := (\lambda[T], \lambda[T+1], \ldots, \lambda[-1])$.

Step 3. Start two chains from $x_{\max}$ and $x_{\min}$, respectively, at time period T, and run each chain to time period 0 according to the update function ϕ with the sequence of numbers in $\boldsymbol{\lambda}$. (Here we note that every chain uses the common sequence $\boldsymbol{\lambda}$.)

Step 4. [Coalescence check] The state obtained at time period 0 is denoted by $\Phi_T^0(x, \boldsymbol{\lambda})$.

(a) If $\exists y \in \Omega$, $y = \Phi_T^0(x_{\max}, \boldsymbol{\lambda}) = \Phi_T^0(x_{\min}, \boldsymbol{\lambda})$, then return y.

(b) Else, update the starting time period $T := 2T$, and go to Step 2.

Theorem 2.1 (Monotone CFTP Theorem [23]) *Suppose that a Markov chain defined by an update function ϕ is monotone with respect to a partially ordered set of states $(\Omega, \succeq)$, and $\exists x_{\max}, \exists x_{\min} \in \Omega$, $\forall x \in \Omega$, $x_{\max} \succeq x \succeq x_{\min}$. Then the monotone CFTP algorithm (Algorithm 1) terminates with probability 1, and obtained value is a realization of a random variable exactly distributed according to the stationary distribution.* ■

Theorem 2.1 gives a (probabilistically) finite time algorithm for infinite time simulation.

2.3 Path Coupling

Given a pair of probability distributions ν_1 and ν_2 on a finite state space Ω, the *total variation distance* between ν_1 and ν_2 is defined by $\mathrm{d_{TV}}(\nu_1, \nu_2) \stackrel{\text{def.}}{=} \frac{1}{2} \sum_{x \in \Omega} |\nu_1(x) - \nu_2(x)|$. The *mixing rate* of an ergodic Markov chain is defined by $\tau \stackrel{\text{def.}}{=} \max_{x \in \Xi} \{\min\{t \mid \forall s \geq t,\ \mathrm{d_{TV}}(\pi, P_x^s) \leq 1/\mathrm{e}\}\}$ where π is the stationary distribution and P_x^s is the probability distribution of the chain at time period $s \geq 0$ with initial state x at time period 0.

The Path Coupling Theorem proposed by Bubbly and Dyer is a useful technique for bounding the mixing rate.

Theorem 2.2 (Path Coupling Theorem [4]) *Let $\mathcal{M}$ be a finite ergodic Markov chain with state space Ω. Let $H = (\Omega, \mathcal{E})$ be a connected undirected graph with vertex set Ω and edge set $\mathcal{E} \subseteq \binom{\Omega}{2}$. Let $l : \mathcal{E} \to \mathbb{R}_{++}$ be a positive length defined* *on the edge set. For any pair of vertices $\{x, y\}$ of H, the distance between x and y, denoted by $d(x, y)$ and/or $d(y,x)$, is the length of a shortest path between x and y, where the length of a path is the sum of the lengths of edges in the path. Suppose that there exists a joint process $(X, Y) \mapsto (X', Y')$ with respect to $\mathcal{M}$ whose marginals are a faithful copy of $\mathcal{M}$ and satisfying*

$$\exists\beta,\ 0 < \beta < 1,\ \forall\{X, Y\} \in \mathcal{E},\ \mathrm{E}[d(X', Y')] \le \beta d(X, Y).$$

Then the mixing rate τ of the Markov chain $\mathcal{M}$ satisfies $\tau \le (1-\beta)^{-1}(1+\ln(D/d))$ where $d \stackrel{\text{def.}}{=} \min\{d(x, y) \mid \forall x, \forall y \in \Omega\}$ and $D \stackrel{\text{def.}}{=} \max\{d(x, y) \mid \forall x, \forall y \in \Omega\}$. ■

The above theorem differs from the original theorem in [4] since the integrality of the edge length is not assumed. We drop the integrality and introduced the minimum distance d. Theorem 2.2 can be proved by a slight modification of the original proof.

3 Perfect Sampler

In the following we consider a closed Jackson network with n nodes, K customers and parameters $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Z}_{++}$, which has the product form solution (1) for any $\boldsymbol{x} \in \Xi$.

3.1 Monotone Markov Chain

Now we propose new Markov chain $\mathcal{M}_{\mathrm{P}}$. with state space Ξ. A transition of $\mathcal{M}_{\mathrm{P}}$ from a current state $X \in \Xi$ to a next state X' is defined as follows. First, we choose a pair of consecutive indices $\{j, j+1\}$ ($j \in \{1, 2, \ldots, n-1\}$) uniformly at random. Next, put $k = X_j + X_{j+1}$, and choose $l \in \{0, 1, \ldots, k\}$ with probability

$$\frac{\alpha_j^l \alpha_{j+1}^{k-l}}{\sum_{s=0}^{k} \alpha_j^s \alpha_{j+1}^{k-s}} \qquad \left(\equiv \frac{\alpha_j^l \alpha_{j+1}^{k-l} \prod_{i \notin \{j,j+1\}} \alpha_i^{X_i}}{\sum_{s=0}^{k} \alpha_j^s \alpha_{j+1}^{k-s} \prod_{i \notin \{j,j+1\}} \alpha_i^{X_i}} \right)$$

and set

$$X'_i = \begin{cases} l & (\text{for } i = j), \\ k - l & (\text{for } i = j+1), \\ X_i & (\text{otherwise}). \end{cases}$$

The Markov chain $\mathcal{M}_{\mathrm{P}}$ is irreducible and aperiodic, so ergodic, hence has a unique stationary distribution. Also, $\mathcal{M}_{\mathrm{P}}$ satisfies the detailed balance equation, thus the stationary distribution is the product form solution $J(\boldsymbol{x})$.

Here, we consider the cumulative distribution function $g_{ij}^k : \{0, 1, \ldots, k\} \to \mathbb{R}_+$ defined by

$$g_{ij}^k(l) \stackrel{\text{def.}}{=} \frac{\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}}{A_{ij}^k} = \begin{cases} \frac{\alpha_i^{l+1} - \alpha_j^{l+1}}{\alpha_i^{k+1} - \alpha_j^{k+1}} \cdot \alpha_j^{k-l} & (\alpha_i \ne \alpha_j), \\ \frac{l}{k+1} & (\alpha_i = \alpha_j), \end{cases}$$

for $l \in \{0, 1, \ldots, k\}$, where $A^k_{ij} \stackrel{\text{def.}}{=} \sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}$ is a normalizing constant. We also define $g^k_{ij}(-1) \stackrel{\text{def.}}{=} 0$, for convenience. For the chain $\mathcal{M}_{\mathrm{P}}$, we define an update function $\phi : \Xi \times [1, n) \to \Xi$ as follows. For a current state $X \in \Xi$, the next state $X' = \phi(X, \lambda) \in \Xi$ with respect to a random number $\lambda \in [1, n)$ is defined by

$$X'_i = \begin{cases} l & (\text{for } i = \lfloor\lambda\rfloor), \\ k - l & (\text{for } i = \lfloor\lambda\rfloor + 1), \\ X_i & (\text{otherwise}), \end{cases}$$

where $k = X_{\lfloor\lambda\rfloor} + X_{\lfloor\lambda\rfloor+1}$ and $l \in \{0, 1, \ldots, k\}$ satisfies

$$g^k_{\lfloor\lambda\rfloor(\lfloor\lambda\rfloor+1)}(l-1) < \lambda - \lfloor\lambda\rfloor \leq g^k_{\lfloor\lambda\rfloor(\lfloor\lambda\rfloor+1)}(l).$$

In the following, we show the monotonicity of $\mathcal{M}_{\mathrm{P}}$. Here we introduce a partial order "$\succeq$" on Ξ. For any state $\boldsymbol{x} \in \Xi$, we introduce *cumulative sum vector* $c_{\boldsymbol{x}} = (c_{\boldsymbol{x}}(0), c_{\boldsymbol{x}}(1), \ldots, c_{\boldsymbol{x}}(n)) \in \mathbb{Z}_+^{n+1}$ defined by

$$c_{\boldsymbol{x}}(i) \stackrel{\text{def.}}{=} \begin{cases} 0 & (\text{for } i = 0), \\ \sum_{j=1}^{i} x_j & (\text{for } i \in \{1, 2, \ldots, n\}). \end{cases}$$

For any pair of states $\boldsymbol{x}, \boldsymbol{y} \in \Xi$, we say $\boldsymbol{x} \succeq \boldsymbol{y}$ if and only if $c_{\boldsymbol{x}} \geq c_{\boldsymbol{y}}$. Next, we define two special states $x_{\max}, x_{\min} \in \Xi$ by $x_{\max} \stackrel{\text{def.}}{=} (K, 0, \cdots, 0)$ and $x_{\min} \stackrel{\text{def.}}{=} (0, \ldots, 0, K)$. Then we can see easily that $\forall \boldsymbol{x} \in \Xi$, $x_{\max} \succeq \boldsymbol{x} \succeq x_{\min}$.

Theorem 3.1 *Markov chain $\mathcal{M}_{\mathrm{P}}$ is monotone on the partially ordered set $(\Xi, \succeq)$, i.e., $\forall \lambda \in [1, n)$, $\forall X, \forall Y \in \Xi$, $X \succeq Y \Rightarrow \phi(X, \lambda) \succeq \phi(Y, \lambda)$.*

Proof: We say that a state $X \in \Xi$ *covers* $Y \in \Xi$ (at j), denoted by $X \cdot\!\succ Y$ (or $X \cdot\!\succ_j Y$), when

$$X_i - Y_i = \begin{cases} +1 & (\text{for } i = j), \\ -1 & (\text{for } i = j + 1), \\ 0 & (\text{otherwise}). \end{cases}$$

We show that if a pair of states $X, Y \in \Xi$ satisfies $X \cdot\!\succ_j Y$, then $\forall \lambda \in [1, n)$, $\phi(X, \lambda) \succeq \phi(Y, \lambda)$. We denote $\phi(X, \lambda)$ by X' and $\phi(Y, \lambda)$ by Y' for simplicity. For any index $i \neq \lfloor\lambda\rfloor$, it is easy to see that $c_X(i) = c_{X'}(i)$ and $c_Y(i) = c_{Y'}(i)$, and so $c_{X'}(i) - c_{Y'}(i) = c_X(i) - c_Y(i) \geq 0$ since $X \succeq Y$. In the following, we show that $c_{X'}(\lfloor\lambda\rfloor) \geq c_{Y'}(\lfloor\lambda\rfloor)$.

Case 1: In case that $\lfloor\lambda\rfloor \neq j - 1$ and $\lfloor\lambda\rfloor \neq j + 1$. If we put $k = X_{\lfloor\lambda\rfloor} + X_{\lfloor\lambda\rfloor+1}$, then it is easy to see that $Y_{\lfloor\lambda\rfloor} + Y_{\lfloor\lambda\rfloor+1} = k$. Accordingly $X'_{\lfloor\lambda\rfloor} = Y'_{\lfloor\lambda\rfloor} = l$ where l satisfies

$$g^k_{\lfloor\lambda\rfloor(\lfloor\lambda\rfloor+1)}(l-1) \leq \lambda - \lfloor\lambda\rfloor < g^k_{\lfloor\lambda\rfloor(\lfloor\lambda\rfloor+1)}(l),$$

hence $c_{X'}(\lfloor\lambda\rfloor) = c_{Y'}(\lfloor\lambda\rfloor)$.

0	$\alpha_i^0\alpha_j^k/A$	$\alpha_i^1\alpha_j^{k-1}/A$	$\cdots$		$\alpha_i^k\alpha_j^0/A$	1
0	$\alpha_i^0\alpha_j^{k+1}/A'$	$\alpha_i^1\alpha_j^k/A'$	$\alpha_i^2\alpha_j^{k-1}/A'$	$\cdots$	$\alpha_i^{k+1}\alpha_j^0/A'$	1

Figure 1: A figure of alternating inequalities for a pair of indices (i, j) and a non-negative integer k. In the figure, $A \stackrel{\text{def.}}{=} \sum_{s=0}^{k} \alpha_i^s\alpha_j^{k-s}$ and $A' \stackrel{\text{def.}}{=} \sum_{s=0}^{k+1} \alpha_i^s\alpha_j^{k+1-s}$ are normalizing constants.

Case 2: Consider the case that $\lfloor\lambda\rfloor = j-1$. Let $k+1 = X_{j-1} + X_j$. Then $Y_{j-1} + Y_j = k$, since $X \cdot\!\succ_j Y$. From the definition of cumulative sum vector,

$$\begin{aligned} & c_{X'}(\lfloor\lambda\rfloor) - c_{Y'}(\lfloor\lambda\rfloor) = c_{X'}(j-1) - c_{Y'}(j-1) \\ = \;& c_{X'}(j-2) + X'_{j-1} - c_{Y'}(j-2) - Y'_{j-1} = c_X(j-2) + X'_{j-1} - c_Y(j-2) - Y'_{j-1} \\ = \;& X'_{j-1} - Y'_{j-1}. \end{aligned}$$

Thus, it is enough to show that $X'_{j-1} \geq Y'_{j-1}$. Now suppose that $l \in \{0, 1, \ldots, k\}$ satisfies $g^k_{(j-1)j}(l-1) \leq \lambda - \lfloor\lambda\rfloor < g^k_{(j-1)j}(l)$ for λ. Then $g^{k+1}_{(j-1)j}(l-1) \leq \lambda - \lfloor\lambda\rfloor < g^{k+1}_{(j-1)j}(l+1)$, since the *alternating inequalities* $g^{k+1}_{(j-1)j}(l-1) \leq g^k_{(j-1)j}(l-1) < g^{k+1}_{(j-1)j}(l) \leq g^{k+1}_{(j-1)j}(l+1)$, which we will show in the next, hold. Thus we have that if $Y'_{j-1} = l$ then $X'_{j-1} = l$ or $l+1$. In other words,

$$\begin{pmatrix} X'_{j-1} \\ Y'_{j-1} \end{pmatrix} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} k \\ k \end{pmatrix}, \begin{pmatrix} k+1 \\ k \end{pmatrix} \right\}$$

and $X'_{j-1} \geq Y'_{j-1}$ in all cases. Accordingly, we have that $c_{X'}(\lfloor\lambda\rfloor) \geq c_{Y'}(\lfloor\lambda\rfloor)$.

Case 3: Consider the case that $\lfloor\lambda\rfloor = j+1$. We can show $c_{X'}(\lfloor\lambda\rfloor) \geq c_{Y'}(\lfloor\lambda\rfloor)$ in a similar way to Case 2.

For any pair of states X, Y satisfying $X \succeq Y$, it is easy to see that there exists a sequence of states $Z_1, Z_2, \ldots, Z_r$ with appropriate length satisfying $X = Z_1 \cdot\!\succ Z_2 \cdot\!\succ \cdots \cdot\!\succ Z_r = Y$. Then applying the above claim repeatedly, we obtain that $\phi(X, \lambda) = \phi(Z_1, \lambda) \succeq \phi(Z_2, \lambda) \succeq \cdots \succeq \phi(Z_r, \lambda) = \phi(Y, \lambda)$. □

Lemma 3.2 *The function g^k_{ij} satisfies the* alternating inequalities*,*

$$g^{k+1}_{ij}(l) \leq g^k_{ij}(l) \leq g^{k+1}_{ij}(l+1), \quad \forall k \in \{1, \ldots, K\},\ \forall l \in \{1, \ldots, k\}.$$

Proof: First, we prove the former inequality $g^{k+1}_{ij}(l) \leq g^k_{ij}(l)$ as follows,

$$\begin{aligned} \frac{g^k_{ij}(l)}{g^{k+1}_{ij}(l)} &= \frac{\sum_{s=0}^{l} \alpha_i^s\alpha_j^{k-s}}{A^k_{ij}} \frac{A^{k+1}_{ij}}{\sum_{s=0}^{l} \alpha_i^s\alpha_j^{k+1-s}} = \frac{A^{k+1}_{ij}}{\alpha_j A^k_{ij}} \\ &= \frac{\sum_{s=0}^{k+1} \alpha_i^s\alpha_j^{k+1-s}}{\alpha_j \sum_{s=0}^{k} \alpha_i^s\alpha_j^{k-s}} = \frac{\sum_{s=0}^{k+1} \alpha_i^s\alpha_j^{k+1-s}}{\sum_{s=0}^{k} \alpha_i^s\alpha_j^{k+1-s}} \geq 1. \end{aligned}$$

Next, we prove the latter inequality $g_{ij}^k(l) \leq g_{ij}^{k+1}(l+1)$ as follows,

$$\frac{g_{ij}^{k+1}(l+1)}{g_{ij}^k(l)} = \frac{A_{ij}^k}{A_{ij}^{k+1}} \frac{\sum_{s=0}^{l+1} \alpha_i^s \alpha_j^{k+1-s}}{\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}} = \frac{\left(\sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}\right)\left(\sum_{s=0}^{l+1} \alpha_i^s \alpha_j^{k+1-s}\right)}{\left(\sum_{s=0}^{k+1} \alpha_i^s \alpha_j^{k+1-s}\right)\left(\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}\right)}$$

$$= \frac{\left(\sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}\right)\left(\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k+1-s} + \alpha_i^{l+1}\alpha_j^{k-l}\right)}{\left(\sum_{s=0}^{k} \alpha_i^s \alpha_j^{k+1-s} + \alpha_i^{k+1}\right)\left(\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}\right)}$$

$$= \frac{\left(\sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}\right)\left(\alpha_j \sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}\right) + \alpha_i^{l+1}\alpha_j^{k-l}\left(\sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}\right)}{\left(\alpha_j \sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}\right)\left(\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}\right) + \alpha_i^{k+1}\left(\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}\right)}$$

$$= \frac{\left(\alpha_i^{l+1}\alpha_j^{k-l}\right)^{-1}\alpha_j\left(\sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}\right)\left(\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}\right) + \sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}}{\left(\alpha_i^{l+1}\alpha_j^{k-l}\right)^{-1}\alpha_j\left(\sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}\right)\left(\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}\right) + \alpha_i^{k-l}\alpha_j^{l-k}\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}}$$

$$= \frac{\left(\alpha_i^{l+1}\alpha_j^{k-l}\right)^{-1}\alpha_j\left(\sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}\right)\left(\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}\right) + \sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}}{\left(\alpha_i^{l+1}\alpha_j^{k-l}\right)^{-1}\alpha_j\left(\sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}\right)\left(\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}\right) + \sum_{s=k-l}^{k} \alpha_i^s \alpha_j^{k-s}} \geq 1.$$

Thus we obtain the claim. □

Since $\mathcal{M}_{\mathrm{P}}$ is a monotone chain, we can design a perfect sampler based on monotone CFTP. We could also employ Wilson's read once algorithm [24] and Fill's interruptible algorithm [9, 10], each of which also gives a perfect sampler.

3.2 Expected Running Time

Here, we assume a condition, which gives expected polynomial time monotone CFTP algorithm.

Condition 1 *Parameters are arranged in non-increasing order i.e.,* $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$.

The following is a main result of this paper.

Theorem 3.3 *Under Condition 1, the expected running time of our perfect sampler is bounded by* $\mathrm{O}(n^3 \ln K)$*, where* n *is the number of nodes and* K *is the number of customers in a closed Jackson network.*

We can show Theorem 3.3 by estimating the expectation of *coalescence time* $T_* \in \mathbb{Z}_{++}$ defined by $T_* \stackrel{\text{def.}}{=} \min\{t > 0 \mid \exists y \in \Xi,\ \forall x \in \Xi,\ y = \Phi_{-t}^0(x, \mathbf{\Lambda})\}$. Note that T_* is a random variable.

First, we show the following lemma.

Lemma 3.4 *Under Condition 1, the mixing rate* τ *of our Markov chain* $\mathcal{M}$ *satisfies*

$$\tau \leq n(n-1)^2(1 + \ln Kn).$$

Proof: Let $G = (\Xi, \mathcal{E})$ be an undirected simple graph with vertex set Ξ and edge set $\mathcal{E}$ defined as follows. A pair of vertices $\{X, Y\}$ is an edge if and only if $(1/2)\sum_{i=1}^{n} |X_i - Y_i| = 1$. Clearly, the graph G is connected. For each edge $e = \{X, Y\} \in \mathcal{E}$, there exists a unique pair of indices $j_1, j_2 \in \{1, \ldots, n\}$, called the *supporting pair* of e, satisfying

$$|X_i - Y_i| = \begin{cases} 1 & (i = j_1, j_2), \\ 0 & \text{(otherwise)}. \end{cases}$$

We define the length $l(e)$ of an edge $e = \{X, Y\} \in \mathcal{E}$ by

$$l(e) \stackrel{\text{def.}}{=} (1/(n-1)) \sum_{i=1}^{j^*-1} (n - i)$$

where $j^* = \max\{j_1, j_2\} \geq 2$ and $\{j_1, j_2\}$ is the supporting pair of e. Note that $1 \leq \min_{e\in\mathcal{E}} l(e) \leq \max_{e\in\mathcal{E}} l(e) \leq n/2$. For each pair $X, Y \in \Xi$, we define the distance $d(X, Y)$ be the length of a shortest path between X and Y on G. Clearly, the diameter of G, i.e., $\max_{(X,Y)\in\Xi^2} d(X, Y)$, is bounded by $Kn/2$, since $d(X, Y) \leq (n/2)\sum_{i=1}^{n}(1/2)|X_i - Y_i| \leq (n/2)K$ for any $(X, Y) \in \Xi^2$. The definition of edge length implies that for any edge $\{X, Y\} \in \mathcal{E}$, $d(X, Y) = l(\{X, Y\})$.

We define a joint process $(X, Y) \to (X', Y')$ as $(X, Y) \to (\phi(X, \Lambda), \phi(Y, \Lambda))$ with uniform real random number $\Lambda \in [1, n)$ and the update function ϕ defined in the previous subsection. Now we show that

$$\mathrm{E}[d(X', Y')] \leq \beta \cdot d(X, Y) \text{ where } \beta = 1 - 1/(n(n-1)^2), \tag{2}$$

for any pair $\{X, Y\} \in \mathcal{E}$. In the following, we denote the supporting pair of $\{X, Y\}$ by $\{j_1, j_2\}$. Without loss of generality, we can assume that $j_1 < j_2$, and $X_{j_2} + 1 = Y_{j_2}$.

Case 1: When $\lfloor\Lambda\rfloor = j_2 - 1$, we will show that

$$\mathrm{E}[d(X', Y') \mid \lfloor\Lambda\rfloor = j_2 - 1] \leq d(X, Y) - (1/2)(n - j_2 + 1)/(n - 1).$$

In case $j_1 = j_2 - 1$, $X' = Y'$ with conditional probability 1. Hence $d(X', Y') = 0$. In the following, we consider the case $j_1 < j_2 - 1$. Put $k' = X_{j_2-1} + X_{j_2}$ and $k'' = Y_{j_2-1} + Y_{j_2}$. Since $X_{j_2} + 1 = Y_{j_2}$, $k' + 1 = k''$ holds. From the definition of the update function of our Markov chain, we have the followings,

$$\begin{aligned} X'_{j_2-1} = l \quad &\Leftrightarrow \quad [g^{k'}_{(j_2-1)j_2}(l-1) \leq \Lambda - \lfloor\Lambda\rfloor < g^{k'}_{(j_2-1)j_2}(l)] \\ Y'_{j_2-1} = l \quad &\Leftrightarrow \quad [g^{k'+1}_{(j_2-1)j_2}(l-1) \leq \Lambda - \lfloor\Lambda\rfloor < g^{k'+1}_{(j_2-1)j_2}(l)]. \end{aligned}$$

Now, the alternating inequalities

$$\begin{aligned} 0 \quad < \quad & g^{k'+1}_{(j_2-1)j_2}(0) = g^{k'}_{(j_2-1)j_2}(0) \leq g^{k'+1}_{(j_2-1)j_2}(1) \leq g^{k'}_{(j_2-1)j_2}(1) \leq \cdots \\ \leq \quad & g^{k'+1}_{(j_2-1)j_2}(k') \leq g^{k'}_{(j_2-1)j_2}(k') = g^{k'+1}_{(j_2-1)j_2}(k'+1) = 1, \end{aligned}$$

hold. Thus we have

$$\begin{pmatrix} X'_{j_2-1} \\ Y'_{j_2-1} \end{pmatrix} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ldots, \begin{pmatrix} k' \\ k' \end{pmatrix}, \begin{pmatrix} k' \\ k'+1 \end{pmatrix} \right\}.$$

If $X'_{j_2-1} = Y'_{j_2-1}$, the supporting pair of $\{X', Y'\}$ is $\{j_1, j_2\}$ and so $d(X', Y') = d(X, Y)$. If $X'_{j_2-1} \neq Y'_{j_2-1}$, the supporting pair of $\{X', Y'\}$ is $\{j_1, j_2 - 1\}$ and so $d(X', Y') = d(X, Y) - (n - j_2 + 1)/(n - 1)$.

Lemma 3.5 (proved later) implies that if $\alpha_{j_2-1} \geq \alpha_{j_2}$, then

$$\begin{aligned} &\Pr[X'_{j_2-1} \neq Y'_{j_2-1} \mid \lfloor \Lambda \rfloor = j_2 - 1] - \Pr[X'_{j_2-1} = Y'_{j_2-1} \mid \lfloor \Lambda \rfloor = j_2 - 1] \\ &= \textstyle\sum_{l=0}^{k'} \left(g^{k'}_{(j_2-1),j_2}(l) - g^{k'+1}_{(j_2-1),j_2}(l) \right) \\ &\quad - \textstyle\sum_{l=1}^{k'} \left(g^{k'+1}_{(j_2-1),j_2}(l) - g^{k'}_{(j_2-1),j_2}(l-1) \right) - g^{k'+1}_{(j_2-1),j_2}(0) \geq 0. \end{aligned}$$

Hence

$$\begin{aligned} \Pr[X'_{j_2-1} = Y'_{j_2-1} \mid \lfloor \Lambda \rfloor = j_2 - 1] &\leq (1/2), \\ \Pr[X'_{j_2-1} \neq Y'_{j_2-1} \mid \lfloor \Lambda \rfloor = j_2 - 1] &\geq (1/2). \end{aligned}$$

Thus we obtain that

$$\begin{aligned} \mathrm{E}[d(X', Y') | \lfloor \Lambda \rfloor = j_2 - 1] &\leq (1/2)d(X, Y) + (1/2)(d(X, Y) - (n - j_2 + 1)/(n - 1)) \\ &= d(X, Y) - (1/2)(n - j_2 + 1)/(n - 1). \end{aligned}$$

Case 2: When $\lfloor \Lambda \rfloor = j_2$, we can show that $\mathrm{E}[d(X', Y') | \lfloor \Lambda \rfloor = j_2] \leq d(X, Y) + (1/2)(n - j_2)/(n - 1)$ in a similar way to Case 1.

Case 3: When $\lfloor \Lambda \rfloor \neq j_2 - 1$ and $\lfloor \Lambda \rfloor \neq j_2$, it is easy to see that the supporting pair $\{j'_1, j'_2\}$ of $\{X', Y'\}$ satisfies $j_2 = \max\{j'_1, j'_2\}$. Thus $d(X, Y) = d(X', Y')$.

The probability of appearance of Case 1 is equal to $1/(n-1)$, and that of Case 2 is less than or equal to $1/(n-1)$. From the above,

$$\begin{aligned} \mathrm{E}[d(X', Y')] &\leq d(X, Y) - \frac{1}{n-1} \cdot \frac{1}{2} \cdot \frac{n - j_2 + 1}{n - 1} + \frac{1}{n-1} \cdot \frac{1}{2} \cdot \frac{n - j_2}{n - 1} = d(X, Y) - \frac{1}{2(n-1)^2} \\ &\leq \left(1 - \frac{1}{2(n-1)^2} \cdot \frac{1}{\max_{\{X,Y\} \in \mathcal{E}} \{d(X, Y)\}} \right) d(X, Y) = \left(1 - \frac{1}{n(n-1)^2} \right) d(X, Y). \end{aligned}$$

Since the diameter of G is bounded by $Kn/2$, Theorem 2.2 implies that the mixing rate τ satisfies $\tau \leq n(n-1)^2(1 + \ln(Kn/2))$. □

Lemma 3.5 *When $\alpha_i \geq \alpha_j > 0$, the inequality*

$$\sum_{l=0}^{k} \left(g^k_{ij}(l) - g^{k+1}_{ij}(l) \right) - \sum_{l=1}^{k} \left(g^{k+1}_{ij}(l) - g^k_{ij}(l-1) \right) - g^{k+1}_{ij}(0) \geq 0.$$

holds.

Proof: We can transform the left-hand side as

$$\begin{aligned}
&\textstyle\sum_{l=0}^{k} \left(g_{ij}^{k}(l) - g_{ij}^{k+1}(l)\right) - \sum_{l=1}^{k} \left(g_{ij}^{k+1}(l) - g_{ij}^{k}(l-1)\right) - g_{ij}^{k+1}(0) \\
&= \textstyle\sum_{l=0}^{k} \left(g_{ij}^{k}(l) - g_{ij}^{k+1}(l)\right) - \sum_{l=0}^{k-1} \left(g_{ij}^{k+1}(k-l) - g_{ij}^{k}(k-l-1)\right) - g_{ij}^{k+1}(0) \\
&= \textstyle\sum_{l=0}^{k-1} \left(g_{ij}^{k}(l) - g_{ij}^{k+1}(l) - g_{ij}^{k+1}(k-l) + g_{ij}^{k}(k-l-1)\right) + 1 - g_{ij}^{k+1}(k) - g_{ij}^{k+1}(0),
\end{aligned}$$

and we can see that,

$$\begin{aligned}
1 - g_{ij}^{k+1}(k) - g_{ij}^{k+1}(0) &= 1 - \frac{\sum_{s=0}^{k} \alpha_i^s \alpha_j^{k+1-s}}{\sum_{s=0}^{k+1} \alpha_i^s \alpha_j^{k+1-s}} - \frac{\sum_{s=0}^{0} \alpha_i^s \alpha_j^{k+1-s}}{\sum_{s=0}^{k+1} \alpha_i^s \alpha_j^{k+1-s}} \\
&= \frac{\alpha_i^{k+1}}{\sum_{s=0}^{k+1} \alpha_i^s \alpha_j^{k+1-s}} - \frac{\alpha_j^{k+1}}{\sum_{s=0}^{k+1} \alpha_i^s \alpha_j^{k+1-s}} \geq 0,
\end{aligned}$$

since $\alpha_i \geq \alpha_j$ (Condition 1). Thus it is enough to show that

$$g_{ij}^{k}(l) - g_{ij}^{k+1}(l) - g_{ij}^{k+1}(k-l) + g_{ij}^{k}(k-l-1) \geq 0.$$

for any l $(0 \leq l \leq k-1)$. The above inequalities are obtained as follows,

$$\begin{aligned}
&g_{ij}^{k}(l) - g_{ij}^{k+1}(l) - g_{ij}^{k+1}(k-l) + g_{ij}^{k}(k-l-1) \\
&= g_{ij}^{k}(l) - g_{ij}^{k+1}(l) - \frac{\sum_{s=0}^{k-l} \alpha_i^s \alpha_j^{k+1-s}}{\sum_{s=0}^{k+1} \alpha_i^s \alpha_j^{k+1-s}} + \frac{\sum_{s=0}^{k-l-1} \alpha_i^s \alpha_j^{k-s}}{\sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}} \\
&= g_{ij}^{k}(l) - g_{ij}^{k+1}(l) - \left(1 - \frac{\sum_{s=k-l+1}^{k+1} \alpha_i^s \alpha_j^{k+1-s}}{\sum_{s=0}^{k+1} \alpha_i^s \alpha_j^{k+1-s}}\right) + \left(1 - \frac{\sum_{s=k-l}^{k} \alpha_i^s \alpha_j^{k-s}}{\sum_{s=0}^{k} \alpha_i^s \alpha_j^{k-s}}\right) \\
&= \frac{\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}}{A_{ij}^{k}} - \frac{\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k+1-s}}{A_{ij}^{k+1}} + \frac{\sum_{s=k-l+1}^{k+1} \alpha_i^s \alpha_j^{k+1-s}}{A_{ij}^{k+1}} - \frac{\sum_{s=k-l}^{k} \alpha_i^s \alpha_j^{k-s}}{A_{ij}^{k}} \\
&= \frac{\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}}{A_{ij}^{k}} - \frac{\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k+1-s}}{A_{ij}^{k+1}} + \frac{\sum_{s=0}^{l} \alpha_i^{k+1-s} \alpha_j^{s}}{A_{ij}^{k+1}} - \frac{\sum_{s=0}^{l} \alpha_i^{k-s} \alpha_j^{s}}{A_{ij}^{k}} \\
&= \left(\frac{1}{A_{ij}^{k}} - \frac{\alpha_j}{A_{ij}^{k+1}}\right) \textstyle\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s} + \left(\frac{\alpha_i}{A_{ij}^{k+1}} - \frac{1}{A_{ij}^{k}}\right) \sum_{s=0}^{l} \alpha_i^{k-s} \alpha_j^{s} \\
&= \frac{\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s}}{A_{ij}^{k} A_{ij}^{k+1}} \left(\textstyle\sum_{s=0}^{k+1} \alpha_i^s \alpha_j^{k+1-s} - \sum_{s=0}^{k} \alpha_i^s \alpha_j^{k+1-s}\right) \\
&\quad + \frac{\sum_{s=0}^{l} \alpha_i^{k-s} \alpha_j^{s}}{A_{ij}^{k} A_{ij}^{k+1}} \left(\textstyle\sum_{s=1}^{k+1} \alpha_i^s \alpha_j^{k+1-s} - \sum_{s=0}^{k+1} \alpha_i^s \alpha_j^{k+1-s}\right) \\
&= \frac{1}{A_{ij}^{k} A_{ij}^{k+1}} \left(\alpha_i^{k+1} \textstyle\sum_{s=0}^{l} \alpha_i^s \alpha_j^{k-s} - \alpha_j^{k+1} \sum_{s=0}^{l} \alpha_i^{k-s} \alpha_j^{s}\right) \\
&= \frac{1}{A_{ij}^{k} A_{ij}^{k+1}} \textstyle\sum_{s=0}^{l} \left(\alpha_i^{k+1+s} \alpha_j^{k-s} - \alpha_i^{k-s} \alpha_j^{k+1+s}\right) \\
&= \frac{1}{A_{ij}^{k} A_{ij}^{k+1}} \textstyle\sum_{s=0}^{l} \left(\alpha_i^{k-s} \alpha_j^{k-s} \left(\alpha_i^{2s+1} - \alpha_j^{2s+1}\right)\right) \geq 0,
\end{aligned}$$

since $\alpha_i \geq \alpha_j$ (Condition 1). □

Next we estimate the expectation of the coalescence time of $\mathcal{M}_{\mathrm{P}}$.

Lemma 3.6 *Under Condition 1, the coalescence time T_* of $\mathcal{M}_{\mathrm{P}}$ satisfies* $\mathrm{E}[T_*] = \mathrm{O}(n^3 \ln(Kn))$.

Proof: Let $G = (\Xi, \mathcal{E})$ be the undirected graph and $d(X, Y)$, $\forall X, \forall Y \in \Xi$, be the metric on G, both of which are defined in the proof of Lemma 3.4. We define $D \stackrel{\text{def.}}{=} d(x_{\max}, x_{\min})$ and $\tau_0 \stackrel{\text{def.}}{=} n(n-1)^2(1 + \ln D)$. By using the inequality (2) obtained in the proof of Lemma 3.4, we have

$$\begin{aligned}
\Pr[T_* > \tau_0] &= \Pr\left[\Phi^0_{-\tau_0}(x_{\max}, \mathbf{\Lambda}) \neq \Phi^0_{-\tau_0}(x_{\min}, \mathbf{\Lambda})\right] = \Pr\left[\Phi^{\tau_0}_0(x_{\max}, \mathbf{\Lambda}) \neq \Phi^{\tau_0}_0(x_{\min}, \mathbf{\Lambda})\right] \\
&\leq \textstyle\sum_{(X,Y)\in \Xi^2} d(X, Y)\Pr\left[X = \Phi^{\tau_0}_0(x_{\max}, \mathbf{\Lambda}), Y = \Phi^{\tau_0}_0(x_{\min}, \mathbf{\Lambda})\right] \\
&= \mathrm{E}\left[d\left(\Phi^{\tau_0}_0(x_{\max}, \mathbf{\Lambda}), \Phi^{\tau_0}_0(x_{\min}, \mathbf{\Lambda})\right)\right] \leq \left(1 - \frac{1}{n(n-1)^2}\right)^{\tau_0} d(x_{\max}, x_{\min}) \\
&= \left(1 - \frac{1}{n(n-1)^2}\right)^{n(n-1)^2(1+\ln D)} D \leq \mathrm{e}^{-1}\mathrm{e}^{-\ln D} D = \frac{1}{\mathrm{e}}.
\end{aligned}$$

By The *submultiplicativity* of coalescence time ([23]), for any $k \in \mathbb{Z}_+$, $\Pr[T_* > k\tau_0] \leq (\Pr[T_* > \tau_0])^k \leq (1/\mathrm{e})^k$. Thus

$$\begin{aligned}
\mathrm{E}[T_*] &= \textstyle\sum_{t=0}^{\infty} t\Pr[T_* = t] \leq \tau_0 + \tau_0\Pr[T_* > \tau_0] + \tau_0\Pr[T_* > 2\tau_0] + \cdots \\
&\leq \tau_0 + \tau_0/\mathrm{e} + \tau_0/\mathrm{e}^2 + \cdots = \tau_0/(1 - 1/\mathrm{e}) \leq 2\tau_0.
\end{aligned}$$

Clearly $D \leq Kn$, then we obtain the result that $\mathrm{E}[T_*] = \mathrm{O}(n^3 \ln(Kn))$. □

Proof of Theorem 3.3 Let T_* be the coalescence time of our chain. Clearly T_* is a random variable. Put $m = \lceil \log_2 T_* \rceil$. Algorithm 2 terminates when we set the starting time period $T = -2^m$ at $(m+1)$st iteration. Then the total number of simulated transitions is bounded by $2(2^0 + 2^1 + 2^2 + \cdots + 2^K) < 2 \cdot 2 \cdot 2^m \leq 8T_*$, since we need to execute two chains from both $x_{\max}$ and $x_{\min}$. Thus the expectation of total number of transitions of $\mathcal{M}$ is bounded by $\mathrm{O}(\mathrm{E}[8T_*]) = \mathrm{O}(n^3 \ln Kn)$. □

4 Concluding Remarks

We proposed a perfect sampler based on monotone CFTP for closed Jackson networks with single servers. We estimated the mixing rate of our chain and showed the running time of the perfect sampler is $\mathrm{O}(n^3 \ln(Kn))$. One of future works is extension to closed Jackson networks with multiple serves. Extension to closed BCMP networks is another.

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